ON AN ANGULAR DERIVATIVE RESULT OF FAN

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Abstract

The Julia–Wolff–Carathéodory result on angular derivatives in the complex plane was proved for operator-valued holomorphic functions by Fan and, shortly after, for more general operator spaces by Włodarczyk. Here we generalise the result to include functions that take values in a complex Banach space whose open unit ball is a bounded symmetric domain.

1. Introduction

We begin with the original Julia–Wolff–Carathéodory (JWC) theorem regarding the limit of a holomorphic function and its derivative along a non-tangential approach to the boundary. Indeed we will state two versions of this classical result: one for the open unit disc Δ and one for the open right half-plane $\Pi = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. The two versions are equivalent via the Cayley transformation of Δ onto Π given by $\varphi(z) = \dfrac{1+z}{1-z}$. The notion of angular or non-tangential approach is formed by consideration of 'angular regions' in Δ and Π, which are defined thus:

$$\Pi_k = \{z \in \Pi : |\operatorname{Im} z| < k \operatorname{Re} z\},$$
$$\Delta_k = \{z \in \Delta : |1 - z| < k(1 - |z|^2)\}.$$

The domains so defined do not correspond exactly via the Cayley map ($0 \in \partial\Pi_k$ but $-1 \notin \partial\Delta_k$), but they are equivalent in regard to approach paths to the boundary points $1 \in \partial\Delta$ and $\infty \in \partial\Pi$.

For $f : \Delta \to \Delta$ we say that f has an *angular limit* of a at 1 if $\lim_{z\to 1, z\in\Delta_k} f(z) = a$ for every $k > 0$, and then one writes $\angle\text{-}\lim_{z\to 1} f(z) = a$. The analogous definition for a function $g : \Pi \to \Pi$ is self-evident, and we write $\angle\text{-}\lim_{z\to\infty} g(z)$ for this limit.

Theorem 1.1 (JWC in Π). *Let $F \in \operatorname{Hol}(\Pi, \Pi)$, and let $a = \inf_\Pi \dfrac{\operatorname{Re} F(z)}{\operatorname{Re} z}$. Then*

$$\begin{aligned} a &= \angle\text{-}\lim_{z\to\infty} \frac{F(z)}{z} \\ &= \angle\text{-}\lim_{z\to\infty} \frac{\operatorname{Re} F(z)}{\operatorname{Re} z} \\ &= \angle\text{-}\lim_{z\to\infty} F'(z). \end{aligned}$$

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Theorem 1.2 (JWC in Δ). *Let $f \in \mathrm{Hol}(\Delta, \Delta)$, and let*

$$\alpha := \liminf_{\zeta \to 1} \frac{1 - |f(\zeta)|^2}{1 - |\zeta|^2}.$$

Assume that there exists a sequence $(\zeta_n)_n \subset \Delta$ with $f(\zeta_n) \to 1$ as $\zeta_n \to 1$ and such that

$$\lim_{n\to\infty} \frac{1 - |f(\zeta_n)|^2}{1 - |\zeta|_n^2} = \alpha.$$

Then

$$\angle\text{-}\lim_{\zeta\to 1} f(\zeta) = 1 \quad \textit{and} \quad \angle\text{-}\lim_{\zeta\to 1} f'(\zeta) = \alpha.$$

Let us make precise our notion of an angular limit for a vector-valued function on Π.

Definition 1.3. For $F \in \mathrm{Hol}(\Pi, U)$, where U is an open subset of a complex Banach space, we write $\angle\text{-}\lim_{z\to 0} F(z) = l$ if $\lim_{z\to\infty, z\in\Pi_k} F(z)$ exists and equals l for all $k > 0$.

In this paper we concentrate on an analogue of Theorem 1.1—a generalisation of Theorem 1.2 to arbitrary bounded symmetric domains appears in [17]. The following extension of Theorem 1.1 to $\mathscr{L}(H)$-valued analytic maps is due to Fan [3]. Throughout this paper, H and K will denote complex Hilbert spaces.

