GROUPS WHOSE PROPER QUOTIENTS ARE NILPOTENT-BY-FINITE

By Maria De Falco*
Dipartimento di Matematica e Applicazioni, Università degli Studi di Napoli
Federico II, Napoli, Italy

[Received 18 July 2000. Read 25 June 2001. Published 31 December 2002.]

Abstract

If $\mathfrak{X}$ is a class of groups, a group G is said to be just-non-$\mathfrak{X}$ if it is not in $\mathfrak{X}$ but all its proper quotients are $\mathfrak{X}$-groups. A description of just-non-(nilpotent-by-finite) groups and just-non-(supersoluble-by-finite) groups is given in this paper.

1. Introduction

Let $\mathfrak{X}$ be a class of groups. A group G is said to be *just-non-*$\mathfrak{X}$ if it is not in $\mathfrak{X}$ but all its proper quotients are $\mathfrak{X}$-groups. Many authors have investigated the structure of just-non-$\mathfrak{X}$ groups for several choices of the group class $\mathfrak{X}$. In particular M.F. Newman [8; 9] considered just-non-abelian groups, while groups whose proper quotients are nilpotent were studied by S. Franciosi and F. de Giovanni [2], and more recently L.A. Kurdachenko and I. Subbotin [4] decribed just-non-hypercentral groups. In another direction, groups whose proper quotients satisfy a certain finiteness condition have been studied in a series of articles (see for instance [1; 3; 6; 7; 12; 13]). Moreover, some recent papers deal with the structure of just-non-$\mathfrak{X}$ groups, where $\mathfrak{X}$ is a property generalising both finiteness and nilpotency; in particular, groups whose proper quotients are finite-by-nilpotent have been considered by Z. Zhang [15].

The aim of this article is to study groups whose proper quotients are nilpotent-by-finite. It turns out that the Fitting subgroup of a just-non-(nilpotent-by-finite) group is either torsion-free abelian or abelian of prime exponent. Obviously every infinite simple group is just-non-(nilpotent-by-finite), and hence in this context it is natural to restrict our investigation to just-non-(nilpotent-by-finite) groups with non-trivial Fitting subgroup (i.e. soluble-by-finite groups). The structure of these groups is described in Section 2, with special attention to the case of groups with a unique minimal normal subgroup. Finally, in Section 3 we characterise just-non-(supersoluble-by-finite) groups (with non-trivial Fitting subgroup).

Most of our notation is standard and can be found in [10].

2. Just-non-(nilpotent-by-finite) groups

We say that a group G is a *JNNF-group* if it is a just-non-(nilpotent-by-finite) group, i.e. G is not nilpotent-by-finite but all its proper quotients have this property. Our first lemma describes some elementary properties of $JNNF$-groups. Recall that the *FC-centre* of a group G is the subgroup consisting of all elements of G with finitely many conjugates.

*E-mail: defalco@matna2.dma.unina.it

Lemma 2.1. *Let G be a JNNF-group. Then the following properties hold:*

(a) *If N_1 and N_2 are non-trivial normal subgroups of G, then $N_1 \cap N_2$ is non-trivial.*
(b) *If K is a normal subgroup of finite index in G, then $C_G(K) = \{1\}$, and in particular K has trivial centre.*
(c) *The FC-centre of G is trivial.*

Proof. (a) Since G/N_1 and G/N_2 are nilpotent-by-finite, the group $G/N_1 \cap N_2$ is also nilpotent-by-finite, and so $N_1 \cap N_2 \neq \{1\}$.

(b) Assume that $C_G(K) \neq \{1\}$, so that $G/C_G(K)$ is nilpotent-by-finite. Then K is nilpotent-by-finite and hence G itself is nilpotent-by-finite, a contradiction.

(c) Let x be any element of the FC-centre of G. Then $C = C_G(\langle x \rangle^G)$ is a normal subgroup of finite index in G and it follows from (b) that $C_G(C) = \{1\}$. Therefore $x = 1$ and G has trivial FC-centre. ■

Theorem 2.2. *Let G be a JNNF-group. Then the Fitting subgroup A of G is abelian and either it is torsion-free or it has prime exponent p. Moreover, if A is not trivial, then $C_G(A) = A$.*

Proof. Let N be any nilpotent normal subgroup of G. Then the factor group G/N' is not nilpotent-by-finite, and hence $N' = \{1\}$. Therefore every nilpotent normal subgroup of G is abelian and so the Fitting subgroup A is abelian.

