THE E. STUDY MAPS OF CIRCLES ON DUAL HYPERBOLIC AND LORENTZIAN UNIT SPHERES H_0^2 AND S_1^2

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Abstract

H.H. Hacısalihoğlu ('Study map of a circle', *Journal of the Faculty of Science of Karadeniz Technical University* **1** (7) (1977), 69–80) shows that the E. Study map of a circle on a dual unit sphere S^2 is a family of hyperboloids of one sheet with two parameters. In this paper we calculate and discuss the E. Study maps of circles which lie on the dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2 at the dual Lorentzian space D_1^3. Furthermore, we examine some special cases, each of which is a geometrical result.

1. Introduction

Dual numbers were introduced by W.K. Clifford (1849–79) as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [5]. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of space of lines R^3.

If we take the Minkowski 3-space R_1^3 instead of R^3 the E. Study mapping can be stated as follows. The dual time-like and space-like unit vectors of dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2 at the dual Lorentzian space D_1^3 are in one-to-one correspondence with the directed time-like and space-like lines of the space of Lorentzian lines R_1^3, respectively [6]. Then a differentiable curve on H_0^2 corresponds

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to a time-like ruled surface at R^3_1. Similarly the time-like (resp. space-like) curve on S^2_1 corresponds to any space-like (resp. time-like) ruled surface at R^3_1.

2. Basic concept

If a and a^* are real numbers and $\varepsilon^2 = 0$, the combination $A = a + \varepsilon a^*$ is called a *dual number*, where ε is dual unit.

The set of all dual numbers forms a commutative ring over the real number field and is denoted by D. Then the set

$$D^3 = \{\widetilde{a} = (A_1, A_2, A_3) | \quad A_i \in D, \ \ 1 \leq i \leq 3\}$$

is a module over the ring D which is called a *D-module* or *dual space*. The elements of D^3 are called *dual vectors*. Thus a dual vector $\widetilde{a}$ can be written

$$\widetilde{a} = a + \varepsilon a^*$$

where a and a^* are real vectors at R^3.

The *Lorentzian inner product* of dual vectors $\widetilde{a}$ and $\widetilde{b}$ is defined by

$$\langle \widetilde{a}, \widetilde{b} \rangle = \langle a, b \rangle + \varepsilon \left(\langle a, b^* \rangle + \langle a^*, b \rangle \right)$$

with $\widetilde{a} = a + \varepsilon a^*$ and $\widetilde{b} = b + \varepsilon b^*$. A dual vector $\widetilde{a}$ is said to be *time-like* if $\langle \widetilde{a}, \widetilde{a} \rangle < 0$, *space-like* if $\langle \widetilde{a}, \widetilde{a} \rangle > 0$ and *light-like* (or *null*) if $\langle \widetilde{a}, \widetilde{a} \rangle = 0$, where $\langle , \rangle$ is a Lorentzian inner product with signature $(+, +, -)$. The set of all dual vectors such that $\langle \widetilde{a}, \widetilde{a} \rangle = 0$ is called the dual *light-like cone* and is denoted by $\wedge$. The norm of a dual vector $\widetilde{a}$ is defined to be

$$|\widetilde{a}| = |a| + \varepsilon \frac{\langle a, a^* \rangle}{|a|^2}.$$

We also consider the time orientation as follows. A dual time-like vector $\widetilde{a}$ is *future-pointing* (resp. *past-pointing*) if and only if a is *future-pointing* (resp. *past-pointing*). We denote the set of all dual Lorentzian vectors by D^3_1. Then we have the following definition.

