CONVERGENCE PROPERTIES OF POSITIVE ELEMENTS IN BANACH ALGEBRAS

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Abstract

We recall the definition and properties of an algebra cone in an ordered Banach algebra and continue to develop spectral theory for the positive elements. If (a_n) is a sequence of positive elements converging to a, then an interesting question is that of which properties of the spectral radius $r(a)$ of a are 'inherited' by $r(a_n)$. We show that under suitable circumstances if $r(a)$ is a Riesz point of the spectrum $\sigma(a)$ of a (relative to some inessential ideal), then $r(a_n) \to r(a)$ and, for all n big enough, $r(a_n)$ is a Riesz point of $\sigma(a_n)$. If the Laurent series of the corresponding resolvents are then investigated, some conclusions can be drawn regarding the convergence of the spectral idempotents, as well as the positive eigenvectors associated with a_n. Some of these results are applicable to certain types of operators.

1. Introduction

In [16] and [13] some spectral theory of positive elements in ordered Banach algebras was developed. An interesting problem in this theory is that of investigating a convergent sequence (a_n) of positive elements: for instance, what can be said about the sequence $(r(a_n))$? Does $r(a_n)$ inherit certain properties of $r(a)$? This problem was originally studied in the context of positive operators on Banach lattices, and results were obtained by Aràndiga and Caselles in [1], [2] and [3]. In this paper we consider the problem in the context of an ordered Banach algebra and show that some interesting answers can be obtained by the use of Newburgh's theorem about the continuity of the spectral radius in some cases.

We shall first provide some preliminary notation and definitions in Section 2 and then explain the concept of an ordered Banach algebra in Section 3. In Section 4 we show that under suitable circumstances, which include some monotonicity assumptions on the spectral radius, if (a_n) is a sequence of positive elements such that $a_n \to a$, and $r(a)$ is a Riesz point of the spectrum $\sigma(a)$ of a (relative to some inessential ideal), then $r(a_n) \to r(a)$, and, for all n big enough, $r(a_n)$ is a Riesz point of $\sigma(a_n)$. We also apply these results to the positive regular operators on a Dedekind complete Banach lattice and to the positive operators on a Hilbert space. In Section 5 we investigate the Laurent series of the resolvents of a and a_n and show that some conclusions can be drawn regarding the convergence of the coefficients of these series, which include results concerning the spectral idempotents, as well as the positive Laurent eigenvectors associated with a_n, whose existence is guaranteed by the well-known Krein–Rutman theorem.

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2. Preliminaries

Throughout, A will be a complex Banach algebra with unit 1, unless otherwise stated. If A and B are Banach algebras, then a linear operator $T : A \to B$ is called a *homomorphism* if $T(ab) = TaTb$ $(a, b \in A)$ and $T1 = 1$. The spectrum of an element a in A will be denoted by $\sigma(a)$, the resolvent set of a in A by $\rho(a)$, and the spectral radius of a in A by $r(a)$ (or by $\sigma(a, A)$, $\rho(a, A)$ and $r(a, A)$ if necessary to avoid confusion). The *peripheral spectrum* $\operatorname{psp}(a)$ of a is the set $\sigma(a) \bigcap \{\lambda \in \mathbb{C} : |\lambda| = r(a)\}$. It is a non-void closed subset of the spectrum. The boundary of the unbounded connected component of the resolvent set of a will be denoted by $\partial_\infty \sigma(a)$. We denote the radical of A by Rad A, and A is *semisimple* if Rad $A = \{0\}$. For an ideal F the closed ideal kh (F) is defined by kh $(F) := \{a \in A : a + \overline{F} \in \text{Rad } (A/\overline{F})\}$, and F and kh (F) have the same set of idempotents [5, proof of lemma 2.1]. Recall that an isolated point $\lambda \in \sigma(a)$ is a *Riesz point of* $\sigma(a)$ relative to F if the corresponding spectral projection $p(a, \lambda)$ belongs to F. Following [5], we define for an element $a \in A$ the set $\mathscr{D}(a, F)$ as

$$\mathscr{D}(a, F) = \sigma(a) \backslash \{\lambda \in \sigma(a) \text{: } \lambda \text{ is a Riesz point of } \sigma(a) \text{ relative to } F\}.$$

If it is clear what ideal is meant, we shall just say 'λ is a Riesz point of $\sigma(a)$' and write $\mathscr{D}(a)$. An ideal I in A is called *inessential* whenever the spectrum in A of every element in I is either finite or a sequence converging to zero [5].

If A has minimal left ideals (resp. minimal right ideals), then its *socle* Soc A is defined as the sum of the minimal left ideals (it is also equal to the sum of the minimal right ideals, so it is a (two-sided) ideal). From the multiplicative characterisation of Soc A (in the case that A is semisimple) in [7, theorem 2.1], it is clear that, if A is semisimple, then Soc A exists and is inessential—in fact, $\sigma(a)$ is finite for all $a \in$ Soc A. In a semisimple Banach algebra A we refer to the elements of Soc A as *finite* elements since, for $a \in$ Soc A, we have that dim $aAa < \infty$ [15, p. 659 and corollary 3.5].

If $a \in A$, then an isolated point α of $\sigma(a)$ is a *pole* of the resolvent $(\lambda - a)^{-1}$ of a if $(\alpha - a)p(a, \alpha)$ is nilpotent, and α is a *pole of order* k of $(\lambda - a)^{-1}$ if k is the smallest natural number such that $(\alpha - a)^k p(a, \alpha) = 0$.