Suppose now that A is not torsion-free, and let S be the socle of the torsion subgroup of A. Clearly S is an elementary abelian p-group for some prime p. Assume by contradiction that $S \neq A$, and let K/S be the Fitting subgroup of G/S. Since K/S is nilpotent, there exists a non-trivial element aS in $A/S \cap Z(K/S)$. If x is any element of K, then $[a, x]$ belongs to S, so that

$$[a^p, x] = [a, x]^p = 1;$$

hence a^p belongs to $Z(K)$. On the other hand, $Z(K) = \{1\}$ by Lemma 2.1, since K is a normal subgroup having finite index in G. Thus $a^p = 1$ and a belongs to S, a contradiction. Therefore $A = S$ is an elementary abelian p-group.

Suppose now that $A \neq \{1\}$ and put $C = C_G(A)$. Since $A \leq Z(C)$, the group $G/Z(C)$ is nilpotent-by-finite and so C is also nilpotent-by-finite. Then C/A is finite, and in particular $C/Z(C)$ is finite. It follows that C' is finite, so that $C' = \{1\}$ and C is abelian. Therefore $C = A$. ■

As a consequence of Theorem 2.2 we can now prove that the class of (soluble-by-finite) $JNNF$-groups does not contain locally nilpotent groups.

Corollary 2.3. *Let G be a locally nilpotent group with non-trivial Fitting subgroup. If every proper quotient of G is nilpotent-by-finite, then G itself is nilpotent-by-finite.*

Proof. Assume by contradiction that G is not nilpotent-by-finite, so that it is a $JNNF$-group, and in particular its Fitting subgroup A is abelian. The locally nilpotent group G/A is nilpotent-by-finite, so that it is hypercentral. Let zA be a non-trivial element of $Z(G/A)$ and let θ be the endomorphism of A defined by $\theta(a) = [a, z]$ for every a in A; thus $\theta(a^g) = \theta(a)^g$ for all $a \in A$ and $g \in G$, so that $K = ker\theta = C_A(z)$ is a normal subgroup of G. Let a be a non-trivial element of A, so that $\langle a, z\rangle$ is a nilpotent group and hence $A \cap Z(\langle a, z\rangle) \neq \{1\}$. Therefore K is not trivial, and G/K is nilpotent-by-finite. Since $C_G(A) = A$ by Theorem 2.2, K is properly contained in A, and so A/K contains a non-trivial element bK having finitely many conjugates under the action of G. Let $\{b^{g_1}K, \dots, b^{g_t}K\}$ be the conjugacy class of bK in G/K. If g is any element of G, there exists $i \leq t$ such that $b^g = b^{g_i}k$ with $k \in K$; thus $\theta(b)$ belongs to the FC-centre of G, so that $\theta(b) = 1$ by Lemma 2.1, and b belongs to K. This contradiction proves the statement. ■

In the statement of Corollary 2.3 the assumption that the group has non-trivial Fitting subgroup cannot be omitted. In fact, there exist locally nilpotent just-non-(abelian-by-finite) groups with trivial Fitting subgroup, as the following example shows.

Let Λ be the set of positive integers with the inverse order, and consider the standard wreath product $G = WrC_p^\Lambda$, where C_p is a group of prime order p. Then G is a countably infinite locally finite p-group with no abelian non-trivial normal subgroups (see [10], part 2, p. 22). Let g be any element of G. Then there exists a bisection $\Lambda = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \ll \Gamma_2$, such that, if $G = W^{(1)}wrW^{(2)}$ is the corresponding decomposition of G, then $W^{(2)}$ is finite and $g \in W^{(2)}$. Let B be the base group of $G = W^{(1)}wrW^{(2)}$. Then B' is contained in the normal subgroup $\langle g\rangle^G$ (see [10], part 2, lemma 6.26), and hence the factor group $G/\langle g\rangle^G$ is abelian-by-finite. Therefore G is just-non-(abelian-by-finite).

We also need the following slight generalisation of a well-known result concerning the upper central series of a group (see [10], part 1, theorem 2.23).

Lemma 2.4. *Let G be a nilpotent group, and let N be a normal subgroup of G such that $N \cap Z(G)$ has finite exponent. Then N also has finite exponent.*

Let Q be a group and let A be a Q-module (i.e. a module over the integral group ring $\mathbb{Z}Q$). We say that A is *just-infinite* if A is infinite but all its proper factor modules are finite. Just-infinite modules over nilpotent-by-finite groups arise naturally in the study of $JNNF$-groups.