The *hyperbolic and Lorentzian unit spheres* are

$$H^2_0 = \left\{ \widetilde{a} = a + \varepsilon a^* \in D^3_1 \middle| \ \langle \widetilde{a}, \widetilde{a} \rangle = -1; \ \ a, a^* \in R^3_1 \right\}$$

and

$$S^2_1 = \left\{ \widetilde{a} = a + \varepsilon a^* \in D^3_1 \middle| \ \langle \widetilde{a}, \widetilde{a} \rangle = 1; \ \ a, a^* \in R^3_1 \right\}$$

respectively. There are two components of H^2_0. We call the components of H^2_0 passing through $(0, 0, 1)$ and $(0, 0, -1)$ a *future-pointing dual hyperbolic unit sphere* and a *past-pointing dual hyperbolic unit sphere*, and denote them by $H^{2^+}_0$ and $H^{2^-}_0$ respectively. With respect to this definition, we can write

$$H^{2^+}_0 = \left\{ \widetilde{a} = a + \varepsilon a^* \in H^2_0 \middle| \, a \textit{ is a future-pointing time-like unit vector} \right\}$$

and

$$H_0^{2^-} = \left\{ \widetilde{a} = a + \varepsilon a^* \in H_0^2 \middle| \, a \text{ is a past-pointing time-like unit vector} \right\}.$$

The *dual Lorentzian cross-product* of $\widetilde{a}$ and $\widetilde{b}$ is defined as

$$\widetilde{a} \wedge \widetilde{b} = a \wedge b + \varepsilon (a \wedge b^* + a^* \wedge b)$$

with the *Lorentzian cross-product* a and b

$$a \wedge b = (a_1, a_2, a_3) \wedge (b_1, b_2, b_3) = (a_3 b_2 - a_2 b_3, a_1 b_3 - a_3 b_1, a_1 b_2 - a_2 b_1).$$

(See [1], [2], [4] and [7] for Lorentzian basic concepts.)

3. E. Study mapping

Theorem 3.1 (E. Study map). *There exists one-to-one correspondence between directed time-like (resp. space-like) lines of R_1^3 and an ordered pair of vectors (a, a^*) such that $\langle a, a \rangle = -1$ (resp. $\langle a, a \rangle = +1$) and $\langle a, a^* \rangle = 0$* [6].

Definition 3.2. A directed time-like line in R_1^3 may be given by two points on it, p and q. If λ is any non-zero constant, the parametric equation of the line is $q = p + \lambda x$. In this case, the vector given by

$$x^* = p \wedge x = q \wedge x$$

is called *the moment* of the vector x with respect to the origin O.

This means that the direction vector x of the time-like line and its moment vector x^* are independent of the choice of the points $p, q, r, \ldots$ on the line. The vectors x and x^* are not independent of one another. They satisfy the following equations:

$$\langle x, x \rangle = 1, \quad \langle x, x^* \rangle = 0.$$

Definition 3.3. The six components x_i and x_i^* $(i = 1, 2, 3)$ of x and x^* are called *Plückerian homogeneous coordinates* of the directed time-like line ℓ_1.

Let H_0^2, 0 and $\{0; \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3 \text{ (time-like)}\}$ denote the dual hyperbolic unit sphere, the centre of H_0^2 and the dual orthonormal system at 0 respectively, where we have

$$\widetilde{e}_i = e_i + \varepsilon e_i^*, \quad 1 \leq i \leq 3, \tag{1}$$

$$\widetilde{e}_1 \wedge \widetilde{e}_2 = \widetilde{e}_3, \quad \widetilde{e}_2 \wedge \widetilde{e}_3 = -\widetilde{e}_1, \quad \widetilde{e}_3 \wedge \widetilde{e}_1 = -\widetilde{e}_2, \tag{2}$$

and

$$e_1 \wedge e_2 = e_3, \quad e_2 \wedge e_3 = -e_1, \quad e_3 \wedge e_1 = -e_2. \tag{3}$$

In this case the orthonormal system $\{0; \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ is the system of the space of lines R_1^3. The moment vectors e_i^* can be written

$$e_i^* = M0 \wedge e_i \ , \quad 1 \leq i \leq 3. \tag{4}$$

Since these moment vectors are the vectors of R_1^3 we may write

$$e_i^* = \sum_{j=1}^{3} \lambda_{ij} e_j, \qquad \lambda_{ij} \in R, \quad 1 \leq i \leq 3. \tag{5}$$

