ON FRÉCHET SPACES WITH A DOMINANT NORM

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Abstract

We characterise Fréchet spaces with a dominant norm in terms of plurisubharmonicity.

A Fréchet space has a dominant norm, or is a DN space, if its topology can be induced by norms $\| \ \|_t$, $t \in \mathbb{N}$, satisfying

$$\|x\|_t^2 \le \|x\|_{t-1}\|x\|_{t+1}, \qquad t = 2, 3, \dots; \tag{1}$$

see [7, Satz 2.1]. These spaces have been introduced by Vogt, and their relevance to infinite-dimensional complex analysis was explored by himself, Meise, Dineen and others; see [2, chapter 4] and [5; 6]. (DN spaces occur in finite-dimensional complex analysis as well; see the survey [1].) Condition (1) and a related if rather stronger convexity condition appear in [3; 4], which also deal with complex analysis in Fréchet spaces. Our purpose here is to characterise DN spaces in terms of plurisubharmonicity. Given that by now the importance of plurisubharmonicity for complex analysis has been firmly established, the characterisation below will explain, in a way, the ubiquity of condition (1) in complex analytical investigations.

Let X be a complex Fréchet space. An upper semicontinuous function $u : X \to [-\infty, \infty)$ is plurisubharmonic if the function $v(\lambda) = u(x + \lambda y)$, $\lambda \in \mathbb{C}$, is subharmonic for all $x, y \in X$. Recall further the notion of a pseudonorm. This is a function $p : X \to [0, \infty)$ that satisfies: $p(x) = 0$ precisely when $x = 0$; $p(\lambda x) \le p(x)$ if $\lambda \in \mathbb{C}$, $|\lambda| \le 1$; and $p(x + y) \le p(x) + p(y)$.

Theorem. *The following three conditions are equivalent for a complex Fréchet space X:*

(i) *X has a dominant norm;*

(ii) *the topology of X can be induced by a pseudonorm p with convex sublevel sets such that $\log p$ is plurisubharmonic;*

(iii) *the topology of X can be induced by a metric d such that $\log d$ is plurisubharmonic.*

Important examples of DN spaces are spaces of rapidly decreasing vectors. Let $(Z, \| \ \|)$ be a complex Banach space; for maps $x : \mathbb{N} \to Z$ define

$$\|x\|_\theta = \sup_n n^\theta \|x(n)\| \le \infty, \quad \theta \in \mathbb{R}, \quad \text{and put}$$
$$s(Z) = \{x : \mathbb{N} \to Z \mid \|x\|_\theta < \infty \ \text{ for all } \theta\}.$$

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It is immediate to check that $s(Z)$ with the norms $\|\ \|_t$, $t \in \mathbb{N}$, is a DN space, and $\|x\|_\theta$ depends continuously on $(x, \theta) \in X \times \mathbb{R}$. A special case is the space $s(\mathbb{C}) = s$ of rapidly decreasing sequences. First we verify (ii) for the space $X = s(Z)$.

Proposition. *The formula*

$$p(x) = \inf\{\theta > 0 : \|x\|_{1/\theta} \leq \theta\}, \quad x \in s(Z) \tag{2}$$

defines a pseudonorm with convex sublevel sets that induces the topology of $s(Z)$. *Further,* $\log p$ *is plurisubharmonic.*

Proof. The first half of the claim is staightforward and holds in greater generality, cf. [3, proposition 5.1]. For example, to verify the triangle inequality, observe that by the intermediate value theorem for $x \neq 0$ there is a unique $\alpha > 0$ with $\|x\|_{1/\alpha} = \alpha$, and this $\alpha = p(x)$. Suppose that $y \neq 0$ and let $\beta = p(y)$. Then

$$\|x + y\|_{1/(\alpha+\beta)} \leq \|x\|_{1/(\alpha+\beta)} + \|y\|_{1/(\alpha+\beta)} \leq \|x\|_{1/\alpha} + \|y\|_{1/\beta} = \alpha + \beta.$$

Hence (2) implies that $p(x + y) \leq \alpha + \beta = p(x) + p(y)$. This inequality also holds when x or y is 0.