Lemma 2.1. *Let A be a semisimple Banach algebra, I an inessential ideal of A, and $a \in A$. Then a point α in $\sigma(a)$ is a Riesz point of $\sigma(a)$ relative to I if and only if α is a pole of the resolvent of a and $p := p(a, \alpha) \in I$.*

Proof. For the non-trivial implication, let α be a Riesz point of $\sigma(a)$ relative to I, i.e. α is an isolated point of $\sigma(a)$ and $p \in I$. By [12, theorem 1.4] $p \in$ kh (Soc A), and, since kh (Soc A) and Soc A have the same idempotents, $p \in$ Soc A. Since Soc A is an ideal, $(\alpha - a)p \in$ Soc A. As $(\alpha - a)p$ is also quasi-nilpotent [8, proposition 9, p. 36], $(\alpha - a)p$ is nilpotent [13, lemma 3.10], so that α is a pole of the resolvent of a. ■

If K is a compact subset of the complex plane $\mathbb{C}$, then the *connected hull* ηK of

K has as its complement the unbounded connected component of $\mathbb{C}\backslash K$. Thus ηK is the union of K and its holes, where a *hole* of K is a bounded component of $\mathbb{C}\backslash K$.

An indispensable tool in this paper will be the important theorem of Newburgh.

Theorem 2.2 (J.D. Newburgh [6, theorem 3.4.4]). *Let A be a Banach algebra and $x \in A$. Suppose that U and V are two disjoint open sets such that $\sigma(x) \subset U \bigcup V$ and $\sigma(x) \bigcap U \neq \emptyset$. Then there exists $r > 0$ such that $\|x - y\| < r$ implies that $\sigma(y) \bigcap U \neq \emptyset$.*

This implies in particular that the spectral radius is continuous at all elements having finite or countable spectrum (see [6, pp 51–2]). It is well known, however, that the spectral radius is always upper semicontinuous (see [6, theorem 3.4.2].

In order to obtain some interesting examples, recall the following. Let E be a Dedekind complete Banach lattice, and denote by $\mathscr{L}(E)$ the space of bounded linear operators on E. An operator $T : E \to E$ is *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The space of all regular operators on E is denoted by $\mathscr{L}^r(E)$ and is a subspace of $\mathscr{L}(E)$. When $\mathscr{L}^r(E)$ is provided with the r-norm

$$||T||_r := \inf\{||S||: S \in \mathscr{L}(E), |Tx| \leq S|x| \text{ for all } x \in E\},$$

it becomes a Banach algebra that contains the unit of $\mathscr{L}(E)$ ([18, IV, § 1] and [4]). The spectrum of T in $\mathscr{L}(E)$ is denoted by $\sigma(T, \mathscr{L}(E))$, and, if T is regular, then the spectrum of T in $\mathscr{L}^r(E)$ is denoted by $\sigma_o(T) = \sigma(T, \mathscr{L}^r(E))$ and is called the *o-spectrum* of T. This concept was introduced by Schaefer in [19]. The *peripheral order spectrum* of a regular operator T is the peripheral spectrum of T considered as an element of $\mathscr{L}^r(E)$. We denote the ideal of compact operators on E by $\mathscr{K}(E)$. Now, the set of finite-rank operators on E is an ideal in $\mathscr{L}^r(E)$, and, since for a finite-rank operator on E the usual spectrum coincides with the o-spectrum (see [4, theorem 2.6]), it is an inessential ideal. Let $\mathscr{K}^r(E)$ denote the closure in $\mathscr{L}^r(E)$ of the ideal of finite-rank operators on E. The ideal $\mathscr{K}^r(E)$ in $\mathscr{L}^r(E)$ is called the ideal of *r-compact* operators [4], and, by [5, corollary 2.6], it is a closed inessential ideal.

Let H be a Hilbert space. An operator T on H is called *positive* (and we write $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. Denote by $\mathscr{L}(H)$ the set of bounded linear operators on H and by $\mathscr{K}(H)$ the set of compact operators on H. Then $\mathscr{K}(H)$ is a closed inessential ideal of $\mathscr{L}(H)$.

3. Ordered Banach algebras

In [16, section 3] we defined an algebra cone C of a complex Banach algebra A and showed that C induced on A an ordering that was compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra. We recall those definitions now and also the additional properties that C may have. Of these properties, normality is the most significant, as it reconciles the order structure and the topology of A.

Let A be a complex Banach space. We call a non-empty subset C of A a *cone* of A if C satisfies the following:

(1) $C + C \subseteq C$;
(2) $\lambda C \subseteq C$ for all $\lambda \geq 0$.

If in addition C satisfies $C \cap -C = \{0\}$, then C is called a *proper* cone.

Any cone C of A induces an *ordering* '$\leq$' on A in the following way:

$$a \leq b \text{ if and only if } b - a \in C$$

$(a, b \in A)$. It can be shown that this ordering is a partial order on A, i.e. for every $a, b, c \in A$

(a) $a \leq a$ ($\leq$ is *reflexive*),
(b) if $a \leq b$ and $b \leq c$, then $a \leq c$ ($\leq$ is *transitive*).

Furthermore, C is proper if and only if this partial order has the additional property of being *antisymmetric*, i.e. if $a \leq b$ and $b \leq a$ then $a = b$. Considering the partial order that C induces, we find that $C = \{a \in A \colon a \geq 0\}$, and therefore we call the elements of C *positive*.

A cone C is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leq a \leq b$ in A that $\|a\| \leq \alpha \|b\|$. It is obvious that, if C is a normal cone, then C is proper.

Now let A be a complex Banach algebra with unit 1. A cone C of A is called an *algebra cone* of A if C satisfies:

(3) $C.C \subseteq C$,
(4) $1 \in C$.

Motivated by this concept, we call a complex Banach algebra with unit 1 an *ordered Banach algebra* (OBA) if A is partially ordered by a relation '$\leq$' in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$

(1′) $a, b \geq 0 \Rightarrow a + b \geq 0$,
(2′) $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$,
(3′) $a, b \geq 0 \Rightarrow ab \geq 0$,
(4′) $1 \geq 0$.

Therefore, if A is ordered by an algebra cone C, then A, or more specifically (A, C), is an OBA.

An algebra cone C of A is called *proper* (*normal*) if C is a proper (normal) cone of A and *closed* if it is a closed subset of A. If an algebra cone C has the property that $r(a) \leq r(b)$ whenever $0 \leq a \leq b$, then we say that the spectral radius in (A, C) is *monotone*. It is always the case that if C is normal then the spectral radius is monotone [16, theorem 4.1]. If C is closed and the spectral radius in (A, C) is monotone, then the spectral radius of every positive element is contained in its spectrum [16, theorem 5.2].