Hence (4) and (5) give us

$$\lambda_{ii} = 0, \ \lambda_{12} = -\lambda_{21}, \ \lambda_{13} = -\lambda_{31}\, , \ \lambda_{23} = \lambda_{32},$$

and so we may denote the scalars λ_{ij} by λ_i, that is $\lambda_{ij} = \lambda_i$. Then (5) reduces to

$$\begin{bmatrix} e_1^* \\ e_2^* \\ e_3^* \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 & \lambda_2 \\ -\lambda_1 & 0 & \lambda_3 \\ \lambda_2 & \lambda_3 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{6}$$

Hence the E. Study mapping can be given as a mapping from the dual Lorentzian orthogonal system, in H_0^2, to the real Lorentzian orthogonal system, in R_1^3. Using the relations (1) and (6) we can express the E. Study mapping in the matrix form as follows:

$$\begin{bmatrix} \widetilde{e}_1 \\ \widetilde{e}_2 \\ \widetilde{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1\varepsilon & \lambda_2\varepsilon \\ -\lambda_1\varepsilon & 1 & \lambda_3\varepsilon \\ \lambda_2\varepsilon & \lambda_3\varepsilon & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \tag{7}$$

Now we give the definition of dual Lorentzian orthogonal matrix.

Definition 3.4. Let A be a matrix with dual coefficient. A is said to be a *dual Lorentzian orthogonal matrix* if

$$A^{-1} = SA^tS\, ; \qquad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where S is a signature matrix at D_1^3 and $1 = (1, 0)$ (see [4] for semi-orthogonal matrix).

We know that the linear mappings are in one-to-one correspondence with the matrices. Then we may give the following theorem.

Theorem 3.5. *The E. Study mapping is a linear isometry.*

Since the Lorentzian motions in R_1^3 leave unchanged the hyperbolic angle [6] and the Lorentzian distance between two time-like lines, the correspondence mappings in the dual Lorentzian space D_1^3 leave the Lorentzian inner product invariant.

Theorem 3.6. *The Lorentzian motions in R_1^3 are in one-to-one correspondence with the dual Lorentzian orthogonal matrices.*

A *ruled surface* is a surface generated by the motion of a straight line in R^3. This line is the generator of the surface. Now we have the following definition.

Definition 3.7. A ruled surface is said to be *time-like* if the normal of surface at every point is space-like, and *space-like* if the normal of surface at every point is time-like.

A differentiable curve

$$t \in R \to \widetilde{x}(t) \in H_0^2$$

depending on a real parameter t represents a differentiable family of straight-like lines of R_1^3 which is a time-like ruled surface. The $\widetilde{x}(t)$ are the generators of the time-like surface.

Let $\widetilde{x}$ and $\widetilde{y}$ denote two different points at H_0^2 and $\widetilde{\varphi}$ denote the dual hyperbolic angle $(\widetilde{x}, \widetilde{y})$. The hyperbolic angle $\widetilde{\varphi}$ has a value $\varphi + \varepsilon\varphi^*$ which is a dual number, where φ and φ^* are the hyperbolic angle and the minimal Lorentzian distance between directed lines $\widetilde{x}$ and $\widetilde{y}$ respectively.

Theorem 3.8. *Let $\widetilde{x}, \widetilde{y} \in H_0^{2^+}$. Then we have*

$$\langle \widetilde{x}, \widetilde{y} \rangle = -\cosh \widetilde{\varphi}$$

where $\cosh \widetilde{\varphi} = -\cosh \varphi - \varepsilon\varphi^* \sinh \varphi$.

PROOF. Moment vectors x^* and y^* are independent of the choice of the points p and q on the directed time-like lines ℓ_1 and ℓ_2 that correspond to $\widetilde{x}$ and $\widetilde{y}$ in R_1^3 space of Lorentzian lines. Thus p and q points can be thought of as feet of common perpendicular of ℓ_1 and ℓ_2.