Lemma 2.5. *Let G be a JNNF-group whose Fitting subgroup A is a non-trivial group with finite Prüfer rank. Then A is a faithful just-infinite G/A-module.*

Proof. Obviously the subgroup A is infinite, and $C_G(A) = A$ by Theorem 2.2, so that A is a faithful G/A-module. Assume by contradiction that A contains a non-trivial normal subgroup B of G such that A/B is infinite. Since G/B is nilpotent-by-finite, its Fitting subgroup K/B is nilpotent and G/K is finite. Clearly $C/B = A/B \cap Z(K/B)$ is a non-trivial G-invariant subgroup of A/B, and $[C, K]$ is contained in B. Moreover, $C_C(K) = \{1\}$ by Lemma 2.1 and hence $C/[C, K]$ is finite (see [5], proposition 1). It follows that C/B is finite, and hence A/B has finite exponent by Lemma 2.4.

Therefore A/B is finite, and this contradiction proves that A is a just-infinite G/A-module. ■

By Theorem 2.2 there is a natural dichotomy in the study of just-non-(nilpotent-by-finite) groups. Let G be a $JNNF$-group with non-trivial Fitting subgroup A; we say that G has *characteristic* 0 if A is torsion-free, and that G has prime *characteristic* p if A has exponent p. The following result gives a description of $JNNF$-groups of characterisic 0 when the Fitting subgroup has finite Prüfer rank.

Theorem 2.6. *Let G be a group whose Fitting subgroup has finite Prüfer rank. Then G is a JNNF-group of characteristic* 0 *if and only if it contains non-trivial torsion-free abelian subgroups A and X satisfying the following conditions:*

(a) *A is normal in G, and it is a faithful just-infinite G/A-module;*

(b) $A \cap X = \{1\}$*;*

(c) *the subgroup AX has finite index in G.*

Proof. Suppose first that G is a $JNNF$-group of characteristic 0, so that by Theorem 2.2 the Fitting subgroup A of G is a torsion-free abelian group. Moreover, A is a faithful just-infinite G/A-module by Lemma 2.5. The factor group G/A is nilpotent-by-finite, and hence in particular FC-hypercentral, so that, since $C_G(A) = A$, G/A contains a torsion-free abelian normal subgroup H/A of finite index (see [3], theorem 4.4). Since $C_A(H) = \{1\}$ by Lemma 2.1, H contains a subgroup X such that $A \cap X = \{1\}$ and the index $|H : XA|$ is finite (see [5], proposition 2). Then X is a non-trivial torsion-free abelian group and XA has finite index in G.

Conversely, suppose that the group G satisfies the conditions of the statement, and let N be a non-trivial normal subgroup of G. If $A \cap N = \{1\}$, we have also $[A, N] = \{1\}$, so that $N \leq C_G(A) = A$, a contradiction. Therefore $A \cap N \neq \{1\}$, and hence $AN/N \cong A/A \cap N$ is finite. It follows that G/N is finite-by-abelian-by-finite, and so also nilpotent-by-fnite (see [10], part 1, theorem 4.25). Assume by contradiction that G is nilpotent-by-finite, so that its Fitting subgroup F has finite index in G and is nilpotent. Thus $A \cap Z(F) \neq \{1\}$ and therefore $A/A \cap Z(F)$ is finite, so that there exists a positive integer n such that A^n is contained in $Z(F)$. For all elements $a \in A$ and $x \in F$ we obtain that

$$[a, x]^n = [a^n, x] = 1,$$

so that $[a, x] = 1$ and F is contained in $C_G(A) = A$, a contradiction since G/A is infinite. Therefore G is a $JNNF$-group and it has obviously characteristic 0, since A is contained in F. ■

Recall that a group G is *monolithic* if it has a unique minimal normal subgroup, which is called the *monolith* of G. In the proof of our next lemma $H^n(Q, A)$ denotes, as usual, the nth *cohomology group* of the group Q with coefficients in the Q-module A.