Theorem 1.4. *Let $F \in \mathrm{Hol}(\Pi, \Pi_{L(H)})$ where $\Pi_{\mathscr{L}(H)} = \{T \in \mathscr{L}(H) : \mathrm{Re}\, T > 0\}$. Suppose that there exists $A = A^* \in \mathscr{L}(H)$ with $\dfrac{\mathrm{Re}\, F(z)}{\mathrm{Re}\, z} > A$ for all $z \in \Pi$ and*

$$\inf_{\Pi} \left\| \frac{\mathrm{Re}\, F(z)}{\mathrm{Re}\, z} - A \right\| = 0.$$

Then

$$\begin{aligned} A &= \angle\text{-}\lim_{z\to\infty} \frac{F(z)}{z} \\ &= \angle\text{-}\lim_{z\to\infty} \frac{\mathrm{Re}\, F(z)}{\mathrm{Re}\, z} \\ &= \angle\text{-}\lim_{z\to\infty} F'(z). \end{aligned}$$

This was improved by Włodarczyk [19], who extended the result from $\mathscr{L}(H)$ to J^*-algebras containing a partial isometry (that is, a non-zero operator V with $VV^*V = V$). Recall that a J^*-algebra is a subspace of $\mathscr{L}(H, K)$ that is closed with respect to the map $A \mapsto AA^*A$. For a J^*-algebra contained in $\mathscr{L}(H, K)$, the right half-plane of $\mathscr{L}(H)$ is used crucially in [19]. In this paper we prove the corresponding result for unital JB^*-algebras and thereby extend the result for J^*-algebras to any

JB^*-triple containing a tripotent. A candidate for the role of $\Pi_{\mathscr{L}(H)}$ will lead us to a brief introduction to Siegel domains.

2. JB^*-triples

Definition 2.1. A JB^*-triple is a complex Banach space Z with a continuous map $\{\cdot,\cdot,\cdot\} : Z^3 \to Z$, $(x, y, z) \to \{x, y, z\}$, that is complex linear and symmetric in x and z and anti-linear in y and satisfies

(i) the operator $x\square x$ has spectrum in $[0, \infty)$,
(ii) $\exp(ix\square x)$ is both an algebraic automorphism and an isometry,
(iii) $\|\{x, x, x\}\| = \|x\|^3$,

for all $x \in Z$, where $x\square y$ denotes the linear map $z \mapsto \{x, y, z\}$.

The equality

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}, \tag{2.1}$$

which ensues from (ii) for all a, b, x, y and $z \in Z$, is known as the Jordan triple identity and provides a weak form of associativity for the triple product. The inequality

$$\|\{x, y, z\}\| \leq \|x\|\,\|y\|\,\|z\| \tag{2.2}$$

is proved in [5].

Any C^*-algebra, and more generally any J^*-algebra, is a JB^*-triple with triple product given by $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ where x^* denotes the usual operator adjoint of x. In particular, every complex Hilbert space is a JB^*-triple whose triple product is given by $\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$. Every JB^*-algebra is a JB^*-triple. We recall that a JB^*-algebra is a commutative (non-associative) Banach $*$-algebra whose product satisfies

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$$

and†

$$\|U_x(x^*)\| = \|x\|^3 \tag{2.3}$$

for all x and y in A, where $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$. For example, a C^*-algebra is a JB^*-algebra under the Jordan product

$$x \circ y = \tfrac{1}{2}(xy + yx).$$

A JB^*-algebra becomes a JB^*-triple when given the triple product

$$\{x, y, z\} = (x \circ y^*) \circ z - (x \circ z) \circ y^* + (z \circ y^*) \circ x.$$

In a unital JB^*-algebra A, we say that $x \in A$ is invertible if U_x is an invertible linear

†Condition (2.3) is the Jordan analogue of the associative C^*-condition $\|x^*x\| = \|x\|^2$. As pointed out by the referee, the condition $\|x^* \circ x\| = \|x\|^2$ for every x in a JB^*-algebra A actually implies that A is associative and hence a commutative C^*-algebra, cf. [7; 21].

operator. The inverse is then given by $x^{-1} = U_x^{-1}(x)$. This allows one to consider the spectrum of an element and thereby define a positive cone C in A, which is open relative to the self-adjoint elements, by

$$C := \{x \in A : x = x^* \text{ and the spectrum of } x \text{ is contained in } (0, \infty)\}.$$

There is then an order structure defined on A by $y > x$ if $y - x \in C$ (cf. [6]). Note that C is not, in general, a (Jordan) subalgebra of A. Indeed, C is a subalgebra precisely when A is associative, cf. [9]. However, if $x > 0$ and $y > 0$, then $U_x(y) > 0$ (see [6, prop. 3.3.6] and [8, I.11]).