Let (A, C) be an OBA. If F is a closed ideal in A and if $\pi : A \rightarrow A/F$ is the canonical homomorphism, then $(A/F, \pi C)$ is an OBA. We say that the spectral radius in $(A/F, \pi C)$ is monotone if $0 \leq a \leq b$ in A relative to C implies that $r(a + F, A/F) \leq r(b + F, A/F)$.

It is well known that a pole of the resolvent of an element a in a Banach algebra is an eigenvalue of a. In the case of a positive element, even more can be said.

Theorem 3.1 ([13, theorem 3.2]).

(1) *Let A be a Banach algebra and $a \in A$. If α is a pole of the resolvent of a of order k, so that*

$$(\lambda - a)^{-1} = \sum_{j=-k}^{\infty} (\lambda - \alpha)^j b_j$$

and $0 \neq u := b_{-k}$, then $au = \alpha u = ua$.

(2) *Let (A, C) be an OBA with C closed and $a \in C$. If $r(a)$ is a pole of the resolvent of a of order k, so that*

$$(\lambda - a)^{-1} = \sum_{j=-k}^{\infty} (\lambda - r(a))^j b_j$$

and $0 \neq u := b_{-k}$, then $u \in C$ and $au = r(a)u = ua$.

The second part of the above theorem (called the Krein–Rutman theorem) says that, if the spectral radius $r(a)$ of a positive element a is a pole of the resolvent of a, then $r(a)$ is an eigenvalue of a, with a positive eigenvector u. To distinguish u from possible other eigenvectors, we shall call u the (positive) *Laurent eigenvector* of the eigenvalue $r(a)$ of a.

We proceed to give some important examples of ordered Banach algebras.

Example 3.2. Let E be a Dedekind complete Banach lattice, $C = \{x \in E : x \geq 0\}$ and $K = \{T \in \mathscr{L}(E) : TC \subset C\}$. Then $(\mathscr{L}^r(E), K)$ is an OBA with a closed normal algebra cone and $(\mathscr{L}^r(E)/\mathscr{K}^r(E), \pi K)$ is an OBA such that the spectral radius in $(\mathscr{L}^r(E)/\mathscr{K}^r(E), \pi K)$ is monotone.

Proof. It follows from the remarks after [16, theorem 6.1] that $(\mathscr{L}^r(E), K)$ is an OBA with a closed normal algebra cone. Since E is Dedekind complete, it follows from [11, theorem 2.8] that the spectral radius in the OBA $(\mathscr{L}^r(E)/\mathscr{K}^r(E), \pi K)$ is monotone. ■

Example 3.3. Let A be a commutative C^*-algebra, $C = \{x \in A : x = x^* \text{ and } \sigma(x) \subset [0, \infty)\}$ and F a closed ideal of A. Then (A, C) is an OBA with a closed normal algebra cone and $(A/F, \pi C)$ is an OBA such that πC is normal in A/F.

Proof. The results follow from [9, proposition 3.7, p. 247], [14, proposition 1.3.5, p. 7], and [9, theorem 4.6, p. 252]. ■

Example 3.4. Let H be a Hilbert space, $C = \{T \in \mathscr{L}(H) : T \geq 0\}$ and B a commutative subset of $\mathscr{L}(H)$ consisting of positive operators. Then there exists a commutative C^*-algebra M that is a closed subalgebra of $\mathscr{L}(H)$ and that contains B and the identity operator. Furthermore,

(1) $\sigma(T, \mathscr{L}(H)) = \sigma(T, M)$ for all $T \in M$.

(2) If $C_1 = \{T \in M : T = T^* \text{ and } \sigma(T, M) \subset [0, \infty)\}$, then $C_1 = C \cap M$.

(3) (M, C_1) is an OBA with C_1 closed and normal, and πC_1 is normal in M/I, where $I = \mathscr{K}(H) \cap M$.
(4) $I = \mathscr{K}(H) \cap M$ is an inessential ideal of M.

Proof. If $C = \{T \in \mathscr{L}(H) : T \geq 0\}$, it follows from [17, theorem 12.32] that $C = \{T \in \mathscr{L}(H) : T = T^*$ and $\sigma(T, \mathscr{L}(H)) \subset [0,\infty)\}$. Furthermore, $\mathscr{L}(H)$ is a C^*-algebra (see [9, example 1.2, p. 238]). Since $B \subset C$, it follows that B is a normal subset of $\mathscr{L}(H)$ and hence, by [8, proposition 7, p. 190], contained in a maximal normal subset M of $\mathscr{L}(H)$, which is a commutative C^*-algebra and a closed subalgebra of $\mathscr{L}(H)$ containing the identity operator.

(1) This is [8, theorem 8, p. 190].
(2) It is clear from (1) that $C_1 = C \cap M$.
(3) This follows from Example 3.3, since $I = \mathscr{K}(H) \cap M$ is a closed ideal of M.
(4) This follows from (1). ■

4. Properties of the peripheral spectrum and spectral radius

We prepare for the proofs of our main results by providing two important lemmas.

Lemma 4.1. *Let A be a Banach algebra and I an ideal of A. Suppose that (a_n) is a sequence in A such that $a_n \to a \in A$.*

(1) *If (α_n) is a sequence such that $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$, then $\alpha \in \sigma(a)$.*
(2) *If $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$ relative to I, then the following properties hold:*
  (a) *There are finitely many (Riesz) points in $\operatorname{psp}(a)$.*
  (b) *$r(a_n) \to r(a)$ as $n \to \infty$.*
  (c) *If (α_n) is a sequence such that $\alpha_n \in \operatorname{psp}(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$, then $\alpha \in \operatorname{psp}(a)$.*

Proof. (1) Suppose that (α_n) is a sequence such that $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$. If $\alpha \notin \sigma(a)$, then there exists a $\delta > 0$ such that $\sigma(a) \subset U$, with $U := \mathscr{C}\overline{B}(\alpha, \delta)$. Since $a_n \to a$, it follows from the upper semicontinuity of the spectrum that there exists an $N \in \mathbb{N}$ such that $\sigma(a_n) \subset U$, for all $n \geq N$. Hence $\alpha_n \notin B(\alpha, \delta)$, for all $n \geq N$, which contradicts $\alpha_n \to \alpha$.

