ON GENERALISED QUASI-PERIODIC RINGS

By

Howard E. Bell*
Department of Mathematics, Brock University, St Catharines, Canada

and

Adil Yaqub
Department of Mathematics, University of California, Santa Barbara

[Received 14 August 2000. Read 25 June 2001. Published 31 December 2002.]

Abstract

We call a ring R generalised quasi-periodic if for each $x \in R$ there exist distinct positive integers m, n and relatively prime non-zero integers r, s such that $rx^m = sx^n$. We investigate when such rings must be periodic, and we establish commutativity or near-commutativity results. In the final section we give applications to a larger class of rings.

A ring R is called periodic (resp. quasi-periodic) if for each $x \in R$ there exist integers n, m with $n > m \geq 1$ such that $x^n = x^m$ (resp. $x^n = kx^m$ for some integer k). Periodic rings have been studied by various authors over several decades; quasi-periodic rings have been studied in [3], [6], [10] and [11]. In this paper we study a larger class of rings R called *generalised quasi-periodic* rings, defined by the property that for each $x \in R$ there exist distinct positive integers m, n and non-zero integers r, s, with $(r, s) = 1$, for which $rx^m = sx^n$. To avoid notational awkwardness we will use the term G-rings as a synonym for generalised quasi-periodic rings.

Our first principal section gives theorems stating that certain G-rings are in fact periodic; the next section is devoted to commutativity and near-commutativity results for G-rings; the final section gives applications to a larger class of rings called pseudo-periodic rings.

1. Preliminaries

The symbols $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^*$ will denote respectively the ring of integers, the set of positive integers and the set of non-zero integers. The symbol Q will, as usual, denote the ring of rational numbers; and, for a division ring Δ, $M_n(\Delta)$ will denote the ring of $n \times n$ matrices over Δ.

If R is any ring, we will denote by Z, T and N the centre, the ideal of torsion elements and the set of nilpotent elements respectively; $C(R)$, $J(R)$ and $\mathscr{P}(R)$ will be the commutator ideal, the Jacobson radical and the prime radical respectively. The ring R will be called *reduced* if $N = \{0\}$.

An element $x \in R$ is called a *periodic* element if there exist distinct $m, n \in \mathbb{Z}^+$

*Corresponding author, e-mail: hbell@spartan.ac.brocku.ca

such that $x^n = x^m$, so that R is periodic if each of its elements is periodic. A J-ring is a reduced periodic ring or, alternatively, a ring R such that for each $x \in R$ there exists an integer $n = n(x) > 1$ for which $x^n = x$. We will make use of two easily verified facts: (i) if x is a periodic element of R, some power of x is idempotent; (ii) if x is a periodic element of $J(R)$, then $x \in N$.

At one point we will mention weakly periodic rings, which are defined as rings R such that each $x \in R$ can be written in the form $a + u$, where $u \in N$ and $a^n = a$ for some $n = n(a) > 1$. It is known that all periodic rings are weakly periodic, but whether all weakly periodic rings are periodic appears to be an open problem.

We conclude this section with four elementary lemmas.

Lemma 1.1. *Let $r, s \in \mathbb{Z}^*$ with $(r, s) = 1$. Let R be a ring, and let x be an element of R such that $rx^m = sx^n$, $n > m$. Then*

(i) $rx - sx^{n-m+1} \in N$;

(ii) *for each $j \in \mathbb{Z}^+$, $r^j x^m = s^j x^{m+j(n-m)}$;*

(iii) *there exists $t \in \mathbb{Z}^+$ such that $y = x^t$ satisfies $r_1 y = s_1 y^2$ for some $r_1, s_1 \in \mathbb{Z}^*$ with $(r_1, s_1) = 1$.*

Proof. (i) Let $y = rx - sx^{n-m+1}$. Then $0 = x^{m-1}y = x^{m-2}rxy = x^{m-2}sx^{n-m+1}y = x^{m-2}(rx - sx^{n-m+1})y = x^{m-2}y^2$; and, proceeding in the obvious way, we ultimately obtain $y^m = 0$.

(ii) The proof is by induction, the $j = 1$ case being obvious. If $r^k x^m = s^k x^{m+k(n-m)}$, then $r^{k+1}x^m = s^k r x^{m+k(n-m)} = s^k s x^{n+k(n-m)} = s^{k+1}x^{m+(k+1)(n-m)}$.

