

NIL R-MOD ABELIAN GROUPS

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ABSTRACT

We study mixed abelian groups such that $\text{End}(M)$, the endomorphism ring of M , has no nilpotent elements other than 0. These are the abelian groups M that only admit the trivial R -module structure for any zero-ring R .

1. Introduction

Given a ring R and an abelian group M , one can turn M into a left R -module ${}_R M$ by defining $r \bullet m = \varphi(r)(m)$ where φ is a ring homomorphism from R into $\text{End}(M)$, the endomorphism ring of M . We call ${}_R M$ the trivial module if $\varphi = 0$, i.e. $rm = 0$ for all $r \in R$, $m \in M$. In this paper we concern ourselves with the question: for which rings R and abelian groups M is the trivial module the *only* left R -module with additive group M ? We call such an M a *nil R -mod group*. This appears to be too general a problem to expect a reasonable answer, and so we restrict our investigation to the following: for which rings R is $M = 0$ the only nil R -mod group? We call such a ring a *0-nil ring*. Moreover, what can be said about an abelian group M that is nil R -mod for every zero-ring R ? We call such an M a *0-nil group*. We will show that R is 0-nil if and only if R has the following (ring) epimorphic images: $n\mathbb{Z}$ for some $n \in \mathbb{N}$, and $\mathbb{Z}/p\mathbb{Z}$ for each prime p . We are able to obtain a description of 0-nil groups M if M is torsion or mixed. Let P be the set of primes such that M_p , the *p -primary part* of M , is not equal to zero and $H = \bigcap_{p \in P} pM$. Then, if M is reduced, the *torsion subgroup* $t(M)$ of M is isomorphic to $\bigoplus_{p \in P} \mathbb{Z}(p)$ and M/H is isomorphic to a subgroup of $\prod_{p \in P} \mathbb{Z}(p)$. We provide examples to show that H may or may not be a 0-nil group and that there are 0-nil mixed groups of arbitrarily large cardinalities. Finally, we show that the only 0-nil almost completely decomposable torsion-free groups are the rigid ones, cf. [3]. Our undefined abelian group notations are standard as in [4].

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2. Notations and Preliminaries

Let R be a ring and M an additive abelian group. If there is a bilinear map $\bullet : R \times M \rightarrow M$ such that $r \bullet (s \bullet m) = (rs) \bullet m$ for all $r, s \in R$ and $m \in M$, then $M = {}_R M$ is a left R -module with module multiplication \bullet . Let $Mult(R, M)$ denote the set of all such module multiplications and $hom(R, S)$ the set of all ring homomorphisms from the ring R into the ring S . We say that ${}_R M$ is a *trivial left R -module* if \bullet is the zero map, i.e. $r \bullet m = 0$ for all $r \in R$ and $m \in M$. We are interested in pairs R, M such that all left R -modules ${}_R M$ are trivial. If M is an abelian group, then $End(M) = E(M)$ denotes the ring of endomorphism of M . The following is easy to prove.

Lemma 1. *The function $\hat{\cdot} : hom(R, E(M)) \rightarrow Mult(R, M)$ with $\hat{\varphi} = \bullet_\varphi$ for all $\varphi \in hom(R, E(M))$ and $r(\bullet_\varphi)m = (\varphi(r))(m)$ for all $r \in R$ and $m \in M$ is bijective.*

Note that \bullet_0 produces the trivial module. We call M a nil R -mod group if $Mult(R, M) = \{\bullet_0\}$.

If R is any ring, R^+ denotes the additive group of R . Forgetting multiplication, we get

Corollary 2. *If $Hom(R^+, E(M)^+) = 0$, then M is a nil mod- R group.*

Recall that $Hom(R^+, E(M)^+) \cong Hom(R^+ \otimes M, M)$, cf. [5, theorem 5.10]. If A is an abelian group, then $t(A)$ denotes the torsion subgroup of A and A_p denotes the p -primary part of $t(A)$. The following is a consequence of Corollary 2:

Corollary 3. *Each of the following conditions implies that M is a nil R -mod group:*

- (1) R^+ is divisible and M is either reduced or a torsion group;
- (2) R^+ is a torsion group and $M_p = 0$ whenever $R_p \neq 0$;
- (3) R^+ is torsion and M is torsion-free.