Lemma 2.7. *Let G be a monolithic JNNF-group with non-trivial Fitting subgroup, and* *let M be the monolith of G. Then every subgroup of G containing M splits conjugately on M.*

Proof. It follows from Lemma 2.1 that M is contained in the Fitting subgroup A of G so that M is abelian. As G/M is nilpotent-by-finite, its Fitting subgroup F/M is nilpotent, and the index $|G:F|$ is finite, so that $Z(F) = \{1\}$. Then $H^0(F/M, M) = \{0\}$ and hence

$$H^1(L/M, M) = H^2(L/M, M) = \{0\}$$

for each subgroup L of G containing M (see [11]). Therefore every subgroup of G containing M splits conjugately on M. ■

Our next result provides a satisfactory description of monolithic $JNNF$-groups of characteristic 0.

Theorem 2.8. *Let G be a monolithic group with monolith M, and suppose that the Fitting subgroup of G has finite torsion-free rank. Then G is a $JNNF$-group of characteristic* 0 *if and only if there exists a subgroup X of G such that $G = X \ltimes M$ and the following conditions hold:*

(a) *M is a torsion-free abelian group and $C_G(M) = M$;*
(b) *X is a finite extension of a torsion-free abelian group.*

Proof. Suppose first that G is a $JNNF$-group of characteristic 0. By Lemma 2.7 every subgroup of G containing M splits over M, and in particular there exists a subgroup X of G such that $G = X \ltimes M$. Let A be the Fitting subgroup of G. Then M is contained in A and A/M is finite by Lemma 2.5, so that $M = A$ is the Fitting subgroup of G. Therefore $C_G(M) = M$ and M is torsion-free abelian. Moreover, it follows from Theorem 2.6 that X is a finite extension of a torsion-free abelian group.

Conversely, every group satisfying the conditions of the statement is a $JNNF$-group of characteristic 0 by Theorem 2.6. ■

Lemma 2.9. *Let G be a $JNNF$-group of prime characteristic p, and let A be the Fitting subgroup of G. Then the Hirsch–Plotkin radical of G/A has no elements of order p.*

Proof. Let R/A be the Hirsch–Plotkin radical of G/A. Then R/A is a locally nilpotent group that is also nilpotent-by-finite, and hence it is hypercentral. Assume by contradiction that R/A contains elements of order p, so that there is an element xA of order p in $Z(R/A)$. As G/A is nilpotent-by-finite, the index $|G:R|$ is finite and hence xA has finitely many conjugates. Therefore $\langle x\rangle^G A/A$ is a finite p-group and so the normal subgroup $\langle x\rangle^G A$ is nilpotent (see [10], part 2, lemma 6.34). This contradiction proves the lemma. ■

Lemma 2.10. *Let G be a periodic $JNNF$-group with non-trivial Fitting subgroup A, and let a be any non-trivial element of A. Then the normal closure $\langle a\rangle^G$ is a just-infinite $G/C_G(\langle a\rangle^G)$-module.*

Proof. Let p be the characteristic of G. Consider any non-trivial normal subgroup N of G that is contained in $\langle a\rangle^G$ and let K/N be the Fitting subgroup of G/N. It follows from Lemma 2.9 that K/A has no elements of order p, so that A/N is the Sylow p-subgroup of the nilpotent group K/N, and hence it is contained in $Z(K/N)$. In particular, the coset aN has finitely many conjugates in G/N, and so $\langle a\rangle^G/N$ is finite. ■

We can now prove the following.

Theorem 2.11. *Let G be a periodic JNNF-group with non-trivial Fitting subgroup A. Then G is monolithic.*

Proof. Let p be the characteristic of G, so that the Fitting subgroup K/A of G/A is a p'-group by Lemma 2.9. Consider any non-trivial element a of A and put $A_1 = \langle a\rangle^G$. Then A is contained in $C = C_G(A_1)$, and hence $X/C = KC/C$ is a p'-group. Assume now by contradiction that G is not monolithic, so that there exists a proper non-trivial subgroup U of A_1 that is normal in G. Then A_1/U is finite by Lemma 2.10, so that $X/C_X(A_1/U)$ is also finite. Thus $Y = C_X(A_1/U)$ is a normal subgroup of finite index in G. As C has non-trivial centre, the factor group G/C is infinite by Lemma 2.1, and hence $Y/C \neq \{1\}$. Therefore Y/C contains a finite non-trivial normal subgroup H/C of G/C. Since H/C is a finite p'-group, it follows from Maschke's Theorem that $A_1 = U \times V$, where V is a non-trivial normal subgroup of H. Then $[V,H] \leq [A_1,Y] \leq U$, so that $[V,H]$ is contained in $U \cap V = \{1\}$ and hence $C_{A_1}(H) \neq \{1\}$.