(iii) By (ii) we have $r_1, s_1 \in \mathbb{Z}^*$ with $(r_1, s_1) = 1$ and $n_0 > m$ such that $r_1 x^m = s_1 x^{n_0}$ and $n_0 - m > m$. Multiplying this equality by $x^{n_0 - 2m}$, we get $r_1 x^{n_0 - m} = s_1 x^{2(n_0 - m)}$.

Lemma 1.2. *Let R be a ring. Let $x \in R$ be such that $rx^n = sx^m$, where $n > m$ and $r, s \in \mathbb{Z}^*$ with $(r, s) = 1$. If $x \in T$, then x is periodic.*

Proof. Write $T = R_{p_1} \oplus R_{p_2} \oplus \ldots \oplus R_{p_n} \oplus \ldots$, where $p_1 < p_2 < \ldots$ is the sequence of primes and R_{p_i} is the p_i-primary component of R. Since x has non-zero component in only finitely many R_{p_i}, we may assume that $p^k x = 0$ for some prime p and $k \in \mathbb{Z}^+$. Let $t = \phi(p^k)$. If $(r, p) = (s, p) = 1$, then $(rx^m)^t = (sx^n)^t$ gives $x^{mt} = x^{nt}$ by the Euler–Fermat theorem. If one of r, s, say s, is divisible by p, then $(rx^m)^t = (sx^n)^t$ yields the result that $x^{mt} \in N$. In any event, x is periodic.

Lemma 1.3. *If R is any torsion-free G-ring with 1, then R is reduced. Hence, if R is any G-ring with 1, then $N \subseteq T$.*

Proof. Let R be a torsion-free G-ring with 1. Suppose that $u \in R$ with $u^2 = 0 \neq u$. Since $1 + u$ is a unit, there exist $n > 1$ and $r, s \in \mathbb{Z}^*$ such that $r(1 + u)^n = s1$. Thus, $r(1 + nu) = s1$, i.e. $rnu = (s - r)1$. Multiplying by u gives $(s - r)u = 0$, hence $s = r$. But this implies that $rnu = 0$, which is not possible. Therefore R is reduced.

Now let R be any G-ring with 1. If $1 \in T$, then $R = T$; hence we may assume that R/T is a torsion-free G-ring with 1, which must be reduced. Thus, $N \subseteq T$.

Our final lemma in this section is a useful result from Chacron [5]. It also appears, with a different proof, as theorem 1 of [2].

Lemma 1.4. *Let R be a ring such that for each $x \in R$ there exist $m \in \mathbb{Z}^+$ and $p(X) \in \mathbb{Z}[X]$ for which $x^m = x^{m+1}p(x)$. Then R is periodic.*

2. Certain *G*-rings are periodic

It is clear that a G-ring need not be periodic. Indeed the ring Q is a G-ring, as are all subrings of Q and all rings of the form $Q \oplus S$, where S is a nil ring. In this section we identify some conditions that are sufficient for a G-ring to be periodic.

Theorem 2.1. *Let $r, s \in \mathbb{Z}^*$ with $(r, s) = 1$. Let R be a ring such that for each $x \in R$ there exist distinct positive integers m, n and non-negative integers j, k such that*

$$r^j x^m = s^k x^n. \tag{$*$}$$

Then R is periodic.

PROOF. Call $x \in R$ a type 1 element (resp. type 2 element) if x satisfies $(*)$ with $n > m$ (resp. $n < m$). Certain elements may be of both types, but this does not create a problem.

By Lemma 1.2, it is sufficient to show that each element of R has a power that is in T. Let $x \in R$, and let p and q be distinct positive primes, neither of which divides r or s, for which px and qx have the same type. We may assume without loss of generality that both are of type 1; hence it follows easily from Lemma 1.1 (ii) that there exist positive integers m, t and non-negative integers j, j_1, k, k_1 such that

$$r^j (px)^m = s^k (px)^{m+t} \tag{2.1}$$

and

$$r^{j_1} (qx)^m = s^{k_1} (qx)^{m+t}. \tag{2.2}$$

We may assume that $j_1 \geq j$. Multiplying (2.1) by $r^{j_1 - j} q^m$ and (2.2) by p^m and subtracting the resulting equalities, we obtain

$$(s^k r^{j_1 - j} q^m p^{m+t} - s^{k_1} p^m q^{m+t}) x^{m+t} = 0. \tag{2.3}$$

By unique factorisation in $\mathbb{Z}$, the coefficient of x^{m+t} in (2.3) is not zero, hence $x^{m+t} \in T$ as required.