Lemma 4. *Let $R = \bigoplus_{i \in I} R_i$ be a ring direct sum. Then the elements of $hom(R, S)$ are in a bijective correspondence with the elements of $\prod_{i \in I} hom(R_i, S)$.*

Corollary 5. *Let R be as in Lemma 4. Then M is a nil R -mod group if and only if M_i is a nil R -mod group for each $i \in I$.*

The proofs of Corollary 3, Lemma 4 and Corollary 5 are easy and left to the reader. A consequence of Lemma 1 is the following.

Corollary 6. *If R is a zero-ring, and if M is not a nil R -mod group, then $E(M)$ contains a subring $0 \neq S$ such that S is a zero-ring.*

Example Let R be the zero-ring with $R^+ \cong \mathbb{Z}/4\mathbb{Z}$ and $M = (\mathbb{Z}/4\mathbb{Z})^+$. Define $\bullet : R \times M \rightarrow M$ by $r \bullet m = 2m$ for all $r \in R$ and $m \in M$. The fact that $\bullet \neq \bullet_0$ shows that M is not a nil R -mod group.

Definition. An abelian group M is called *0-nil* if M is a nil R -mod group for every zero-ring R .

By Lemma 1, a group M is 0-nil if and only if $\text{hom}(R, E(M)) = 0$ for each zero-ring R . Dually, a ring R is called a *0-nil ring* if $M = \{0\}$ is the *only* nil R -mod abelian group.

Example The groups $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}(n)$ with n square free, \mathbb{Q}, J_p are all nil R -mod groups for every zero-ring R , in fact for every nil ring R , because their endomorphism rings do not contain nilpotent elements other than 0. Since each endomorphism ring contains the identity, \mathbb{Z} is a 0-nil ring.

Theorem 7. A ring R is 0-nil if and only if R has the following rings as epimorphic images: $n\mathbb{Z}$ for some positive integer n , and $\mathbb{Z}/p\mathbb{Z}$ for all primes p .

PROOF. Let R be 0-nil. Then there is a non-zero-ring homomorphism $\varphi : R \rightarrow E(\mathbb{Z}) \cong \mathbb{Z}$, so $\varphi(R) = n\mathbb{Z}$ for some positive integer n . Moreover, for each prime p there exists a non-zero homomorphism $\phi_p : R \rightarrow \mathbb{Z}/p\mathbb{Z}$. Therefore $\mathbb{Z}/p\mathbb{Z}$ is an epimorphic image of R .

Conversely, suppose that $n\mathbb{Z}$, n a positive integer, and $\mathbb{Z}/p\mathbb{Z}$ are epimorphic images of R for each prime p . Let M be a group. If M is not bounded, then $\langle id_M \rangle \subseteq E(M)$ is a subring isomorphic to \mathbb{Z} and $n\mathbb{Z}$ embeds as a subring in $E(M)$ and $\text{hom}(R, E(M)) \neq 0$. Now assume that M is bounded. Then $\mathbb{Z}/(p^k\mathbb{Z})$ is a subring of $E(M)$, for some prime p and positive k . If $k = 1$, then $\text{hom}(R, E(M)) \neq 0$, since $\mathbb{Z}/p\mathbb{Z}$ is an epimorphic image of R . If $k \geq 2$, define $\psi : n\mathbb{Z} \rightarrow \mathbb{Z}/p^k\mathbb{Z}$ by $\psi(xn) = xp^{k-1} + p^k\mathbb{Z}$. Then ψ is a non-zero-ring homomorphism, which, composed with φ , yields a non-zero homomorphism $\psi \circ \varphi : R \rightarrow E(M)$, where $\varphi : R \rightarrow n\mathbb{Z}$ is an epimorphism. ■

The following lemma will help us describe mixed 0-nil groups.