For elements x and y of a JB^*-triple Z, the Bergman operator $B(x, y) \in \mathscr{L}(Z)$, defined by

$$B(x, y) = \mathsf{Id} - 2x\Box y + Q_x Q_y$$

where $Q_x(z) = \{x, z, x\}$, is an important tool. Note that, for elements of a JB^*-algebra, $Q_x(y) = U_x(y^*)$. For a C^*-algebra, the Bergman operator takes the form $B(x, y)z = (1 - xy^*)z(1 - y^*x)$.

An element $e \in Z$ for which $\{e, e, e\} = e$ is called a tripotent, and, from (iii) above, a non-zero tripotent has norm one. For example, a tripotent of a C^*-algebra is just a partial isometry. Each tripotent induces a splitting of Z, called the Peirce decomposition, into $Z = Z_1 \oplus Z_{\frac{1}{2}} \oplus Z_0$ where Z_k is the k-eigenspace of $e\Box e$, with mutually orthogonal projections P_k onto the subspaces Z_k,

$$\begin{aligned} P_1 &= Q_e Q_e, \\ P_{\frac{1}{2}} &= 2(e\Box e - Q_e Q_e), \\ P_0 &= B(e, e), \end{aligned}$$

satisfying $P_1 + P_{\frac{1}{2}} + P_0 = \mathsf{Id}$. Notice that the Peirce 1-space P_1 is itself a unital JB^*-algebra with respect to the product

$$x \circ y := \{x, e, y\}$$

and involution $x^* := \{e, x, e\}$. Indeed, we have $\{Z_i, Z_j, Z_k\} \subset Z_{i-j+k}$ for $i, j, k \in \{0, \frac{1}{2}, 1\}$ where $Z_{i-j+k} = \{0\}$ if $i - j + k \notin \{0, \frac{1}{2}, 1\}$. The tripotent e is called *maximal* if $Z_0 = \{0\}$, and this is the case precisely when e is an extreme point of the unit ball of Z [13]. The tripotent is called *unitary* if $P_0 = P_{\frac{1}{2}} = 0$.

A seminal result of Kaup [11] asserts that a Banach space Z is a JB^*-triple if and only if its open unit ball B is a bounded symmetric domain, that is, if the group of biholomorphic automorphisms of B act transitively. In this case, every automorphism of B is of the form $g = kg_c$ for some linear isometry k of Z and generalised Möbius map g_c for some $c \in B$. Every such Möbius map can be factored thus:

$$g_c(z) = c + B(c, c)^{\frac{1}{2}} \tilde{t}_{-c}(z), \quad z \in B,$$

where $\tilde{t}_y(x) = \sum_{k=0}^{\infty} (x\Box y)^k(x)$ is the *quasi-inverse* map. This definition of the quasi-inverse map is valid, of course, when $\|x\Box y\| < 1$ and, in particular, when $\|x\| \|y\| < 1$.

For a more complete definition and discussion of the quasi-inverse see [2] or [15]. The notation x^y is used for $\tilde{t}_y(x)$.

Associated to each JB^*-triple Z, and hence to each bounded symmetric domain, there exists a unique simply connected symmetric manifold of 'compact type', which we write as $M_K(Z)$. We refer to [10; 11] for details. For the purpose of this paper, it suffices to think of the compact-type manifold as a higher dimensional analogue of the Riemann sphere, providing a compact-type enveloping Banach manifold of the JB^*-triple. For $c \in Z$, the fractional linear map

$$\hat{g}_c(z) = c + B(c, -c)^{\frac{1}{2}} \tilde{t}_c(z), \quad z \in Z,$$

extends to a biholomorphic isometry of $M_K(Z)$. Note that $B(c, -c)^{\frac{1}{2}}$ exists by virtue of $B(c, -c)$ having spectrum contained in $(0, \infty)$, see [11, corollary 3.4]. For example, with $Z = \mathbb{C}$, $\hat{g}_1$ is nothing but the Cayley map φ.