Theorem 2.2. *Let R be a ring satisfying an identity $rx^m = sx^n$, where n and m are positive integers with $n > m$ and $r, s \in \mathbb{Z}^*$ with $(r, s) = 1$. Then R is periodic. Moreover, if R has 1 and m and n are of opposite parity, R is a J-ring satisfying the identity $x^{n-m+1} = x$.*

PROOF. The first conclusion is immediate from Theorem 2.1, so we assume that R has 1 and m and n are of opposite parity. Substituting 1 and -1 shows that $r1 = s1$

and $r1 = -s1$, hence $2r1 = 2s1 = 0$; and, since $(r, s) = 1$, we conclude that $2x = 0$ for all $x \in R$. If one of r and s was even, then R would be nil; hence r and s are both odd and R satisfies the identity $x^m = x^n$. Now if $u \in N$, substituting $1 + u$ for x yields $1 = (1 + u)^{n-m} = 1 + u + \binom{n-m}{2}u^2 + \cdots + u^{n-m}$ and hence $u = u^2p(u)$ for some $p(X) \in \mathbb{Z}[X]$. But this implies that $u = 0$, so R is reduced. Therefore, R satisfies the identity $x^{n-m+1} = x$ by Lemma 1.1 (i).

Theorem 2.3. *If R is a weakly periodic G-ring with* 1*, then R is periodic.*

Proof. By Lemma 1.3, $N \subseteq T$. If $x \in R$ and we write $x = a + u$, where $a^n = a$ for some $n > 1$ and $u \in N$, we see that $x^n - x \in T$; therefore, by Lemma 1.2 there exist distinct $j, k \in \mathbb{Z}^+$ for which $(x^n - x)^j = (x^n - x)^k$. It follows by Lemma 1.4 that R is periodic.

3. Commutativity results for G-rings

Of course, G-rings need not be commutative, nor even nearly commutative in the sense that $C(R)$ is nil; there exist periodic counter-examples. Hence, we seek conditions under which G-rings are commutative or have restricted commutativity behaviour. Our next theorem is central to this study.

Theorem 3.1. *Let R be a reduced G-ring. Then R is commutative; moreover, R is either a J-ring or isomorphic to a subring of the ring Q.*

Proof. By Lemma 1.1 (i), every reduced G-ring is a generalised quasi-Jacobson ring as defined by Wang and Luh in [11]. Thus, our conclusions follow from theorems 3, 5 and 7 of [11].

It was proved long ago by Herstein that a periodic ring must be commutative if $N \subseteq Z$ [8]; and in [3] the same was shown for quasi-periodic rings. We now provide an extension to G-rings.

Theorem 3.2. *If R is any G-ring with $N \subseteq Z$, then R is commutative.*

Proof. Since $N \subseteq Z$, N is an ideal; and, by Theorem 3.1, $\bar{R} = R/N$ is either a J-ring or isomorphic to a subring of Q. In the first case, Lemma 1.4 shows that R is periodic and hence commutative by Herstein's result. The ring Q has the property that if $u, v \neq 0$ there exist $r, s \in \mathbb{Z}^*$ with $(r, s) = 1$ such that $ru = sv$. Therefore, if $\bar{R} = R/N$ is isomorphic to a subring of Q, and if $x, y \in R \backslash N$, there exist relatively prime $r, s \in \mathbb{Z}^*$ such that $rx - sy \in N \subseteq Z$. It follows that $[rx - sy, y] = r[x, y] = 0$ and $[x, sy - rx] = s[x, y] = 0$; and, since $(r, s) = 1$, $[x, y] = 0$. Therefore, R is commutative.

A notion of near-commutativity introduced in [4] is that $C(R)$ is periodic. In this sense, G-rings with 1 are nearly commutative.

Theorem 3.3. *If R is any G-ring with 1, then $C(R)$ is periodic.*

PROOF. If $1 \in T$, then $R = T$ and, by Lemma 1.2, R is periodic. If $1 \notin T$, then R/T is reduced by Lemma 1.3 and commutative by Theorem 3.1. Thus $C(R) \subseteq T$, and hence $C(R)$ is periodic by Lemma 1.2.