Lemma 8. The following are equivalent for an abelian group M :

- (1) M is 0-nil.
- (2) For every $\varphi \in E(M)$, $\varphi^2 = 0$ implies that $\varphi = 0$.
- (3) $\text{Hom}(M/K, K) = 0$ for every subgroup K of M .
- (4) For all non-zero $\varphi, \psi \in E(M)$, if $\varphi(M) \subseteq \psi(M)$, then $\varphi \circ \psi \neq 0$.

PROOF. Clearly (1) and (2) are equivalent, because M is 0-nil if $E(M)$ does not contain a non-zero zero-ring. If $\psi \in \text{Hom}(M/K, K)$, then ψ is induced by some $\varphi \in \text{Hom}(M, K) \subseteq \text{Hom}(M, M) = \text{End}(M)$ with $\varphi(M) \subseteq K \subseteq \ker(\varphi)$. Thus $\varphi \circ \varphi = 0$ and $\varphi = 0$ follow from (2). This shows that (2) implies (3). To show that (3) implies (4), let $0 \neq \varphi, \psi \in E(M)$ with $\varphi(M) \subseteq \psi(M)$. Assume that $\varphi \circ \psi = 0$. Then $\varphi(M) \subseteq \psi(M) \subseteq \ker(\varphi)$ and φ induces an element $\hat{\varphi} \in \text{Hom}(M/\ker(\varphi), \ker(\varphi)) = 0$ by (3). Thus $\varphi = 0$, a contradiction. That (4) implies (2) follows from the special case $\varphi = \psi$. ■

Lemma 9. *Let M be 0-nil. Then*

- (1) *If $M = H \oplus K \oplus C$ is a direct sum, then $\text{Hom}(H, K) = 0$.*
- (2) *For any prime p we have $M_p \in \{0, \mathbb{Z}(p), \mathbb{Z}(p^\infty)\}$.*
- (3) *If $M_p \neq 0$, then M/M_p is p-divisible.*
- (4) *If $M_p = \mathbb{Z}(p^\infty)$ for some prime p, then $M = \mathbb{Z}(p^\infty) \oplus K$ where K is a 0-nil, p' -torsion group.*

PROOF. (1) Let $\alpha \in \text{Hom}(H, K)$. Then there is a $\varphi \in E(M)$ such that $\varphi \upharpoonright_{(K \oplus C)} = 0$ and $\varphi \upharpoonright_H = \alpha$. Note that $\varphi^2 = 0$. Since M is 0-nil, we infer $\varphi = 0$ and thus $\alpha = 0$. To show (2), let $M_p \neq 0$. If $M_p = pM_p$, then M_p is divisible and M_p is a direct sum of copies of $\mathbb{Z}(p^\infty)$. By (1) we have $M_p \cong \mathbb{Z}(p^\infty)$. Thus we may assume that $M_p/pM_p \neq 0$. Then M_p has a non-trivial p-basic subgroup B and each bounded summand of B is a summand of M as well. Thus there is a $k \geq 1$ such that M_p has a cyclic summand $\langle a \rangle$ of order p^k , and $M = \langle a \rangle \oplus K$. If $K_p \neq \{0\}$, then $\text{Hom}(\langle a \rangle, K) \neq 0$, a contradiction to (1). Thus $M_p = \langle a \rangle$. If $k \geq 2$, consider $\varphi \in \text{Hom}(\langle a \rangle, \langle p^{k-1}a \rangle)$ with $\varphi(a) = p^{k-1}a$. Then φ extends to $0 \neq \psi : M \rightarrow \langle p^{k-1}a \rangle$ with $\psi^2 = 0$, because $(k-1)^2 \geq k$. This is a contradiction to M 0-nil, and $M_p \cong \mathbb{Z}(p)$ follows.