(2) (a) If $\operatorname{psp}(a)$ contains infinitely many points, they contain a bounded sequence, which—by the Bolzano–Weierstrass theorem—has a convergent subsequence. If λ is the limit of this sequence, then $\lambda \in \operatorname{psp}(a)$, since $\operatorname{psp}(a)$ is closed. But then λ is an accumulation point of $\sigma(a)$ in $\operatorname{psp}(a)$, which is impossible, as $\operatorname{psp}(a)$ consists of Riesz points.

(b) If $\operatorname{psp}(a) = \{\mu_1, \ldots, \mu_k\}$ (see (1)), let $B(0,K)$ and $B(\mu_i, r)$ $(i = 1, \ldots, k)$ be disjoint open balls with $\sigma(a) \backslash \operatorname{psp}(a) \subset B(0,K)$. Let $0 < \epsilon < r$. Then $\sigma(a) \subset B(0, r(a) + \epsilon)$. Since $a_n \to a$, it follows from the upper semicontinuity of the spectrum that there exists an $N_{\epsilon,1} \in \mathbb{N}$ such that $\sigma(a_n) \subset B(0, r(a) + \epsilon)$ and hence $r(a_n) < r(a) + \epsilon$, for all $n \geq N_{\epsilon,1}$. If (say) $|\mu_1| = r(a)$, then $\sigma(a) \subset B(0,K) \cup B(\mu_1, \epsilon) \cup \ldots \cup B(\mu_k, \epsilon)$ and $\sigma(a) \cap B(\mu_1, \epsilon) \neq \emptyset$. From Newburgh's theorem (Theorem 2.2) it follows that there exists an $N_{\epsilon,2} \in \mathbb{N}$ such that $\sigma(a_n) \cap B(\mu_1, \epsilon) \neq \emptyset$, say $\alpha_n \in \sigma(a_n)$ and $|\alpha_n - \mu_1| < \epsilon$, for all $n \geq N_{\epsilon,2}$. Then $r(a_n) \geq |\alpha_n| > r(a) - \epsilon$ for all $n \geq N_{\epsilon,2}$. Let $N := \max\{N_{\epsilon,1}, N_{\epsilon,2}\}$. Then it follows that, if $n \geq N$, then $r(a) - \epsilon < r(a_n) < r(a) + \epsilon$.

(c) This follows from (1) and (2b). ■

Lemma 4.2. *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Let $a \in C$.*

(1) *If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a + I) < r(a)$.*

(2) *If, in addition, the spectral radius in (A, C) is also monotone, then $r(a)$ is a Riesz point of $\sigma(a)$ if and only if $r(a + I) < r(a)$.*

PROOF. (1) If $r(a + I) = r(a)$, then, by [16, theorem 5.3], $r(a) \in \sigma(a + I)$. Therefore, by [6, theorem 5.7.4], $r(a) \in \mathscr{D}(a)$, so that $r(a)$ is not a Riesz point of $\sigma(a)$.

(2) If $r(a)$ is not a Riesz point of $\sigma(a)$, then, by [16, theorem 5.2], $r(a) \in \mathscr{D}(a)$, so that $r(a) \in \eta\sigma(a + I)$, by [6, theorem 5.7.4]. Therefore $r(a) \leq r(a + I)$. ■

In [13, theorem 4.1] it was proved that, under certain circumstances, if the spectral radius of a positive element a is a Riesz point of its spectrum then the peripheral spectrum of a consists of isolated points. Using the above lemma, we formulate a slightly stronger version of this result.

Theorem 4.3. *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. If $a \in C$ is such that $r(a)$ is a Riesz point of $\sigma(a)$, then* psp(a) *consists of Riesz points of $\sigma(a)$.*

PROOF. Let $\lambda \in \text{psp}(a)$. If $\lambda \in \sigma(a+I)$, then $r(a) = |\lambda| \leq r(a+I)$, so that $r(a) = r(a+I)$. But, by Lemma 4.2, this is contradictory to the fact that $r(a)$ is a Riesz point of $\sigma(a)$. Therefore it follows from [10, theorem 6.1] that λ is a Riesz point of $\sigma(a)$. ■

It was further proved in [13, theorem 4.3] that, under suitable circumstances, if $0 \leq a \leq b$ and a and b have the same spectral radius, then, if the spectral radius of b is a Riesz point of the spectrum of b, the spectral radius of a is a Riesz point of the spectrum of a. If we use Theorem 4.3, it is again possible to strengthen this result.

Theorem 4.4. *Let (A, C) be an OBA with C closed and the spectral radius in (A, C) monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a, b \in A$ with $0 \leq a \leq b$ and $r(a) = r(b)$. If $r(b)$ is a Riesz point of $\sigma(b)$, then* psp(a) *consists of Riesz points of $\sigma(a)$.*

PROOF. Since $r(b)$ is a Riesz point of $\sigma(b)$, it follows from Lemma 4.2 that $r(b + I) < r(b)$. By the monotonicity of the spectral radius in $(A/I, \pi C)$ we have that $r(a + I) \leq r(b + I)$, and, since $r(a) = r(b)$, it follows that $r(a + I) < r(a)$. Lemma 4.2 implies that $r(a)$ is a Riesz point of $\sigma(a)$. The result now follows from Theorem 4.3. ■

The above theorem includes the fact that, under the mentioned assumptions, a positive element a inherits from any 'larger' positive element the property of its spectral radius being a Riesz point of its spectrum.