We now define R to be a *special G-ring* if for each $x \in R$ there exist $n, m \in \mathbb{Z}^+$ *of opposite parity* and relatively prime $r, s \in \mathbb{Z}^*$ such that $rx^m = sx^n$.

Theorem 3.4. *If R is a special G-ring with 1, then $C(R)$ is nil.*

PROOF. Consider the ring $\bar{R} = R/J(R)$, which may be expressed as a subdirect product of primitive rings R_α, each of which is a special G-ring. Let $\tilde{R}$ be a typical R_α. Then by the density theorem there exists a division ring Δ such that either $\tilde{R} \cong \Delta$ or $M_2(\Delta)$ is a homomorphic image of a subring of $\tilde{R}$. By substituting the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we see that $M_2(\Delta)$ is not a special G-ring; therefore $\tilde{R} \cong \Delta$, and hence $\tilde{R}$ is commutative by Theorem 3.1. Thus $\bar{R}$ is commutative, and hence $C(R) \subseteq J(R)$. But by Theorem 3.3 $C(R)$ is periodic; and the only periodic elements in $J(R)$ are nilpotent. Consequently, $C(R)$ is nil.

It has been known for some time that if R is a periodic ring in which N is commutative then $C(R)$ is nil [1]. We seek extensions of this result to G-rings. Our key tool is the following lemma.

Lemma 3.5. *If R is a G-ring and N is commutative, then $N \subseteq J(R)$.*

PROOF. Suppose that $N \not\subseteq J(R)$. Then $R/J(R)$ has a non-zero nilpotent element $u + J(R)$, where $u \in N$. Write $R/J(R)$ as a subdirect product of primitive rings R_α, each of which is the image of R under an epimorphism ϕ_α. At least one R_α, say $\tilde{R}$, has a non-zero nilpotent element $\tilde{u} = \phi_\alpha(u)$, where $u \in N$. For arbitrary $r \in R$, denote $\phi_\alpha(r)$ by $\tilde{r}$.

By the density theorem there exists a division ring Δ such that either $\tilde{R} \cong M_n(\Delta)$ for some n or, for every positive integer m, $M_m(\Delta)$ is a homomorphic image of some subring $\tilde{S}_m$ of $\tilde{R}$. If $\operatorname{char} \Delta = 0$, Lemma 1.3 shows that $M_n(\Delta)$ is not a G-ring for $n > 1$; and Δ is reduced. Therefore Δ has prime characteristic, so that $\tilde{R} = T(\tilde{R})$ and hence $\tilde{R}$ is periodic.

Let $\tilde{u} \in \tilde{R}$ such that $\tilde{u}^2 = 0 \neq \tilde{u}$ and $u \in N$, and let $\tilde{x}$ be an arbitrary element of $\tilde{R}$. Then there exists $k \in \mathbb{Z}^+$ such that $(\tilde{u}\tilde{x})^k$ is an idempotent, say $\tilde{e}$. Let $y \in R$ such that $\tilde{y} = \tilde{e}$ and note that $\tilde{y}^t = \tilde{e}$ for any $t \in \mathbb{Z}^+$; hence by Lemma 1.1 (iii) we may assume that there exist $r, s \in \mathbb{Z}^*$ such that $(r, s) = 1$ and $re^2 = se$. Then $r[e^2, x] = s[e, x]$, i.e.

$$re[e, x] + r[e, x]e = s[e, x]. \tag{3.1}$$

Left-multiplyng (3.1) by e and using the fact that $re^2 = se$, we see that $re[e, x]e = 0 = re^2[e, x]e = se[e, x]e$; and because $(r, s) = 1$ we have $e[e, x]e = 0$ and hence

$[e,x]e \in N$. Therefore $[\tilde{e},\tilde{x}]\tilde{e} = \tilde{e}\tilde{x}\tilde{e} - \tilde{x}\tilde{e}$ commutes with $\tilde{u}$. It follows that $\tilde{u}(\tilde{x}\tilde{e} - \tilde{e}\tilde{x}\tilde{e})\tilde{u} = (\tilde{x}\tilde{e} - \tilde{e}\tilde{x}\tilde{e})\tilde{u}^2 = 0$; and, since $\tilde{u}\tilde{e} = 0$, we have $\tilde{u}\tilde{x}\tilde{e}\tilde{u} = 0 = \tilde{u}\tilde{x}(\tilde{u}\tilde{x})^k\tilde{u} = (\tilde{u}\tilde{x})^{k+2}$. Thus, $\tilde{u}\tilde{R}$ is a non-zero nil right ideal in $\tilde{R}$, which is not possible since $\tilde{R}$ is primitive. Hence $N \subseteq J(R)$ as claimed.