Clearly $C_{A_1}(H)$ is normal in G, and so $A_1/C_{A_1}(H)$ is finite. A second application of Maschke's Theorem yields now that $A_1 = C_{A_1}(H) \times W$ where W is a finite normal subgroup of H. Therefore $[A_1,H] = [W,H]$ is contained in W and so it is finite. On the other hand, $[A_1,H]$ is normal in G, so that $[A_1,H] = \{1\}$ and $H \leq C$. This contradiction proves that G is monolithic. ■

Lemma 2.12. *Let G be a monolithic JNNF-group of prime characteristic p. Then the Fitting subgroup A of G is the unique minimal normal subgroup of G.*

Proof. Assume by contradiction that the unique minimal normal subgroup M of G is properly contained in A. Since G/M is nilpotent-by-finite, it follows that A/M contains a finite non-trivial subgroup E/M such that E is normal in G.

Thus E has a finite G-composition series. Since $G/C_G(E)$ is nilpotent-by-finite, we obtain that $E = E_1 \times E_2$, where E_1 is a finite normal subgroup of G and E_2 is a normal subgroup of G having no finite G-composition factors (see [14], corollary 1). Therefore $E_1 = \{1\}$, and $E = E_2$ has no finite G-composition factors. This contradiction proves the lemma. ■

The last theorem of this section describes monolithic *JNNF*-groups of prime characteristic. In particular, it follows from Theorem 2.11 that all periodic *JNNF*-groups with non-trivial Fitting subgroup have such structure.

Theorem 2.13. *Let G be a monolithic group with monolith M. Then G is a JNNF-group of prime characteristic p if and only if there exists a subgroup X of G such that $G = X \ltimes M$ and the following conditions hold:*

(a) *M is an elementary abelian p-group and $C_G(M) = M$;*

(b) *X is an infinite nilpotent-by-finite group.*

Proof. Suppose first that G is a $JNNF$-group of characteristic p. Then G has the required structure by Lemma 2.7 and Lemma 2.12.

Conversely, suppose that $G = X \ltimes M$ satisfies conditions (a) and (b), and let N be any non-trivial normal subgroup of G. Since $C_G(M) = M$, we have $M \cap N \neq \{1\}$, so that M is contained in N and hence G/N is nilpotent-by-finite. Finally, assume by contradiction that G is nilpotent-by-finite and let F be the Fitting subgroup of G. Then $Z(F)$ is non-trivial, and so M is contained in $Z(F)$. Therefore F is contained in $C_G(M) = M$, and G/M is finite. This contradiction completes the proof. ∎

3. Just-non-(supersoluble-by-finite) groups

It is well known that every supersoluble group is nilpotent-by-finite, so that a group is supersoluble-by-finite if and only if it is nilpotent-by-finite and finitely generated. In particular, every finite-by-supersoluble group is supersoluble-by-finite. We say that a group G is a *JNSF-group* if it is just-non-(supersoluble-by-finite), i.e. if G is not supersoluble-by-finite but all its proper quotients have this property. Note that soluble just-non-supersoluble groups are completely described in [12].

Lemma 3.1. *Let G be a JNSF-group. Then G is a JNNF-group. Moreover, if G has characteristic* 0*, the Fitting subgroup A of G has finite rank.*

Proof. Assume by contradiction that G is nilpotent-by-finite, and let x be a non-trivial element of the centre of the Fitting subgroup of G. Then x has finitely many conjugates in G, and hence $\langle x\rangle^G$ is finitely generated. It follows that G is finitely generated and so also supersoluble-by-finite. This contradiction shows that G is not nilpotent-by-finite, so that it is a $JNNF$-group.

Suppose now that G has characteristic 0, and let a be any non-trivial element of A. Since $G/\langle a\rangle^G$ satisfies the maximal condition on subgroups, we have that $A/\langle a\rangle^G$ is finitely generated, and hence A is a finitely generated G/A-module. It follows that A contains a free-abelian subgroup S such that A/S is a π-group, where π is a finite set of prime numbers (see [10], part 2, lemma 9.53). Consider a prime number q that is not in π, so that $A^q \cap S = S^q$. As G/A^q is supersoluble-by-finite, the group A/A^q is finite, so that S/S^q is also finite, and hence A has finite rank. ∎

Lemma 3.1 allows us to use the results of Section 2 to characterise $JNSF$-groups.