The unit vector of common perpendicular is

$$n = \mp \frac{x \wedge y}{\|x \wedge y\|}.$$

If we show the shortest distance between ℓ_1 and ℓ_2 by φ^*,

$$p - q = \mp \frac{x \wedge y}{\|x \wedge y\|} \varphi^*.$$

Then

$$\langle x, y^* \rangle = \langle x, q \wedge y \rangle = -\langle q, x \wedge y \rangle$$

$$\langle x^*, y \rangle = \langle p \wedge a, y \rangle = \langle p, x \wedge y \rangle$$

where $x^* = p \wedge a$ and $y^* = q \wedge y$.

By adding the last two equations, we get

$$\begin{aligned}\langle x, y^*\rangle + \langle x^*, y\rangle &= \langle p-q, x\wedge y\rangle \\ &= \left\langle \mp\frac{x\wedge y}{\|x\wedge y\|}, x\wedge y\right\rangle \\ &= \mp\varphi^*\ \|x\wedge y\| \\ &= \mp\varphi^* \sinh\varphi.\end{aligned}$$

If we choose the signal − (minus), we get

$$\begin{aligned}\langle\widetilde{x},\widetilde{y}\rangle &= -\cosh\varphi - \varphi^*\sinh\varphi \\ &= -(\cosh\varphi + \varphi^*\sinh\varphi).\end{aligned}$$

Thus from the Taylor formula

$$\langle\widetilde{x},\widetilde{y}\rangle = -\cos(\varphi+\varepsilon\varphi^*) = -\cos\widetilde{\varphi}.$$

Now we give the special cases of the theorem.

(a) $\langle\widetilde{x},\widetilde{y}\rangle \neq 0$. This means that the lines $\widetilde{x}$ and $\widetilde{y}$ cannot be orthogonal.

(b) $\langle\widetilde{x},\widetilde{y}\rangle =$ pure real if $\varphi^* = 0$. Then the lines $\widetilde{x}$ and $\widetilde{y}$ intersect each other.

(c) $\langle\widetilde{x},\widetilde{y}\rangle = \mp 1$ if $\varphi = 0$ and $\varphi^* = 0$. This means that the lines $\widetilde{x}$ and $\widetilde{y}$ are coincident. ■

4. The E. Study mapping of a circle on $H_0^{2^+}$

Let ℓ be the straight line corresponding to the dual time-like unit vector $\widetilde{e}_3$. If we choose the point P on ℓ then we have $\lambda_2 = \lambda_3 = 0$ and so the matrix (7) reduces to

$$\begin{bmatrix}\widetilde{e}_1\\ \widetilde{e}_2\\ \widetilde{e}_3\end{bmatrix} = \begin{bmatrix}1 & \lambda_1\varepsilon & 0\\ -\lambda_1\varepsilon & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix}. \tag{8}$$

The inverse of this mapping is

$$\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix} = \begin{bmatrix}1 & -\lambda_1\varepsilon & 0\\ \lambda_1\varepsilon & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{e}_1\\ \widetilde{e}_2\\ \widetilde{e}_3\end{bmatrix}. \tag{9}$$

Let

$$S_0^1 = \left\{\widetilde{x}\in H_0^{2+}\,\middle|\ \langle\widetilde{x},\widetilde{e}_3\rangle = -\cosh\widetilde{\varphi} = cons\tan t\right\}$$

be the circle on the sphere H_0^{2+} and a point of S_0^1 be $\widetilde{x}$. Thus the dual vector $\widetilde{x}$ can be expressed as

$$\widetilde{x} = sh\widetilde{\varphi}\cos\widetilde{\psi}\widetilde{e}_1 + sh\widetilde{\varphi}\sin\widetilde{\psi}\widetilde{e}_2 + ch\widetilde{\varphi}\widetilde{e}_3 \tag{10}$$

where $\widetilde{\varphi} = \varphi + \varepsilon\varphi^*$ and $\widetilde{\psi} = \psi + \varepsilon\psi^*$ are the dual hyperbolic angle and dual angle respectively. Since we have the relations

$$\widetilde{x} = x + \varepsilon x^*$$

$$\left.\begin{array}{ll} sh\widetilde{\varphi} = sh\varphi + \varepsilon\varphi^* ch\varphi, & \sin\widetilde{\psi} = \sin\psi + \varepsilon\psi^*\cos\psi \\ ch\widetilde{\varphi} = ch\varphi + \varepsilon\varphi^* sh\varphi, & \cos\widetilde{\psi} = \cos\psi - \varepsilon\psi^*\sin\psi \end{array}\right\}, \tag{11}$$