3. Siegel domains

It is often useful to transfer the study of holomorphic functions from the open unit disc to the right half-plane via the Cayley transformation. Indeed, this is often the first step in proving fundamental holomorphic and geometric results in the disc. It will be useful for us to observe the same process in higher dimensions. The most common generalisation of the half-plane is the so-called tube domains (see [4]), but we will consider the more general Siegel domains [13; 18]. Let U be a complex Banach space carrying a continuous involution. The self-adjoint part of U is defined in the usual way as $\{x \in U : x^* = x\}$. Let us suppose that U carries sufficient order structure that we may talk about a positive cone $C = \{x \in U : x > 0\}$ of U. Then $P^1 := \{u \in U : \mathrm{Re}\, u \in C\}$ is called the *tube domain* in U or, in more general terms, a *Siegel domain of the first kind.*

Suppose that we have another Banach space V and a continuous sesquilinear map (linear in the first coordinate, antilinear in the second) $\varphi : V \times V \to U$ satisfying $\varphi(v_1, v_2)^* = \varphi(v_2, v_1)$ for all v_1 and v_2 in V. Then

$$P^2 := \{(u, v) \in U \times V : 2\mathrm{Re}\, u - \varphi(v, v) \in C\}$$

is called a *Siegel domain of the second kind.*

Finally, let us suppose that there is another Banach space W and continuous sesquilinear map $\psi : V \times W \to V$ that satisfies $\varphi(\psi(v_1, w), v_2) = \varphi(\psi(v_2, w), v_1)$ for all v_1, v_2 in V and all $w \in W$, and that, for w in the open unit ball B_W of W, $[\mathrm{Id} + \psi(\cdot, w)]$ is an invertible linear operator on V. Then, for

$$\varphi_w(v_1, v_2) := \varphi([\mathrm{Id} + \psi(\cdot, w)]^{-1} v_1, v_2),$$

we call

$$P^3 := \{(u, v, w) \in U \times V \times B_W : \mathrm{Re}\,(2u - \varphi_w(v, v)) \in C\}$$

a *Siegel domain of the third kind.*

Let us make these definitions more concrete. We will think of $U = Z_1$, $V = Z_{\frac{1}{2}}$ and $W = Z_0$ as the Peirce spaces of a JB^*-triple Z with respect to a tripotent e. Of course, Z_1 is a unital JB^*-algebra and, as such, has both natural continuous involution and positive cone. The sesquilinear map φ will be $\varphi : Z_{\frac{1}{2}} \times Z_{\frac{1}{2}} \to Z_1$ given by $\varphi(v_1, v_2) = 2\{e, v_1, v_2\}$. The map $\psi : Z_{\frac{1}{2}} \times Z_0 \to Z_{\frac{1}{2}}$ is given by $\psi(v, w) = 2\{e, v, w\}$. The following theorem appears in [18, theorem 21.25] and is an extension of the corresponding result for maximal tripotents given in [13].

Theorem 3.1. *Let e be a tripotent in the JB^*-triple Z generating the Peirce decomposition $Z = Z_1 \oplus Z_{\frac{1}{2}} \oplus Z_0$. Then $\hat{g}_e$ maps B onto the Siegel domain*

$$P := \{(z_1, z_{\frac{1}{2}}, z_0) \in Z : \|z_0\| < 1, \mathrm{Re}\,(2z_1 - \varphi_{z_0}(z_{\frac{1}{2}}, z_{\frac{1}{2}})) \in C\}$$

where C is the positive cone in Z_1, $\varphi_x(y, z) = \varphi([\mathrm{Id} + \psi(\cdot, x)]^{-1}_{|Z_{\frac{1}{2}}} y, z)$ and $\varphi(u, v) = \psi(u, v) = 2\{e, u, v\}$, $\varphi : Z_{\frac{1}{2}} \times Z_{\frac{1}{2}} \to Z_1$, $\psi : Z_{\frac{1}{2}} \times Z_0 \to Z_{\frac{1}{2}}$.

Note that in general P is a Siegel domain of the third kind, but if e is maximal ($Z_0 = 0$) or unitary ($Z = Z_1$) then it is of the second or first kind respectively. In particular, if A is a unital JB^*-algebra with unit e, then the Siegel domain P determined by e is the natural open right half-plane $P = \{u \in A : \mathrm{Re}\, u > 0\}$.