Theorem 3.6. *If R is any quasi-periodic ring in which N is commutative, then $C(R)^3 = \{0\}$.*

Proof. It is proved in [3] that, if R is quasi-periodic, $J(R) \subseteq N$. Thus, by Lemma 3.5, $N = J(R)$, so that N is an ideal. By Theorem 3.1 R/N is commutative, hence $C(R) \subseteq N$. Now, for any commutative ideal I, $I^2 \subseteq Z$; and any central ideal annihilates $C(R)$. Therefore $C(R)^3 = \{0\}$.

Remarks. 1. Theorem 3.6 is essentially Theorem 4.3 in [3], but the proof in [3] is incorrect.

2. For G-rings R, we cannot prove that $J(R) \subseteq N$, for it is known that Q has a subring Q_0 for which $Q_0 = J(Q_0)$.

Theorem 3.7. *If R is any G-ring with 1 in which N is commutative, then $C(R)^3 = \{0\}$.*

Proof. Let $x \in N$. Then $xR \subseteq T$ by Lemma 1.3, so xR is periodic by Lemma 1.2. But $x \in J(R)$ by Lemma 3.5, hence $xR \subseteq J(R)$; and, since periodic subrings of $J(R)$ are nil, $xR \subseteq N$ and similarly $Rx \subseteq N$. Since N commutative implies that N is an additive subgroup of R, N is an ideal; and, as in the previous proof, R/N is commutative. It follows as above that $C(R)^3 = \{0\}$.

Theorem 3.8. *If R is a G-ring with N finite, then $C(R)$ is finite.*

Proof. Consider $\bar{R} = R/\mathscr{P}(R)$. By a result of Klein and Bell [9], $\bar{R} = S \oplus F$, where S is reduced and F is a direct sum of finitely many total matrix rings over finite fields. Since S is commutative by Theorem 3.1, $C(\bar{R})$ is finite. Let $C(\bar{R}) = \{\overline{y_1}, \overline{y_2}, \ldots, \overline{y_k}\}$. If $y \in C(R)$, then $\bar{y} = y + \mathscr{P}(R) \in C(\bar{R})$; hence $\bar{y} = \overline{y_i}$ for some $i = 1, 2, \ldots, k$ and $y \in y_i + \mathscr{P}(R)$. Since $\mathscr{P}(R)$ is nil, hence finite, $C(R)$ is finite.

The final theorem of this section is representative of a class of theorems obtained by imposing some commutativity-or-finiteness condition on a G-ring. Recently, various group theorists have introduced setwise conditions that imply that a group must be either abelian or finite. For example, Hassanabadi and Rhemtulla [7] defined a group to be an (m,n)-group if

$$(XY)^n = (YX)^n \quad \text{for all subsets } X, Y \text{ of size } m; \tag{†}$$

and they proved that, if $m \geq n \geq 1$, every (m,n)-group is either abelian or finite. Defining an (m,n)-ring to be a ring satisfying (†), we have the following theorem.

Theorem 3.9. *Let R be a G-ring with 1. If R is also an (m,n)-ring with $m \geq n \geq 1$, then either $C(R)$ is finite or $C(R)^3 = \{0\}$.*

PROOF. If N is finite, $C(R)$ is finite by Theorem 3.8. If N is infinite, the group of units of R is infinite, hence abelian by the Hassanabadi–Rhemtulla result; thus, N is commutative and $C(R)^3 = \{0\}$ by Theorem 3.7.

4. Some applications

We define a ring R to be *pseudo-periodic* if for each $x \in R$ there exist distinct $m, n \in \mathbb{Z}^+$ and $r, s \in \mathbb{Z}^*$, not necessarily relatively prime, for which $rx^m = sx^n$.

The class of pseudo-periodic rings, which contains all rings R such that $R = T$, is obviously much larger than the class of G-rings; and in general we cannot expect results for G-rings to extend to pseudo-periodic rings. However, our G-ring results do yield some information about pseudo-periodic rings.

Note that, if $rx^m = sx^n$ for $r, s, \in \mathbb{Z}^*$ and if $d = (r, s)$, then $d(r'x^m - s'x^n) = 0$ for some $r', s' \in \mathbb{Z}^*$ with $(r', s') = 1$. Thus, if R is pseudo-periodic, R/T is a G-ring; hence torsion-free pseudo-periodic rings are G-rings.