We now focus our attention on the following problem: if (a_n) is a sequence of positive elements converging to an element a, what can be said about the sequence $(r(a_n))$? Furthermore, does $r(a_n)$ inherit certain properties of $r(a)$? We begin with the following result, which provides an interesting continuity property of the spectral radius.

Theorem 4.5. *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. If $r(a)$ is a Riesz point of $\sigma(a)$, then $r(a_n) \to r(a)$ as $n \to \infty$.*

Proof. Since C is closed, $a \in C$. If $r(a)$ is a Riesz point of $\sigma(a)$, Theorem 4.3 implies that $\operatorname{psp}(a)$ consists of Riesz points of $\sigma(a)$. The result therefore follows from Lemma 4.1 (2b). ■

It follows from Theorem 4.4 that, if (a_n) is a sequence of positive elements converging to an element a that dominates a_n, i.e. $a_n \leq a$, for all n big enough, and such that $r(a_n) = r(a)$ for all these n, then, if the spectral radius of a is a Riesz point of the spectrum of a, for all n big enough the spectral radius of a_n is a Riesz point of the spectrum of a_n. We now show that the conditions $a_n \leq a$ and $r(a_n) = r(a)$ are not necessary.

Theorem 4.6. *Let (A, C) be an OBA with C closed and the spectral radius in (A, C) monotone. Let I be a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$ and $a_n \to a$ as $n \to \infty$. If $r(a)$ is a Riesz point of $\sigma(a)$, then there is a natural number N such that, for all $n \geq N$, $r(a_n)$ is a Riesz point of $\sigma(a_n)$.*

Proof. By the upper semicontinuity of the spectral radius $\overline{\lim}\, r(a_n + I) \leq r(a + I)$, and by Lemma 4.2 we have $r(a + I) < r(a)$. Let $K \in \mathbb{R}$ be such that $r(a + I) < K < r(a)$. Then there exists an $N_1 \in \mathbb{N}$ such that, for $n \geq N_1$, $r(a_n + I) < K$. It follows from Theorem 4.5 that there exists an $N_2 \in \mathbb{N}$ such that, for $n \geq N_2$, $r(a_n) \geq K$. Therefore, if $n \geq N := \max\{N_1, N_2\}$, then $r(a_n + I) < r(a_n)$, so that, by Lemma 4.2, $r(a_n)$ is a Riesz point of $\sigma(a_n)$. ■

It is interesting to note that Theorem 4.6 can be extended to any sequence (α_n), where each α_n is an element of the boundary of the unbounded connected component of the resolvent set of a_n, and (α_n) converges to an element α in the peripheral spectrum of a. Moreover, this result can be proved without assuming that the spectral radius in (A, C) is monotone.

Theorem 4.7. *Let (A, C) be an OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$*

for all $n \in \mathbb{N}$, *that* $a_n \to a$ *as* $n \to \infty$, *and that* $r(a)$ *is a Riesz point of* $\sigma(a)$. *If* $\alpha \in \operatorname{psp}(a)$, $\alpha_n \in \partial_\infty \sigma(a_n)$ *for all* $n \in \mathbb{N}$ *and* $\alpha_n \to \alpha$ *as* $n \to \infty$, *then there is a natural number* N *such that, for all* $n \geq N$, α_n *is a Riesz point of* $\sigma(a_n)$.

PROOF. Suppose that there exists a subsequence (α_{n_k}) of (α_n) such that $\alpha_{n_k} \in \eta\sigma(a_{n_k}+I)$ for all $k \in \mathbb{N}$. Then $|\alpha_{n_k}| \leq r(a_{n_k}+I)$ for all $k \in \mathbb{N}$. Since $\alpha_n \to \alpha \in \operatorname{psp}(a)$, it follows that $|\alpha_{n_k}| \to r(a)$ as $k \to \infty$, so that, by the upper semicontinuity of the spectral radius, $r(a) \leq \overline{\lim}\, r(a_{n_k}+I) \leq r(a+I)$. Therefore $r(a) = r(a+I)$, which contradicts the fact that $r(a)$ is a Riesz point of $\sigma(a)$ (see Lemma 4.2). Consequently, there exists an $N \in \mathbb{N}$ such that, if $n \geq N$, then $\alpha_n \notin \eta\sigma(a_n+I)$. Since $\alpha_n \in \partial_\infty\sigma(a_n)$, we also have $\alpha_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$. Therefore it follows from [10, theorem 6.1] that, if $n \geq N$, then α_n is a Riesz point of $\sigma(a_n)$. ■

Corollary 4.8. *Let* (A, C) *be an OBA with* C *closed and* I *a closed inessential ideal of* A *such that the spectral radius in* $(A/I, \pi C)$ *is monotone. Suppose that* $a \in A$, $a_n \in C$ *for all* $n \in \mathbb{N}$, *that* $a_n \to a$ *as* $n \to \infty$ *and that* $r(a)$ *is a Riesz point of* $\sigma(a)$. *If* $\alpha \in \mathbb{C}$, $\alpha_n \in \operatorname{psp}(a_n)$ *for all* $n \in \mathbb{N}$ *and* $\alpha_n \to \alpha$ *as* $n \to \infty$, *then there is a natural number* N *such that, for all* $n \geq N$, α_n *is a Riesz point of* $\sigma(a_n)$.

PROOF. The result follows from Theorem 4.7 by noticing that $\operatorname{psp}(a_n) \subset \partial_\infty\sigma(a_n)$ and using Theorem 4.3 and Lemma 4.1 (2c). ■

We are now in a position to give some interesting properties concerning the spectral radii and peripheral spectra of positive regular operators and of positive operators on Hilbert spaces. (Some of these properties are similar to results that have been proved (by different methods) by Aràndiga and Caselles in [1], [2] and [3] for positive operators on Banach lattices.)