4. The main result

In this section we generalise to JB^*-triples the results of Fan [3] and Włodarczyk [19] for operator-valued holomorphic functions on the right half-plane. We begin with a bounding lemma for the compact-type Möbius maps of a JB^*-triple Z, $\{\hat{g}_c : c \in Z\}$.

Lemma 4.1. *Let x and y be elements of a JB^*-triple Z satisfying $\|x\|\,\|y\| < 1$. Then*

$$\|\hat{g}_y(x)\| \le \frac{\|x\| + \|y\|}{1 - \|x\|\,\|y\|}.$$

Proof. Recall that $\hat{g}_c$ may be represented as

$$\hat{g}_c = t_c B(c, -c)^{\frac{1}{2}} \tilde{t}_c$$

where $t_c(z) = z + c$ and $\tilde{t}_c$ is the quasi-inverse map $\tilde{t}_c(z) = z^c$. Since $\|x\|\,\|y\| < 1$, we can write $x^y = \sum_{k=0}^{\infty} (x \Box y)^k x$. In particular,

$$\|x^y\| \le \|x\| \sum_{k=0}^{\infty} (\|x\|\,\|y\|)^k = \frac{\|x\|}{1 - \|x\|\,\|y\|}.$$

Since by [12, corollary 3.8 (i)], $\|B(y,-y)^{\frac{1}{2}}\| \le 1 + \|y\|^2$, we have as a consequence

$$\begin{aligned}\|\hat{g}_y(x)\| &= \|t_y B(y,-y)^{\frac{1}{2}}\tilde{t}_y(x)\| \\ &\le \|y + B(y,-y)^{\frac{1}{2}}x^y\| \\ &\le \|y\| + (1+\|y\|^2)\frac{\|x\|}{1-\|x\|\|y\|} \\ &= \frac{\|x\|+\|y\|}{1-\|x\|\|y\|}\end{aligned}$$

as required. ■

We recall the Schwarz lemma, which lies at the heart of any analysis of holomorphic functions and will be of fundamental importance to us (see [1] for details).

Theorem 4.2 (The Schwarz lemma). *Let B and B' each be the open unit ball of a complex Banach space, and let $f \in \mathrm{Hol}(B, B')$ be holomorphic. Suppose that $f(0) = 0$. Then, for all $x \in B$,*

$$\|f(x)\| \le \|x\|.$$

We use this classical result to make the following estimate, which generalises, for example, [3, lemma 2], [19, cor. 3.3] and [20, thm 2.1].

Lemma 4.3. *Let Z and Z' be JB^*-triples with open unit balls B and B' respectively, and let $e \in Z$ be a tripotent with corresponding Siegel domain $P \subset Z$. Let $f : B' \to P$ be a holomorphic function such that $f(z) = e$ for some $z \in B'$. Then, for all $x \in B'$,*

$$\|f(x)\| \le \frac{1+\|g_{-z}(x)\|}{1-\|g_{-z}(x)\|},$$

where $g_w = t_w B(w,w)^{\frac{1}{2}}\tilde{t}_{-w}$ is an automorphism of the bounded symmetric domain B'.

PROOF. Let us define a holomorphic map $h : B' \to B$ by $h = \hat{g}_e^{-1} \circ f \circ g_z$. Then $h(0) = \hat{g}_e^{-1}(f(z)) = \hat{g}_e^{-1}(e) = 0$, so, by the Schwarz lemma (4.2), $\|h(x)\| \le \|x\|$ for x in B'. Thus,

$$\|\hat{g}_e^{-1} \circ f(x)\| \le \|g_{-z}(x)\| < 1$$

for every $x \in B'$. Since $\|\hat{g}_e^{-1}(f(x))\|\|e\| < 1$, we can apply Lemma 4.1 to obtain

$$\begin{aligned}\|\hat{g}_e(\hat{g}_e^{-1} \circ f(x))\| &\le \frac{\|e\| + \|\hat{g}_e^{-1}(f(x))\|}{1-\|e\|\|\hat{g}_e^{-1}(f(x))\|} \\ &= \frac{1+\|\hat{g}_e^{-1}(f(x))\|}{1-\|\hat{g}_e^{-1}(f(x))\|}.\end{aligned}$$

Now, since $\|\hat{g}_e^{-1}(f(x))\| \leq \|g_{-z}(x)\|$, we have

$$\|f(x)\| \leq \frac{1 + \|g_{-z}(x)\|}{1 - \|g_{-z}(x)\|},$$

which proves the result. ■

We can transfer this result from the unit ball to its unbounded realisation as a Siegel domain using a generalised Cayley transform as follows.