(8) and (10) give us the vectors x and x^* in the matrix form:

$$x = [e_1\ e_2\ e_3]\begin{bmatrix} sh\varphi\cos\psi \\ sh\varphi\sin\psi \\ ch\varphi \end{bmatrix}$$

$$x^* = [e_1\ e_2\ e_3]\begin{bmatrix} \varphi^* ch\varphi\cos\psi - (\psi^* + \lambda_1)sh\varphi\sin\psi \\ \varphi^* ch\varphi\sin\psi + (\psi^* + \lambda_1)sh\varphi\cos\psi \\ \varphi^* sh\varphi \end{bmatrix}. \tag{12}$$

On the other hand, the point $\widetilde{x}$ is on the circle whose centre is a point of the axis $\widetilde{e}_3$. Thus we may write

$$\langle\widetilde{x}, \widetilde{e}_3\rangle = -ch\widetilde{\varphi} = -ch\varphi - \varepsilon\varphi^* sh\varphi = cons\tan t, \tag{13}$$

which means that $\varphi = c_1$ (constant) and $\varphi^* = c_2$ (constant).

The equations (12) and (13) permit us to write the following relations:

$$\begin{aligned} \langle x, x\rangle &= -1, \\ \langle x, x^*\rangle &= 0, \\ \langle x, e_3\rangle + ch\varphi &= 0, \\ \langle x, e_3^*\rangle + \langle x^*, e_3\rangle + \varphi^* sh\varphi &= 0. \end{aligned} \tag{14}$$

Equations (14) have only two parameters, φ and φ^*. So (14) represents a time-like congruence in R_1^3.

Now we may calculate the equations of this congruence in Plücker coordinates. Let y denote a point of congruence. Then we have

$$y = x(\psi, \psi^*) \wedge x^*(\psi, \psi^*) + vx(\psi, \psi^*), \tag{15}$$

where $\wedge$ is the Lorentzian cross-product. If the coordinates of y are (y_1, y_2, y_3) then this gives us

$$\begin{aligned} y_1 &= -\varphi^*\sin\psi - (\psi^* + \lambda_1)ch\varphi sh\varphi\cos\psi + vsh\varphi\cos\psi \\ y_2 &= -\varphi^*\cos\psi - (\psi^* + \lambda_1)sh\varphi ch\varphi\sin\psi + vsh\varphi\sin\psi \\ y_3 &= -(\psi^* + \lambda_1)sh^2\varphi + vch\varphi. \end{aligned} \tag{16}$$

In this case, (16) gives us

$$\frac{y_1^2}{c_2^2} + \frac{y_2^2}{c_2^2} - \frac{\left[y_3 - (\psi^* + \lambda_1)\right]^2}{(c_2\cot gh\varphi)^2} = 1, \tag{17}$$

which has two parameters, ψ^* and λ_1, so it represents a time-like line congruence with degree two. The lines of this congruence are located so that

(a) the shortest Lorentzian distance of these lines and the line ℓ is $\varphi^* = c_2$;

(b) the hyperbolic angle of these lines and the line ℓ is $\varphi = c_1$.

Hence we can say that the lines of this congruence intersect the generators of a cylinder whose radius is constant φ^* and whose axis is ℓ, under the hyperbolic angle $\varphi =$ constant.

Definition 4.1. If all the lines of a time-like congruence have a constant hyperbolic angle with a definite line then the congruence is called a *time-like inclined congruence*.

According to this definition, (16) represents a time-like inclined congruence. Hence, we may give the following theorem.

Theorem 4.2. *Let S_0^1 be a circle with two parameters on the dual hyperbolic unit sphere H_0^2. The E. Study map of S_0^1 is a time-like inclined congruence with two degrees.*

Special cases

(i) The case that $\varphi^* \neq 0$ and $\varphi = 0$.