Corollary 4.9. *If* E *is a Dedekind complete Banach lattice, then the regular operators on* E *have the following properties:*

1. *If* T *is a positive operator on* E *such that* $r_o(T)$ *is a Riesz point of the o-spectrum of* T, *then the peripheral order spectrum of* T *consists of Riesz points of* $\sigma_o(T)$.
2. *If* $0 \leq S \leq T$, $r_o(S) = r_o(T)$ *and* $r_o(T)$ *is a Riesz point of the o-spectrum of* T, *then the peripheral order spectrum of* S *consists of Riesz points of* $\sigma_o(S)$.
3. *If* (T_n) *is a sequence of positive operators converging to an operator* T *in the r-norm and* $r_o(T)$ *is a Riesz point of the o-spectrum of* T, *then*
    - (a) $r_o(T_n) \to r_o(T)$ *as* $n \to \infty$ *and*
    - (b) *for all* n *big enough* $r_o(T_n)$ *is a Riesz point of the o-spectrum of* T_n.
4. *Suppose that* (T_n) *is a sequence of positive operators converging to an operator* T *in the r-norm and* $r_o(T)$ *is a Riesz point of the o-spectrum of* T. *If* $\alpha_n \in \partial_\infty\sigma_o(T_n)$ *for all* $n \in \mathbb{N}$ *and* $\alpha_n \to \alpha$ *as* $n \to \infty$ *where* α *is in the peripheral order spectrum of* T, *then, for all* n *big enough,* α_n *is a Riesz point of the o-spectrum of* T_n.

Proof. The results follow from Example 3.2, together with Theorems 4.3, 4.4, 4.5, 4.6 and 4.7, respectively. ■

Corollary 4.10. *Let H be a Hilbert space. The positive operators on H have the following properties:*

1. *If T is a positive operator on H and $r(T, \mathscr{L}(H))$ is a Riesz point of $\sigma(T, \mathscr{L}(H))$, then the peripheral spectrum of T in $\mathscr{L}(H)$ consists of Riesz points of $\sigma(T, \mathscr{L}(H))$.*
2. *Suppose that $0 \leq S \leq T$ and $ST = TS$. If $r(S, \mathscr{L}(H)) = r(T, \mathscr{L}(H))$ and $r(T, \mathscr{L}(H))$ is a Riesz point of $\sigma(T, \mathscr{L}(H))$, then the peripheral spectrum of S in $\mathscr{L}(H)$ consists of Riesz points of $\sigma(S, \mathscr{L}(H))$.*
3. *Suppose that (T_n) is a sequence of positive operators converging uniformly to an operator T such that the T_n $(n \in \mathbb{N})$ commute mutually. If $r(T, \mathscr{L}(H))$ is a Riesz point of $\sigma(T, \mathscr{L}(H))$, then*
    - (a) *$r(T_n, \mathscr{L}(H)) \to r(T, \mathscr{L}(H))$ as $n \to \infty$ and*
    - (b) *for all n big enough $r(T_n, \mathscr{L}(H))$ is a Riesz point of $\sigma(T_n, \mathscr{L}(H))$.*
4. *Suppose that (T_n) is a sequence of positive operators converging uniformly to an operator T such that the T_n $(n \in \mathbb{N})$ commute mutually. Suppose that $r(T, \mathscr{L}(H))$ is a Riesz point of $\sigma(T, \mathscr{L}(H))$. If $\alpha_n \in \partial_\infty \sigma(T_n, \mathscr{L}(H))$ for all $n \in \mathbb{N}$ and $\alpha_n \to \alpha$ as $n \to \infty$ where α is in the peripheral spectrum of T in $\mathscr{L}(H)$, then, for all n big enough, α_n is a Riesz point of $\sigma(T_n, \mathscr{L}(H))$.*

Proof. In each case we choose a B in Example 3.4 and let M, C_1 and I be as in this example.

1. Let $B = \{T\}$. Then the result follows from Theorem 4.3.
2. Let $B = \{S, T\}$. Then the result follows from Theorem 4.4.
3. Let $B = \{T_n : n \in \mathbb{N}\}$. Then the results follow from Theorems 4.5 and 4.6.
4. Let $B = \{T_n : n \in \mathbb{N}\}$. Then the result follows from Theorem 4.7.

■

5. Convergence properties

Returning to Banach algebras in general, we recall that, under the mentioned assumptions, if $a_n \to a$, then $r(a_n) \to r(a)$, and $r(a)$ and $r(a_n)$ (for all n big enough) are Riesz points of the corresponding spectra and hence (in the semisimple case) poles of the corresponding resolvents. We shall now investigate the Laurent series of these resolvents, making the assumptions as general as possible (see Theorem 5.2) to begin with. We first make the following observation.

Theorem 5.1. *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is a pole of the resolvent of a_n, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is a pole of the resolvent of a. Let $r_n := d(\alpha_n, \sigma(a_n) \backslash \{\alpha_n\})$, for all $n \in \mathbb{N}$ such that $\sigma(a_n) \backslash \{\alpha_n\} \neq \emptyset$. If $r_n \to s$, then $s \neq 0$.*

Proof. For each $n \in \mathbb{N}$, r_n is the largest number such that $B(\alpha_n, r_n) \cap \sigma(a_n) = \{\alpha_n\}$. Hence for each $n \in \mathbb{N}$ and every $m \in \mathbb{N}$ there exists a $\lambda_{n,m} \in B(\alpha_n, r_n + \frac{1}{m}) \cap \sigma(a_n)$

such that $\lambda_{n,m} \neq \alpha_n$ and $\lambda_{n,m} \notin B(\alpha_n, r_n)$. Since $a_n \to a$, the upper semicontinuity of the spectrum implies that each sequence $(\lambda_{n,m}) = (\lambda_{1,m}, \lambda_{2,m}, \ldots)$ is bounded and hence has a convergent subsequence. Denote the limits of these convergent subsequences by λ_m ($m \in \mathbb{N}$). Since $\lambda_{n,m} \in \sigma(a_n)$ for all m and n in $\mathbb{N}$, it follows from Lemma 4.1 (1) that $\lambda_m \in \sigma(a)$ for all $m \in \mathbb{N}$. By letting $n \to \infty$ in $r_n \leq |\lambda_{n,m} - \alpha_n| < r_n + \frac{1}{m}$ (where the convergent subsequence of $(\lambda_{n,m})$ is again denoted by $(\lambda_{n,m})$), it follows that $s \leq |\lambda_m - \alpha| \leq s + \frac{1}{m}$ for all $m \in \mathbb{N}$. Hence if $s = 0$ then $\lambda_m \to \alpha$ as $m \to \infty$, so that α is an accumulation point of $\sigma(a)$. Since α is a pole of $(\lambda - a)^{-1}$, it follows that $s \neq 0$. ■