Corollary 4.4. *Let Z and Z' be JB^*-triples with open unit balls B and B' respectively, and let $e \in Z$, $e' \in Z'$ be tripotents with corresponding Siegel domains $P \subset Z$ and $P' \subset Z'$. Let $F : P' \to P$ be a holomorphic function such that $F(z) = e$ for some $z \in P'$. Then, for all $x \in P'$,*

$$\|F(x)\| \leq \frac{1 + \|g_{-\varphi^{-1}(z)}(\varphi^{-1}(x))\|}{1 - \|g_{-\varphi^{-1}(z)}(\varphi^{-1}(x))\|}$$

where $\varphi = \hat{g}_{e'} : B' \to P'$.

Proof. Simply consider the holomorphic map $f : B' \to P$ given by $f = F \circ \hat{g}_{e'}$ together with the previous lemma. ■

We can now state and prove our desired result, generalising results in [3] and [19]. Elements of the proof date back to that of Landau and Valiron [14] for the corresponding result in the disc. Fan [3] showed how the proof could be extended to an operator setting, and Włodarczyk [19], in extending the result to J^*-algebras, essentially showed that only a triple product was necessary. The extension required here is to transfer the calculus and estimates from $\mathscr{L}(H)$ to a unital JB^*-algebra.

Theorem 4.5. *Let A be a unital JB^*-algebra, and $F \in \mathrm{Hol}(\Pi, A)$. Suppose that there exists $a = a^* \in A$ with*

$$\frac{\operatorname{Re} F(\lambda)}{\operatorname{Re} \lambda} > a \text{ for all } \lambda \in \Pi \tag{4.1}$$

and

$$\inf_{\Pi} \left\| \frac{\operatorname{Re} F(\lambda)}{\operatorname{Re} \lambda} - a \right\| = 0. \tag{4.2}$$

Then

$$a = \angle\text{-} \lim_{\lambda \to \infty} \frac{F(\lambda)}{\lambda} \tag{4.3}$$

$$= \angle\text{-} \lim_{\lambda \to \infty} \frac{\operatorname{Re} F(\lambda)}{\operatorname{Re} \lambda} \tag{4.4}$$

$$= \angle\text{-} \lim_{\lambda \to \infty} F'(\lambda). \tag{4.5}$$

Note 4.6. We have not imposed the condition that F maps into the generalised right half-plane $\Pi_A = \{z \in A : \operatorname{Re} z > 0\}$, which, as pointed out by the referee, is an unnecessary constraint on the hypothesis. However, no increase in generality is gained, as can be seen by replacing the function F with $\lambda \mapsto F(\lambda) - \lambda a$.

PROOF. For a fixed ε choose, by (4.2), $\zeta \in \Pi$ such that $\left\| \frac{\operatorname{Re} F(\zeta)}{\operatorname{Re} \zeta} - a \right\| < \varepsilon$ and use it to define holomorphic functions $E : \Pi \to A$ and $G : \Pi \to A$ by

$$\begin{aligned} E(\lambda) &= F(\lambda) - \lambda a \\ G(\lambda) &= U_{(\operatorname{Re} E(\zeta))^{-\frac{1}{2}}}(E(\lambda) - i \operatorname{Im} E(\zeta)). \end{aligned}$$

Equation (4.1) tells us that for every $\lambda \in \Pi$, $\operatorname{Re} E(\lambda) > 0$. Thus $(\operatorname{Re} E(\zeta))^{-\frac{1}{2}} > 0$ and $\operatorname{Re}(E(\lambda) - i \operatorname{Im} E(\zeta)) > 0$ for $\lambda \in \Pi$. The self-adjoint elements of A are closed under the Jordan product, and it follows that $\operatorname{Re}(U_x(y)) = U_x(\operatorname{Re} y)$ for $x > 0$. Therefore the condition $U_u(v) > 0$ for $u > 0$ and $v > 0$ implies that $\operatorname{Re}(G(\lambda)) > 0$. Therefore G maps Π into Π_A. Since $G(\zeta) = \mathbf{1}_A$ we can apply Corollary 4.4 to obtain

$$\|G(\lambda)\| \leq \frac{1 + |g_{-\varphi^{-1}(\zeta)}(\varphi^{-1}\lambda)|}{1 - |g_{-\varphi^{-1}(\zeta)}(\varphi^{-1}\lambda)|}$$

for all $\lambda \in \Pi$, where φ is the Cayley transform in $\mathbb{C}$ given by $\varphi(\xi) = \dfrac{1+\xi}{1-\xi}$.