Theorem 3.2. *A group G is a JNSF-group of characteristic* 0 *if and only if it contains non-trivial torsion-free abelian subgroups A and X satisfying the following conditions:*

(a) *A is normal in G, and it is a faithful just-infinite G/A-module;*

(b) *X is finitely generated;*
(c) *$A \cap X = \{1\}$ and AX has finite index in G.*

Proof. Suppose that G is a $JNSF$-group of characteristic 0. Then G is $JNNF$ and its Fitting subgroup has finite rank by Lemma 3.1, so that by Theorem 2.6 there exist in G torsion-free abelian subgroups A and X satisfying conditions (a) and (c). Moreover, X is finitely generated since G/A is supersoluble-by-finite.

Conversely, suppose that G contains subgroups A and X satisfying the conditions of the statement. Then G is a $JNNF$-group of characteristic 0 by Theorem 2.6. Let N be any non-trivial normal subgroup of G. Then $A \cap N \neq \{1\}$, and so $A/A \cap N$ is finite. It follows that $G/A \cap N$ is finitely generated, so that G/N is supersoluble-by-finite, and G is a $JNSF$-group. ■

Theorem 3.3. *A group G is a JNSF-group of prime characteristic p if and only if it contains an abelian normal subgroup A of exponent p such that G/A is an infinite supersoluble-by-finite group and A is a faithful just-infinite G/A-module.*

Proof. Suppose first that G is a $JNSF$-group of prime characteristic p, and let A be the Fitting subgroup of G. Then A is an abelian group of exponent p and G/A is infinite and supersoluble-by-finite. Moreover, $C_G(A) = A$ by Theorem 2.2. It is also clear that A is a just-infinite G/A-module, since every proper quotient of G satisfies the maximal condition on subgroups.

Conversely, let the group G satisfy the conditions of the statement, and let N be any non-trivial normal subgroup of G. As $C_G(A) = A$, we have that $A \cap N \neq \{1\}$, and hence $A/A \cap N$ is finite. Therefore G/N is finite-by-supersoluble-by-finite, and so also supersoluble-by-finite. Clearly A is infinite, so that G is not supersoluble-by-finite, and hence it is a $JNSF$-group. ■

References

[1] S. Franciosi and F. de Giovanni, Soluble groups with many Černikov quotients, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Serie VIII* **79** (1985), 19–24.

[2] S. Franciosi and F. de Giovanni, Soluble groups with many nilpotent quotients, *Proceedings of the Royal Irish Academy* **89**A (1989), 43–52.

[3] S. Franciosi, F. de Giovanni and L.A. Kurdachenko, Groups whose proper quotients are FC-groups, *Journal of Algebra* **186** (1996), 544–77.

[4] L.A. Kurdachenko and I.Ya. Subbotin, Groups whose proper quotients are hypercentral, *Journal of the Australian Mathematical Society, Series A* **65** (1998), 224–37.

[5] J.C. Lennox and D.J.S. Robinson, Nearly maximal subgroups of finitely generated soluble groups, *Archiv der Mathematik* **38** (1982), 289–95.

[6] D. McCarthy, Infinite groups whose proper quotients are finite I, *Communications on Pure and Applied Mathematics* **21** (1968), 545–62.

[7] D. McCarthy, Infinite groups whose proper quotients are finite II, *Communications on Pure and Applied Mathematics* **23** (1970), 767–89.

[8] M.F. Newman, On a class of metabelian groups, *Proceedings of the London Mathematical Society. Third Series* **10** (1960), 354–64.

[9] M.F. Newman, On a class of nilpotent groups, *Proceedings of the London Mathematical Society. Third Series* **10** (1960), 365–75.

[10] D.J.S. Robinson, *Finiteness conditions and generalized soluble groups*, Springer, Berlin, 1972.

[11] D.J.S. Robinson, Cohomology of locally nilpotent groups, *Journal of Pure and Applied Algebra* **48** (1987), 281–300.

[12] D.J.S. Robinson and J.S. Wilson, Soluble groups with many polycyclic quotients, *Proceedings of the London Mathematical Society. Third Series* **48** (1984), 193–229.

[13] J.S. Wilson, Groups with every proper quotient finite, *Mathematical Proceedings of the Cambridge Philosophical Society* **69** (1971), 373–91.

[14] D.I. Zaicev, Direct sums of abelian groups with operators, *Ukrainian Mathematical Journal* **40** (1988), 257–63.

[15] Z. Zhang, Groups whose proper quotients are finite-by-nilpotent, *Archiv der Mathematik* **57** (1991), 521–30.