In this case, the lines of the time-like congruence coincide with the generators of the time-like cylinder which is the envelope of the lines' congruence. This means that the E. Study map of S_0^1 reduces to the time-like cylinder whose equations from (16) are

$$\begin{aligned} y_1^2 + y_2^2 &= c_2^2 \\ y_3 &= v. \end{aligned} \tag{18}$$

(ii) The case that $\varphi^* = 0$ and $\varphi = 0$.

In this case, all of the lines of the congruence coincide with the line ℓ. Indeed, (16) reduces to the time-like line ℓ:

$$\begin{aligned} y_1^2 + y_2^2 &= 0 \\ y_3 &= v. \end{aligned} \tag{19}$$

(iii) The case that $\varphi^* = 0$ and $\varphi \neq 0$.

In this case, all of the lines of congruence intersect the axis ℓ under the constant hyperbolic angle φ. We can say that the lines of the congruence are the common time-like lines of two linear line complexes. From (16) the equation of congruence is

$$y_1^2 + y_2^2 - \frac{[y_3 - (\psi^* + \varphi)]^2}{\cot gh^2\varphi} = 0. \tag{20}$$

5. The E. Study mapping of a circle on S_1^2

Let

$$S_1^1 = \left\{ \widetilde{x} \in S_1^2 \middle| \ \langle \widetilde{x}, \widetilde{e}_3 \rangle = -sh\widetilde{\varphi} = cons\tan t \right\}$$

be the circle on the sphere S_1^2 and a point of S_1^1 be $\widetilde{x}$. Thus the dual vector $\widetilde{x}$ can be expressed as

$$\widetilde{x} = ch\widetilde{\varphi}\cos\widetilde{\psi}\widetilde{e}_1 + ch\widetilde{\varphi}\sin\widetilde{\psi}\widetilde{e}_2 + sh\widetilde{\psi}\widetilde{e}_3 \tag{21}$$

where $\widetilde{\varphi} = \varphi + \varepsilon\varphi^*$ and $\widetilde{\psi} = \psi + \varepsilon\psi^*$ are the dual central angle between the vector $\widetilde{x}$ and the space-like plane $(\widetilde{e}_1, \widetilde{e}_2)$, and the dual angle, respectively. The relations (8) and (11) give us the vectors x and x^* in the matrix form:

$$x = [e_1\ e_2\ e_3]\begin{bmatrix} ch\varphi\cos\psi \\ ch\varphi\sin\psi \\ sh\varphi \end{bmatrix},$$
$$x^* = [e_1\ e_2\ e_3]\begin{bmatrix} \varphi^* ch\varphi\cos\psi - (\varphi^* + \lambda_1)ch\varphi\sin\psi \\ \varphi^* sh\varphi\sin\psi + (\varphi^* + \lambda_1)ch\varphi\cos\psi \\ \varphi^* ch\varphi \end{bmatrix}. \tag{22}$$

On the other hand, the point $\widetilde{x}$ is on the circle whose centre is a point of the axis $\widetilde{e}_3$. Thus we may write

$$\langle\widetilde{x}, \widetilde{e}_3\rangle = -sh\widetilde{\varphi} = -sh\varphi - \varepsilon\varphi^* ch\varphi = cons\tan t \tag{23}$$

which means that $\varphi = c_1$ (constant) and $\varphi^* = c_2$ (constant).

The equations (22) and (23) permit us to write the following relations:

$$\begin{aligned} \langle x, x\rangle &= -1 \\ \langle x, x^*\rangle &= 0 \\ \langle x, e_3\rangle + sh\varphi &= 0 \\ \langle x, e_3^*\rangle + \langle x^*, e_3\rangle + \varphi^* ch\varphi &= 0. \end{aligned} \tag{24}$$

The equations (24) have only two parameters, ψ and ψ^*. So (24) represents a space-like congruence in R_1^3.