Theorem 5.2. *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is a pole of the resolvent of a_n of order k_n, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is a pole of the resolvent of a of order k. If*

$$(\lambda - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j \quad (b_{-j} = 0 \textit{ for all } j > k)$$

and

$$(\lambda - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j} \quad (b_{n,-j} = 0 \textit{ for all } j > k_n)$$

are the Laurent series of the resolvents of a and a_n, then $b_{n,j} \to b_j$ as $n \to \infty$, for all $j \in \mathbb{Z}$.

PROOF. Let $r_n := d(\alpha_n, \sigma(a_n) \backslash \{\alpha_n\})$ for all $n \in \mathbb{N}$ such that $\sigma(a_n) \backslash \{\alpha_n\} \neq \emptyset$, and $r := d(\alpha, \sigma(a) \backslash \{\alpha\})$ if $\sigma(a) \backslash \{\alpha\} \neq \emptyset$ (and $r = 1$ (say) if $\sigma(a) \backslash \{\alpha\} = \emptyset$). From Theorem 5.1 it follows that $\inf_{n \in \mathbb{N}} r_n = K_1 > 0$. For $K_2 = \min\{K_1, r\}$ we have $B(\alpha_n, K_2) \cap \sigma(a_n) = \{\alpha_n\}$ for all $n \in \mathbb{N}$ and $B(\alpha, K_2) \cap \sigma(a) = \{\alpha\}$. Take a fixed $K > 0$ with $K < K_2$ and define $\Gamma, \Gamma_n : [0, 2\pi] \to \mathbb{C}$ by $\Gamma_n(t) = \alpha_n + Ke^{it}$ and $\Gamma(t) = \alpha + Ke^{it}$. Then $\Gamma_n^* \subset \rho(a_n)$ for all $n \in \mathbb{N}$ and $\Gamma^* \subset \rho(a)$, where Γ_n^* and Γ^* denote the ranges of Γ_n and Γ, respectively. It follows that, for each $j \in \mathbb{Z}$,

$$\begin{aligned} b_{n,j} &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{(\lambda - a_n)^{-1}}{(\lambda - \alpha_n)^{j+1}} \, d\lambda \\ &= \frac{1}{2\pi i} \int_0^{2\pi} g_{n,j}(t) \, dt \end{aligned}$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} b_j &= \frac{1}{2\pi i} \int_{\Gamma} \frac{(\lambda - a)^{-1}}{(\lambda - \alpha)^{j+1}} \, d\lambda \\ &= \frac{1}{2\pi i} \int_0^{2\pi} g_j(t) \, dt \end{aligned}$$

where $g_{n,j}(t) = (\alpha_n + Ke^{it} - a_n)^{-1} i(Ke^{it})^{-j}$ and $g_j(t) = (\alpha + Ke^{it} - a)^{-1} i(Ke^{it})^{-j}$ are continuous on $[0, 2\pi]$.

Let $f_n(t) = \alpha_n + Ke^{it} - a_n$ and $f(t) = \alpha + Ke^{it} - a$ for all $n \in \mathbb{N}$ and for all $t \in [0, 2\pi]$. Then (f_n) converges to f uniformly on $[0, 2\pi]$ and $f_n(t), f(t) \in B := \bigcup_{n \in \mathbb{N}} (\Gamma_n^* - a_n) \cup (\Gamma^* - a)$ for all $n \in \mathbb{N}$ and for all $t \in [0, 2\pi]$. Since B is compact and contained in the subset of invertible elements of A, the function $x \mapsto x^{-1}$ is uniformly continuous on B, so that $({f_n}^{-1})$ converges to f^{-1} uniformly on $[0, 2\pi]$. Since $|e^{-jit}| = 1$ for all $t \in [0, 2\pi]$ and for all $j \in \mathbb{Z}$, it follows that $(g_{n,j})$ converges to g_j uniformly on $[0, 2\pi]$ for each $j \in \mathbb{Z}$, which yields the result. ■

Notice in the above theorem that $b_{-1} = p(a, \alpha)$ and $b_{n,-1} = p(a_n, \alpha_n)$.

Corollary 5.3. *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is a pole of the resolvent of a_n, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is a pole of the resolvent of a. If $p := p(a, \alpha)$ and $p_n := p(a_n, \alpha_n)$, then $p_n \to p$ as $n \to \infty$.*

Corollary 5.4. *Let A be a Banach algebra and (a_n) a sequence in A such that $a_n \to a \in A$. Suppose that (α_n) is a sequence in $\mathbb{C}$ such that, for each $n \in \mathbb{N}$, α_n is a pole of the resolvent of a_n of order k_n, and $\alpha_n \to \alpha \in \mathbb{C}$ where α is a pole of the resolvent of a of order k. Let the Laurent series of the resolvents of a and a_n be as in Theorem 5.2, and $u := b_{-k}$, $u_n := b_{n,-k_n}$ (where $au = \alpha u = ua$ and $a_n u_n = \alpha_n u_n = u_n a_n$). If there exists an $N \in \mathbb{N}$ such that $k_n \leq k$ for all $n \geq N$, then $u_n \to u$ as $n \to \infty$.*