To prove that $\log p$ is plurisubharmonic, note that $\|x\|_{1/\theta} \leq \theta$ means $\|x(n)\| \leq \theta n^{-1/\theta}$ for all n. Since $\theta n^{-1/\theta}$ increases with θ,

$$p(x) = \sup_n \inf\{\theta > 0 : \|x(n)\| \leq \theta n^{-1/\theta}\}, \quad \text{or}$$
$$\log p(x) = \sup_n \inf\{\omega \geq -\infty : \log \|x(n)\| \leq \omega - e^{-\omega} \log n\}. \tag{3}$$

The map $\omega \mapsto \omega - e^{-\omega} \log n$ being a concave, increasing self-homeomorphism of $[-\infty, \infty)$, its inverse f_n is increasing and convex; also the inf in (3) is achieved for $\omega = f_n(\log \|x(n)\|)$. Therefore $\log p$, the sup of plurisubharmonic functions $f_n(\log \|x(n)\|)$, $n \in \mathbb{N}$, is itself plurisubharmonic.

Proof of the theorem. (i) $\Rightarrow$ (ii): According to [8, lemma 2.1], a DN space is isomorphic to a subspace of $s(Z)$ for some Banach space Z. Since, by the proposition, $s(Z)$ admits a pseudonorm as in (ii), so does X.

(ii) $\Rightarrow$ (iii): The metric $d(x, y) = p(x - y)$ will do.

(iii) $\Rightarrow$ (i): First, observe that balls $\{x \in X : d(0, x) < r\}$ contain no complex lines, for otherwise the restriction of $d(0, \cdot)$ to such a line would be a non-constant, bounded subharmonic function, which is impossible. Second, inductively construct positive numbers $\epsilon_t \to 0$ and continuous seminorms $\|\ \|_t$ such that

$$\text{if } \|y\|_t \leq 1 \ \text{ then } d(0, y) \leq \epsilon_t, \qquad t = 1, 2, \ldots; \tag{4}$$
$$\text{if } d(0, y) \leq (\epsilon_{t-1}\epsilon_{t+1})^{1/2} \ \text{ then } \|y\|_t \leq 1, \qquad t = 2, 3, \ldots. \tag{5}$$

The observation above implies that $\|\ \|_t$ are in fact norms; also they induce the topology of X. The function $\lambda \mapsto \log d(0, x/\lambda)$ being subharmonic on $\mathbb{C} \setminus \{0\}$, by the three-circles theorem we have if $x \neq 0$

$$\max_{|\lambda|=\|x\|_{t-1}} d\left(0, \frac{x}{\lambda}\right) \max_{|\lambda|=\|x\|_{t+1}} d\left(0, \frac{x}{\lambda}\right) \geq d\left(0, \frac{x}{(\|x\|_{t-1}\|x\|_{t+1})^{1/2}}\right)^2.$$

Hence (4) implies that

$$(\epsilon_{t-1}\epsilon_{t+1})^{1/2} \geq d\left(0, \frac{x}{(\|x\|_{t-1}\|x\|_{t+1})^{1/2}}\right),$$

and so $\|x/(\|x\|_{t-1}\|x\|_{t+1})^{1/2}\|_t \leq 1$ by (5), which is equivalent to (1).

What about non-DN spaces? Are their topologies induced by pseudonorms that still have some plurisubharmonicity properties? We cannot give a general answer, but here is an example. Let Z and $\| \ \|_\theta$ be as above, and define

$$X = \{x : \mathbb{N} \to Z \mid \|x\|_\theta < \infty \text{ for all } \theta < 0\}.$$

With the norms $\| \ \|_\theta$, $\theta < 0$, X becomes a Fréchet space. Arguing as in the Proposition, one can verify that $q(x) = \inf\{\tau > 0 : \|x\|_{-\tau} \leq \tau\}$ is a pseudonorm with convex sublevel sets that induces the topology of X, and q *itself* is plurisubharmonic.

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REFERENCES

[1] A. Aytuna, Stein spaces M for which $\mathcal{O}(M)$ is isomorphic to a power series space, in T. Terzioglu (ed.), *Advances in the theory of Fréchet spaces*, NATO ASI series C, vol. 287, Kluwer, Dordrecht, 1989, pp 115–54.

[2] S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer, Berlin, 1999.

[3] L. Lempert, Holomorphic approximation in Fréchet spaces, *Communications in Analysis and Geometry*, to appear.

[4] L. Lempert, Analytic cohomology in Fréchet spaces, *Communications in Analysis and Geometry*, to appear.

[5] R. Meise and D. Vogt, Counterexamples in holomorphic functions on nuclear Fréchet spaces, *Mathematische Zeitschrift* **182** (1983), 167–77.

[6] D. Vogt, Vektorwertige Distributionen als Randverteilungen holomorphen Funktionen, *Manuscripta Mathematica* **17** (1975) 267–90.

[7] D. Vogt, Charakterisierung der Unterräume von *s*, *Mathematische Zeitschrift* **155** (1977), 109–17.

[8] D. Vogt, On two classes of (F)-spaces, *Archiv der Mathematik* **45** (1985), 255–66.