To show (3), notice that (2) implies that M_p is a direct summand of M with $\text{Hom}(M/M_p, M_p) = 0$ by (1). Thus M/M_p is p-divisible. If $M_p \cong \mathbb{Z}(p^\infty)$, then M/M_p is a p' -torsion group and a direct summand of M. This proves (4). ■

Lemma 10. *Let $M = \oplus_{i \in I} M_i$. Then M is 0-nil if and only if M_i is 0-nil for each $i \in I$, and $\text{Hom}(M_i, M_j) = 0$ for all $i \neq j \in I$.*

PROOF. If M is 0-nil, apply Lemma 9 (1) and Lemma 8. For the converse, observe that $\text{End}(M) \cong \prod_{i \in I} \text{End}(M_i)$ and apply Lemma 8. ■

Example The direct sum in Lemma 10 cannot be replaced by a direct product. Let P be an infinite set of primes. If $M = \prod_{p \in P} \mathbb{Z}(p)$, then M is 0-nil, because $\text{End}(M)$ is a ring direct product of fields. However, if $p_0 \in P$ and $M = \mathbb{Z}(p_0^\infty) \oplus \prod_{p \in P - \{p_0\}} \mathbb{Z}(p)$, then M is not 0-nil by Lemma 9 (4).

Corollary 11. *Let M be a 0-nil group that is not reduced.*

- (1) *If \mathbb{Q} is a subgroup of M, then there is a set P of primes and $M = \mathbb{Q} \oplus (\oplus_{p \in P} \mathbb{Z}(p))$.*
- (2) *If there is a prime q such that $\mathbb{Z}(q^\infty)$ is a subgroup of M, then $M = \oplus_p M_p$ is a torsion group and for each prime p we have $M_p \in \{0, \mathbb{Z}(p), \mathbb{Z}(p^\infty)\}$.*
- (3) *If \mathbb{Q}/\mathbb{Z} is a subgroup of M, then $M = \mathbb{Q}/\mathbb{Z}$.*

PROOF. For (1), apply Lemma 9 (1) and Lemma 10. For (2) apply the same and Lemma 9 (4). Item (3) is an immediate consequence of (2). ■

3. The structure of reduced 0-nil groups

Throughout this section we assume that M is a reduced 0-nil group, \mathbb{P} is the set of all prime integers and $P = P(M) = \{p \in \mathbb{P} : M_p \neq 0\}$. Moreover, we set $T = \oplus_{p \in P} \mathbb{Z}(p)$ and $\Pi = \prod_{p \in P} \mathbb{Z}(p)$ and $S = \mathbb{Z}[\frac{1}{p} : p \in P]$ is the subring of \mathbb{Q} generated over \mathbb{Z} by

all $\frac{1}{p} \in P$. For any abelian group A we define $A[S] = \sum_{\varphi \in \text{Hom}(S, A)} \varphi(S)$, which is the largest S -module contained in A . Note that $A[S] \subseteq \bigcap_{p \in P} (\bigcap_{n \in \mathbb{N}} p^n A)$.

Theorem 12. *Let M be 0-nil, reduced and $P = P(M)$.*

(1) *If P is finite, then $M \cong \bigoplus_{p \in P} \mathbb{Z}(p) \oplus K$, such that K is torsion-free, 0-nil and p -divisible for all $p \in P$.*

(2) *If P is infinite, let $H = M[S]$. Then $t(M) = T \subseteq M/H \subseteq \Pi$ and H is fully invariant in M .*

(3) *If $P = \mathbb{P}$ is the set of all primes, then M is a subgroup of Π .*

PROOF. By Lemma 9, $T = t(M)$. If P is finite, $t(M)$ is a summand of M by Lemma 9 (2). Now apply Lemma 8 (3) to show (1). Let $P = \{p_i : i \in \mathbb{N}\}$ be infinite and define $T_k = \bigoplus_{i=1}^k \mathbb{Z}(p_i)$. Then T_k is a summand of M and $M = T_k \oplus A_k$. Note that $\text{Hom}(A_k, T_k) = 0$, because M is 0-nil and Lemma 9 (1) applies. Thus A_k is p_j -divisible for $1 \leq j \leq k$. Thus $p_1 p_2 \dots p_k M = p_1 p_2 \dots p_k A = A_k$ and therefore the A_k 's form a descending chain of summands of M . Note that $t(A_k) = \bigoplus_{j=k+1}^{\infty} \mathbb{Z}(p_j)$. Let $H = \bigcap_{k \in \mathbb{N}} A_k$. Then H is torsion-free, p_j -divisible for all $j \in \mathbb{N}$, and $H = M[S]$ follows. Moreover, M/H is Hausdorff in the topology τ induced by the subgroups $\{A_k/H : k \in \mathbb{N}\}$, and $T \cong (T \oplus H)/H$ is dense in M . Thus M/H is a subgroup of the completion of T with respect to τ , which, restricted to T , coincides with the product topology induced by Π . Thus $T \subseteq M/H \subseteq \Pi$. Clearly, H is fully invariant in M .