Now we may calculate the equations of this congruence in Plücker coordinates. Let y denote a point of this congruence. Then we have

$$y = x(\psi, \psi^*) \wedge x^*(\psi, \psi^*) + vx(\psi, \psi^*), \tag{25}$$

where $\wedge$ is the Lorentzian cross-product.

If the coordinates of y are (y_1, y_2, y_3) then (25) gives us

$$\begin{aligned} y_1 &= -\varphi^*\sin\psi + (\psi^* + \lambda_1)ch\varphi sh\varphi\cos\psi + vch\varphi\cos\psi \\ y_2 &= \varphi^*\cos\psi + (\psi^* + \lambda_1)sh\varphi ch\varphi\sin\psi + vch\varphi\sin\psi \\ y_3 &= (\psi^* + \lambda_1)ch^2\varphi + vsh\varphi. \end{aligned} \tag{26}$$

In this case, by (26) we obtain the equation

$$\frac{y_1^2}{c_2^2} + \frac{y_2^2}{c_2^2} - \frac{\left[y_3 - (\psi^* + \lambda_1)\right]^2}{(c_2 tgh\varphi)^2} = 1 \tag{27}$$

which has two parameters, ψ^* and λ_1. So it represents a space-like line congruence with degree 2. The lines of these congruences are located so that

(a) the shortest Lorentzian distance of these lines and the line ℓ is $\varphi^* = c_2$,

(b) the central angle of these lines and the space-like plane $(\widetilde{e}_1, \widetilde{e}_2)$ is $\varphi = c_1$.

Hence we can say that the lines of this congruence intersect the generators of a time-like cylinder whose radius is $\varphi^* = cons\tan t$ and whose axis is ℓ, under the central angle $\varphi = cons\tan t$.

Definition 5.1. If all the lines of a space-like congruence have a constant central angle with a definite space-like plane, the congruence is called a *space-like inclined congruence.*

According to this definition, (26) represents a space-like inclined congruence. Hence we may give the following theorem.

Theorem 5.2. *Let S_1^1 be a circle with two parameters on the dual Lorentzian unit sphere S_1^2. The E. Study mapping of S_1^1 is a space-like inclined congruence with degree two.*

Special cases

(i) The case that $\varphi^* \neq 0$ and $\varphi = 0$.
In this case, the lines of the space-like congruence orthogonally intersect the generators of the time-like cylinder whose axis is ℓ and whose radius is φ^*. Then (26) reduces to

$$\begin{aligned} y_1^2 + y_2^2 &= \varphi^{*^2} + v^2 \\ y_3 &= \psi^* + \lambda_1. \end{aligned} \tag{28}$$

(ii) The case that $\varphi^* = 0$ and $\varphi = 0$.
In this case, all of the lines of the congruence orthogonally intersect the axis ℓ. This means that the congruence reduces the linear line complex whose axis is ℓ. Equations (26) give us that the equation of congruence is

$$\begin{aligned} y_1^2 + y_2^2 &= v^2 \\ y_3 &= \psi^* + \lambda_1. \end{aligned} \tag{29}$$

(iii) The case that $\varphi^* = 0$ and $\varphi \neq 0$.
In this case, all of the lines of congruence intersect the axis ℓ under the constant central angle φ. Then we can say that the lines of the congruence are the common time-like lines of two linear line complexes. By (26) the equation of congruence is

$$y_1^2 + y_2^2 - \frac{\left[y_3 - (\psi^* + \varphi_1)\right]^2}{tgh^2\varphi} = 0, \tag{30}$$

that is, the equation (30) is a space-like inclined congruence.

Definition 5.3. If all the lines of a space-like congruence orthogonally intersect a constant time-like line the congruence is *called a space-like recticongruence.*

Result 5.4. Let S_1^1 be a circle on S_1^2, that is,

$$S_1^1 = \left\{ \widetilde{x} \in S_1^2 \middle| \ \langle \widetilde{x}, \widetilde{e}_3 \rangle = 0 \right\}.$$

Then the E. Study mapping of S_1^1 is a space-like recticongruence.

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