Now $E(\lambda) = U_{(\operatorname{Re} E(\zeta))^{\frac{1}{2}}}(G(\lambda)) + i \operatorname{Im} E(\zeta)$ since $U_x U_{x^{-1}} = \mathsf{Id}$ for all invertible x in a unital JB^*-algebra (cf. [8, I.11]). Since $\|U_x\| = \|x\|^2$ we have

$$\|E(\lambda)\| \leq \|\operatorname{Re} E(\zeta)\| L + \|\operatorname{Im} E(\zeta)\|$$

where $L = \dfrac{1 + |g_{-\varphi^{-1}(\zeta)}(\varphi^{-1}\lambda)|}{1 - |g_{-\varphi^{-1}(\zeta)}(\varphi^{-1}\lambda)|}$. Thus we can write

$$\begin{aligned} \left\| \frac{E(\lambda)}{\lambda} \right\| &\leq \left\| \frac{\operatorname{Re} E(\zeta)}{\operatorname{Re} \zeta} \right\| \frac{\operatorname{Re} \zeta}{|\lambda|} L + \left\| \frac{\operatorname{Im} E(\zeta)}{\lambda} \right\| \\ &\leq \varepsilon \frac{\operatorname{Re} \zeta}{|\lambda|} L + \left\| \frac{\operatorname{Im} E(\zeta)}{\lambda} \right\|, \end{aligned}$$

while

$$\begin{aligned} \left\| \frac{\operatorname{Re} E(\lambda)}{\operatorname{Re} \lambda} \right\| &\leq \left\| \frac{E(\lambda)}{\operatorname{Re} \lambda} \right\| \\ &\leq \frac{|\lambda|}{\operatorname{Re} \lambda} \left\| \frac{E(\lambda)}{\lambda} \right\| \\ &\leq (1 + k^2)^{\frac{1}{2}} \left\| \frac{E(\lambda)}{\lambda} \right\| \end{aligned}$$

for $\lambda \in \Pi_k := \{x \in \Pi : |\operatorname{Im} x| < k \operatorname{Re} x\} \subset \Pi$. We claim that $\operatorname{Re} \zeta \dfrac{L}{|\lambda|}$ is bounded on Π_k independently of ε as $\lambda \to \infty$ and hence, since ε is arbitrarily small, $\left\| \dfrac{E(\lambda)}{\lambda} \right\| \to 0$, as does $\left\| \dfrac{\operatorname{Re} E(\lambda)}{\operatorname{Re} \lambda} \right\|$, when $\lambda \to \infty$ in Π_k, which is precisely the content of (4.3) and (4.4).

Claim. $\operatorname{Re} \zeta \dfrac{L}{|\lambda|}$ is bounded on Π_k (independently of ε) as $\lambda \to \infty$.

Proof of claim. Let $z = \varphi^{-1}(\zeta) = \dfrac{\zeta - 1}{\zeta + 1} \in \Delta$. Since, for $\mu \in \Delta$ the equality $g_{-z}(\mu) = \dfrac{\mu - z}{1 - \bar{z}\mu}$ holds, for $\lambda \in \Pi_k$ we have

$$\begin{aligned} L &= \frac{(1 + |g_{-z}(\varphi^{-1}\lambda)|)^2}{1 - \|g_{-z}(\varphi^{-1}\lambda)\|^2} \\ &\leq \frac{4}{1 - |g_{-z}(\varphi^{-1}(\lambda))|^2} \\ &= 4 \frac{1}{1 - |\varphi^{-1}(\lambda)|^2} |1 - \varphi^{-1}(\lambda)\bar{z}|^2 \frac{1}{1 - |z|^2}. \end{aligned}$$