Theorem 2.2 yields the following:

Theorem 4.1. *Let R be a ring satisfying an identity of the form $rx^m = sx^n$, where m, n are distinct positive integers and $r, s \in \mathbb{Z}^*$. If T is periodic, then R is periodic.*

PROOF. The ring R/T satisfies the hypotheses of Theorem 2.2, hence is periodic. Therefore, for each $x \in R$ there exist distinct $m, n \in \mathbb{Z}^+$ such that $x^m - x^n \in T$. Since T is periodic, it follows by Lemma 1.4 that R is periodic.

Our final theorem is reminiscent of Theorems 3.2 and 3.7.

Theorem 4.2. *Let R be a pseudo-periodic ring with 1. If T is commutative, then $C(R)^2 = \{0\}$. If $T \subseteq Z$, then R is commutative.*

PROOF. If $1 \in T$, there is nothing to prove; hence, we assume that $1 \notin T$, in which case R/T is a reduced G-ring by Lemma 1.3. Thus, R/T is commutative by Theorem 3.1; and therefore $C(R) \subseteq T$. But Theorem 3.1 also tells us that R/T is isomorphic to a subring of Q; hence for each $x, y \in R \backslash T$ there exist relatively prime $r, s \in \mathbb{Z}^*$ such that $rx - sy \in T$. Since T is commutative, it follows that

$$r[x, z] = s[y, z] \quad \text{for all} \quad z \in T. \tag{4.1}$$

Replacing z by zy, we get $r[x, zy] = s[y, zy]$, so that $rz[x, y] + r[x, z]y = s[y, z]y$; and by (4.1) we have $rz[x, y] = 0$. Similarly, replacing z by zx, we get $sz[x, y] = 0$; and, since $(r, s) = 1$ for each choice of $x, y \in R \backslash T$, we conclude that

$$z[x, y] = 0 \quad \text{for all} \quad z \in T \quad \text{and} \quad x, y \in R \backslash T. \tag{4.2}$$

Since $x \in R \backslash T$ and $u \in T$ implies that $x + u \notin T$, it follows easily from (4.2) that

$T[x, y] = \{0\}$ for all $x, y \in R$. We have already noted that $C(R) \subseteq T$; therefore any product having at least two factors that are commutators is 0. Thus, $C(R)^2 = \{0\}$.

If $T \subseteq Z$, then in the preceding argument z may be any element of R, so for each $x, y \in R \backslash T$ we have $r[x, y] = s[x, y] = 0 = [x, y]$. Therefore R is commutative.

Acknowledgement

This research was supported by the Natural Sciences and Engineering Research Council of Canada, Grant no. 3961.

References

[1] H.E. Bell, Some commutativity results for periodic rings, *Acta Mathematica Academiae Scientiarum Hungaricae* **28** (1976), 279–83.

[2] H.E. Bell, On commutativity of periodic rings and near rings, *Acta Mathematica Academiae Scientiarum Hungaricae* **36** (1980), 293–302.

[3] H.E. Bell, On quasi-periodic rings, *Archiv der Mathematik* **36** (1981), 502–9.

[4] H.E. Bell and A.A. Klein, On rings with Engel cycles, *Canadian Mathematical Bulletin* **34** (1991), 295–300.

[5] M. Chacron, On a theorem of Herstein, *Canadian Journal of Mathematics* **21** (1969), 1348–53.

[6] S. Feigelstock, Rings in which a power of each element is an integral multiple of the element, *Archiv der Mathematik* **32** (1979), 551–4.

[7] A.M. Hassanabadi and A. Rhemtulla, Criteria for commutativity in large groups, *Canadian Mathematical Bulletin* **41** (1998), 65–70.

[8] I.N. Herstein, A note on rings with central nilpotent elements, *Proceedings of the American Mathematical Society* **5** (1954), 620.

[9] A.A. Klein and H.E. Bell, Rings with finitely many nilpotent elements, *Communications in Algebra* **22** (1994), 349–54.

[10] J. Luh, J. Wang and L. Chung, On quasi-boolean rings and generalized quasi-boolean rings, *Mathematica Japonica* **20** (1975), 43–7.

[11] J. Wang and J. Luh, The structure of a certain class of rings, *Mathematica Japonica* **20** (1975), 149–57.