If $P = \mathbb{P}$, then H is divisible and thus $H = 0$. ■

The next theorem will be a tool to construct examples of 0-nil groups.

Theorem 13. *Let P be an infinite set of primes, $T = \bigoplus_{p \in P} \mathbb{Z}(p)$, and $S = \mathbb{Z}[\frac{1}{p} : p \in P]$. Moreover, let G be an abelian group and H a subgroup of G such that H is a torsion-free S -module. If $G/H \cong \bigoplus_{\mu} \mathbb{Q}$, $1 \leq \mu \leq 2^{\aleph_0}$, and $\text{Hom}(G, H) = 0$, then there exists a 0-nil group M with $t(M) = T$, $M[S] \cong H$, and M/H is isomorphic to a pure subgroup of Π .*

PROOF. Since Π/T is isomorphic to a direct sum of 2^{\aleph_0} copies of \mathbb{Q} , there is a pure subgroup L of Π with $G/H \cong \bigoplus_{\mu} \mathbb{Q} \cong L/T$, and $\alpha : G \rightarrow \bigoplus_{\mu} \mathbb{Q}$ and $\beta : L \rightarrow \bigoplus_{\mu} \mathbb{Q}$ are the epimorphisms that induce the given isomorphisms. Thus there is a commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & T & = & T & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & H & \rightarrow & M & \rightarrow & L \rightarrow 0, \text{ where } \alpha, \beta \text{ are as given and } M \text{ is their} \\
 & & \parallel & & \downarrow & & \downarrow^{\beta} \\
 0 & \rightarrow & H & \rightarrow & G & \xrightarrow{\alpha} & \bigoplus_{\mu} \mathbb{Q} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

pull-back, cf. [4, section 10]. Note that $\bigcap_{p \in P} pL = 0$, which implies that $H = \bigcap_{p \in P} pM$

and H is fully invariant in M . Let $\varphi \in \text{End}(M)$ with $\varphi^2 = 0$. Since L is 0-nil, $\varphi(M) \subseteq H$ and H is torsion-free. Thus $T \subseteq \ker(\varphi)$ and φ induces $\hat{\varphi} \in \text{Hom}(M/T, H) \cong \text{Hom}(G, H) = 0$. Thus $\hat{\varphi} = 0$, which implies that $\varphi = 0$. This shows that M is 0-nil with $M[S] = H$. ■

We will now construct some examples of 0-nil groups M , and we will always assume that $P \neq \mathbb{P}$.

Example of a 0-nil group M with $H = M[S]$ not a 0-nil group

Let $P \neq \mathbb{P}$ be an infinite set of primes and $S = \mathbb{Z}[\frac{1}{p} : p \in P]$. Let $H = \bigoplus_{i \in I} e_i S$, $|I| \leq 2^{\aleph_0}$ and fix $\{\pi_i : i \in I\}$, a set of algebraically independent (over \mathbb{Z}_p) elements in J_p , $p \notin P$, the ring of p -adic integers. Let $G_i = \langle e_i S, e_i \pi_i \mathbb{Z}_p \rangle_*$, a p -pure subgroup of J_p . Note that $G_i/e_i S \cong \mathbb{Q}$ and $\text{Hom}(G_i, S) = 0 = \text{Hom}(G, H)$ where $G = \bigoplus_{i \in I} G_i$. Thus there exist 0-nil groups with H a free S -module, which, if $|I| \geq 2$, is not 0-nil.

This example might suggest that 0-nil groups are limited in cardinality. This is not the case.