Expanding $\varphi^{-1}(\lambda)$, one finds that

$$L \leq 4 \frac{|\lambda + 1|^2}{4 \operatorname{Re} \lambda} \left| 1 - \frac{\lambda - 1}{\lambda + 1} \bar{z} \right|^2 \frac{1}{1 - |z|^2},$$

and, since $\lambda \in \Pi_k$ implies $\dfrac{1}{\operatorname{Re} \lambda} < (1 + k^2)^{\frac{1}{2}} \dfrac{1}{|\lambda|}$,

$$\frac{L}{|\lambda|} \leq (1 + k^2)^{\frac{1}{2}} \left| \frac{\lambda + 1}{\lambda} \right|^2 \left| 1 - \frac{\lambda - 1}{\lambda + 1} \bar{z} \right|^2 \frac{1}{1 - |z|^2}.$$

For λ large enough, $\dfrac{L}{|\lambda|} \leq 2(1 + k^2)^{\frac{1}{2}} \dfrac{|1 - \bar{z}|^2}{1 - |z|^2}$. Since $\operatorname{Re} \zeta = \dfrac{1 - |z|^2}{|1 - z|^2}$ we see that $\operatorname{Re} \zeta \dfrac{L}{|\lambda|} \leq 2(1 + k^2)^{\frac{1}{2}}$, which proves the claim.

Next, we prove our angular derivative identity (4.5). Let $h = \dfrac{1}{2(1 + k^2)^{\frac{1}{2}}}$. Then, for $x \in \Pi_k$ and $R := \{\xi \in \mathbb{C} : |\xi - x| < h|x|\}$ (that is, the closed disc centred at x with radius $h|x|$), we see that $R \subset \Pi$. Moreover, assuming for convenience that $|x| > 1$, one checks readily that $R \subset \Pi_{\frac{2(1+k^2)+1}{2(1+k^2)^{\frac{1}{2}}-1}} =: \Pi_{k'}$.

Using the Cauchy integral formula for Banach space valued holomorphic maps, we can write

$$E'(\lambda) = \frac{1}{2\pi i}\int_{\partial R}\frac{E(\lambda+\mu)}{\mu^2}d\mu.$$

Of course, for $y \in R$, $|y| \le (1+h)|x|$ and so

$$\begin{aligned}\|E'(\lambda)\| &\le \frac{1}{2\pi}2\pi\frac{1}{h|x|}\sup_{y\in\partial R}\|E(y)\| \\ &\le \frac{1}{h|x|}|y|\sup_{y\in R}\left\|\frac{E(y)}{y}\right\| \\ &\le \frac{1+h}{h}\sup_{y\in R}\left\|\frac{E(y)}{y}\right\|.\end{aligned}$$

But R is contained in $\Pi_{k'}$, so by the first part of the proof there exists $M > 0$ such that $|y| > M$ implies $\left\|\frac{E(y)}{y}\right\| < \varepsilon$. Choosing x large enough ensures that this is true for all $y \in R$, and so we can conclude that $\|E'(\lambda)\| \to 0$ as $x \to \infty$ in Π_k. Since $E'(\lambda) = (F'(\lambda) - a)$, we have proven the theorem. ■

The following is an immediate corollary of the main result for JB^*-triples containing a tripotent, which we include for completeness (cf. [19, thm 3.4]).

Theorem 4.7. *Let e be a tripotent in a JB^*-triple Z, P be the corresponding Siegel domain and P_i be the Peirce projections associated to e for $i = 0, \frac{1}{2}, 1$. Let $F \in H(\Pi, P)$. Suppose that there exists $a = a^* \in Z_1 = P_1(Z)$ that satisfies*

$$\frac{\operatorname{Re} P_1F(\lambda)}{\operatorname{Re}\lambda} > a \text{ for all } \lambda\in\Pi \quad \text{and} \quad \inf_{\Pi}\left\|\frac{\operatorname{Re} P_1F(\lambda)}{\operatorname{Re}\lambda} - a\right\| = 0.$$

Then

$$a = \angle\text{-}\lim_{\lambda\to\infty}\frac{P_1F(\lambda)}{\lambda} = \angle\text{-}\lim_{\lambda\to\infty}\frac{\operatorname{Re} P_1F(\lambda)}{\operatorname{Re}\lambda} = \angle\text{-}\lim_{\lambda\to\infty}P_1F'(\lambda).$$

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