Proof. Suppose that $k_n \leq k$ for all $n \geq N$, for some $N \in \mathbb{N}$. If there exists no $N_1 \in \mathbb{N}$ such that $k_n = k$ for all $n \geq N_1$, then (k_n) has a subsequence with all its terms smaller than k. This subsequence in turn has a constant subsequence (k_{n_m}), say $k_{n_m} = v < k$ for all $m \in \mathbb{N}$. Let $c_n := (\alpha_n - a_n) p_n$ and $c = (\alpha - a) p$, where $p_n := p(a_n, \alpha_n)$ and $p := p(a, \alpha)$. Then by Corollary 5.3 we have that $p_n \to p$, so that $c_n \to c$ as $n \to \infty$. Therefore, although $c_{n_m}^{k_{n_m}} = 0$ for all $m \in \mathbb{N}$, $c_{n_m}^{k_{n_m}} = c_{n_m}^{v} \to c^{v} \neq 0$. This contradiction proves that there exists an $N_1 \in \mathbb{N}$ such that $k_n = k$ for all $n \geq N_1$. Therefore, for all n large enough, $u_n = b_{n,-k_n} = b_{n,-k}$, and $b_{n,-k} \to b_{-k} = u$ as $n \to \infty$, so that $u_n \to u$ as $n \to \infty$. ■

Keeping Lemma 2.1 in mind, we have the following theorem.

Theorem 5.5. *Let (A, C) be a semisimple OBA with C closed and I a closed inessential ideal of A such that the spectral radius in $(A/I, \pi C)$ is monotone. Suppose that $a \in A$, $a_n \in C$ for all $n \in \mathbb{N}$, that $a_n \to a$ as $n \to \infty$ and that $r(a)$ is a Riesz point of $\sigma(a)$. If $\alpha_n \in \operatorname{psp}(a_n)$ such that $\alpha_n \to \alpha$, then the following hold:*

1. *For all n big enough, α_n is a pole, say of order k_n, of $(\lambda - a_n)^{-1}$, and α is a pole, say of order k, of $(\lambda - a)^{-1}$.*
2. *If*

$$(\lambda - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha)^j b_j \quad (b_{-j} = 0 \textit{ for all } j > k)$$

and for all $n \geq N$

$$(\lambda - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - \alpha_n)^j b_{n,j} \quad (b_{n,-j} = 0 \textit{ for all } j > k_n),$$

then $b_{n,j} \to b_j$ *as* $n \to \infty$, *for all* $j \in \mathbb{Z}$.

(3) *If* $p := p(a, \alpha)$ *and* $p_n := p(a_n, \alpha_n)$, *then* $p_n \to p$ *as* $n \to \infty$.

(4) *If* $k_n \leq k$ *for all* $n \geq N_1$, *for some* $N_1 \in \mathbb{N}$, *and* $u := b_{-k}$, $u_n := b_{n,-k_n}$ (*where* $au = \alpha u = ua$ *and* $a_n u_n = \alpha_n u_n = u_n a_n$), *then* $u_n \to u$ *as* $n \to \infty$.

PROOF. By Theorem 4.3 psp(a) consists of Riesz points of $\sigma(a)$. Therefore, from Lemma 4.1 (2c), $\alpha \in \operatorname{psp}(a)$ and hence α is a Riesz point of $\sigma(a)$ and (by Lemma 2.1) a pole of $(\lambda - a)^{-1}$. By Corollary 4.8 α_n is a Riesz point of $\sigma(a_n)$, and hence a pole of $(\lambda - a_n)^{-1}$, for all n big enough. This proves (1).

Using (1), we obtain (2) from Theorem 5.2, (3) from Corollary 5.3 and (4) from Corollary 5.4. ■

Finally, we can apply our observations to the spectral radii and obtain what is probably the most interesting version (if not the strongest) of the above theorem.

Theorem 5.6. *Let* (A, C) *be a semisimple OBA with* C *closed and the spectral radius in* (A, C) *monotone. Let* I *be a closed inessential ideal of* A *such that the spectral radius in* $(A/I, \pi C)$ *is monotone. Suppose that* $a \in A$, $a_n \in C$ *for all* $n \in \mathbb{N}$, *that* $a_n \to a$ *as* $n \to \infty$ *and that* $r(a)$ *is a Riesz point of* $\sigma(a)$. *Then the following hold:*

(1) *For all* n *big enough,* $r(a_n)$ *is a pole, say of order* k_n, *of* $(\lambda - a_n)^{-1}$, *and* $r(a)$ *is a pole, say of order* k, *of* $(\lambda - a)^{-1}$.

(2) *If*

$$(\lambda - a)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a))^j b_j \quad (b_{-j} = 0 \textit{ for all } j > k)$$

and for all $n \geq N$

$$(\lambda - a_n)^{-1} = \sum_{j=-\infty}^{\infty} (\lambda - r(a_n))^j b_{n,j} \quad (b_{n,-j} = 0 \textit{ for all } j > k_n),$$

then $b_{n,j} \to b_j$ *as* $n \to \infty$, *for all* $j \in \mathbb{Z}$.

(3) *If* $p := p(a, r(a))$ *and* $p_n := p(a_n, r(a_n))$, *then* $p_n \to p$ *as* $n \to \infty$.

(4) *Let* u *denote the positive Laurent eigenvector of the eigenvalue* $r(a)$ *of* a, *and* u_n *the positive Laurent eigenvector of the eigenvalue* $r(a_n)$ *of* a_n. *If* $k_n \leq k$ *for all* $n \geq N_1$, *for some* $N_1 \in \mathbb{N}$, *then* $u_n \to u$ *as* $n \to \infty$.

PROOF. Since the spectral radius in (A, C) is monotone, $r(a_n) \in \sigma(a_n)$, so that $r(a_n) \in \operatorname{psp}(a_n)$, and by Theorem 4.3 psp(a) consists of Riesz points of $\sigma(a)$, so that, by Lemma 4.1 (2b), $r(a_n) \to r(a)$. The results (1)–(4) now follow from Theorem 5.5. ■

Alternatively, the theorem can be proved using Theorems 4.6, 4.5 and 5.2 and Corollaries 5.3 and 5.4.

Since the monotonicity of the spectral radius in (A, C) is needed only to ensure that $r(a_n) \in \sigma(a_n)$, this assumption can be removed if the sequence (a_n) under consideration has the property that $r(a_n) \in \sigma(a_n)$ for all n big enough.

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