Example of 0-nil groups of arbitrarily large cardinality

Let S be as above, p a prime not in P , and H a torsion-free S -module with $\text{End}(H) \cong S$ such that H is an \aleph_1 -free S -module. Of course, H is 0-nil. Such modules H exist in abundance and of arbitrarily large cardinality, cf. [1], [2]. Pick $h \in H$ such that hS is a pure submodule of H , which can be done since H is an \aleph_1 -free S -module. Let π_j be one of the p -adic numbers used in the previous example, and let $= \langle H, h\pi_j \mathbb{Z}_p \rangle_*$, a p -pure subgroup of $H \otimes J_p$. Since $\text{End}(H) = S$, we get $\text{Hom}(G, H) = 0$ and $G/H \cong \mathbb{Q}$. We apply Theorem 13 to obtain a 0-nil group M with $M/(T \oplus H) \cong \mathbb{Q}$.

In light of the previous two examples it might be tempting to conjecture that there is some cardinality restriction on 0-nil groups M with $H = M[S]$ not 0-nil. This is not so. Let R be the ring of all 2×2 -matrices over S , or any other \aleph_1 -free S -algebra with nilpotent elements. Let H be again a torsion-free S -module with $\text{End}(H) \cong R$ and H an \aleph_1 -free R -module. Then H is not 0-nil. Let $h \in H$ such that hS is a pure S -submodule of H . As before, for $p \notin P$, define $= \langle H, h\pi_j \mathbb{Z}_p \rangle_*$. Again, $\text{Hom}(G, H) = 0$, H is not 0-nil, but there is a 0-nil group M with $H = M[S]$ and $M/(T \oplus H) \cong \mathbb{Q}$.

4. Almost completely decomposable 0-nil groups

We now consider torsion-free 0-nil groups M . By Lemma 9 (1), such a group M is either reduced or $M \cong \mathbb{Q}$. We assume that M is reduced. If M is not cotorsion-free, then M has a summand isomorphic to J_p and $M = J_p \oplus A$, where A is p -divisible. Let P be the set of primes such that J_p is contained in M . Then we construct a descending chain of subgroups A_k such that $M = \bigoplus_{i=1}^k \bigoplus_{p \in P} A_k$ and $p_1 p_2 \dots p_k A = A_k$. Let $S = \mathbb{Z}[\frac{1}{p} : p \in P]$. Then $M[S] = \bigcap_{p \in P} pM$ and $H = M[S]$ is cotorsion-free and $\bigoplus_{p \in P} J_p \subseteq M/H \subseteq \prod_{p \in P} J_p$.

Recall that each cotorsion-free ring is the endomorphism ring of a cotorsion-free abelian group, cf. [1]. Thus we can construct many examples of 0-nil, torsion-free, not cotorsion-free abelian groups.

A torsion-free abelian group M of finite rank is almost completely decomposable (acd) if there is some $m \in \mathbb{N}$ and $C = \bigoplus_{i=1}^n A_i$, a subgroup of M , where A_i is a subgroup of \mathbb{Q} for all $1 \leq i \leq n$, such that $mM \subseteq C$. The acd group M is called rigid [3] if $\text{Hom}(A_i, A_j) = 0$ for all $i \neq j$. We conclude this paper with:

Proposition 14. *An acd group M is 0-nil if and only if M is rigid.*

PROOF. If M is rigid, then each A_i , which we may choose to be pure in M , is fully invariant in M , and it follows that $\text{End}(M)$ is a subring of $\bigoplus_{i=1}^n \text{End}(A_i)$, where $\text{End}(A_i)$ is a subring of the field \mathbb{Q} . Thus M is 0-nil.

If M is not rigid, then there is $i \neq j$ and $\text{Hom}(A_i, A_j) \neq 0$. Thus there is $\varphi \in \text{End}(C)$ with $\varphi^2 = 0 \neq \varphi$. Let $\psi = m\varphi \in \text{End}(M)$ and observe that $\psi^2 = 0 \neq \psi$. Thus M is not 0-nil.

(The *uniform*, rigid acd groups have been classified in terms of numerical near-isomorphism invariants in [3].)

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