FURI–PERA-TYPE THEOREMS FOR THE $\mathscr{U}_c^k$-ADMISSIBLE MAPS OF PARK

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Abstract

New fixed-point theorems are presented for $\mathscr{U}_c^k$-admissible and PK-admissible maps. The arguments rely on fixed-point theory for self-maps, together with properties of the Minkowski functional and retractions.

1. Introduction

This paper presents new fixed-point results of Furi–Pera type [1; 2] for the $\mathscr{U}_c^k$-admissible maps of Park [4; 5]. In the process we establish new Leray–Schauder alternatives for $\mathscr{U}_c^k$-admissible maps defined on closed convex subsets of locally convex Hausdorff topological vector spaces. Also we present new fixed-point results for set-valued maps with continuous selections (i.e. the PK-admissible maps).

For the remainder of this section we present some definitions and known results [3; 4; 5]. Let X and Y be Hausdorff topological vector spaces. Recall that a *polytope* P in X is any convex hull of a non-empty finite subset of X. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (the non-empty subsets of Y) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. A class $\mathscr{U}$ of maps [4; 5] is defined by the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single-valued continuous functions;
(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact-valued; and
(iii) for any polytope P, $F \in \mathscr{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathscr{U}$.

Definition 1.1. $F \in \mathscr{U}_c^k(X, Y)$ (i.e. F is $\mathscr{U}_c^k$-admissible) if for any compact subset K of X there is a $G \in \mathscr{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Recall from [5] that $\mathscr{U}_c^k$ is closed under compositions.

Next let Z and W be subsets of Hausdorff topological vector spaces X and Y, and F be a multifunction. We say that $F \in PK(Z, W)$ (i.e. F is PK-admissible) if W is convex and there exists a map $S : Z \to W$ with

$$Z = \bigcup\{int\, S^{-1}(w) : w \in W\},\ co(S(x)) \subseteq F(x) \text{ for } x \in Z,$$

and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$. Finally we recall the following well-known selection theorem.

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Theorem 1.1. *If Z is paracompact, W is convex and $F \in PK(Z, W)$, then there exists a continuous (single-valued) function $f : Z \to W$ with $f(x) \in F(x)$ for each $x \in Z$.*

2. Fixed-point results for $\mathcal{U}_c^k$-admissible maps

In this section we present some Leray–Schauder-type [2] alternatives and two fixed-point results of Furi–Pera type [1; 2] for $\mathcal{U}_c^k$-admissible maps. Our theorems are based on the following three fixed-point results of O'Regan [3].

Theorem 2.1. *Let Ω be a closed convex subset of a locally convex Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose that $F \in \mathcal{U}_c^k(\Omega, \Omega)$ with the following property holding:*

$$A \subseteq \Omega,\ A \subseteq \overline{co}(\{x_0\} \cup F(A)) \text{ implies that } \overline{A} \text{ is compact.} \tag{2.1}$$

Then F has a fixed point in Ω.

Theorem 2.2. *Let Ω be a closed convex subset of a locally convex Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose that $F \in \mathcal{U}_c^k(\Omega, \Omega)$ satisfies the following properties:*

$$A \subseteq \Omega,\ A \subseteq co(\{x_0\} \cup F(A)) \text{ implies that } \overline{A} \text{ is compact} \tag{2.2}$$

and

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } \Omega. \tag{2.3}$$

Then F has a fixed point in Ω.

Theorem 2.3. *Let Ω be a closed convex subset of a locally convex Hausdorff topological vector space E and $x_0 \in \Omega$. Suppose that $F \in \mathcal{U}_c^k(\Omega, \Omega)$ maps compact sets into relatively compact sets and satisfies* (2.3) *and suppose that the following properties hold:*

$$\begin{cases} A \subseteq \Omega,\ A \subseteq co(\{x_0\} \cup F(A)),\ \text{with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies that } \overline{A} \text{ is compact;} \end{cases} \tag{2.4}$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } \Omega \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A} \end{cases} \tag{2.5}$$

and

$$\text{if } A \text{ is a compact subset of } \Omega, \text{ then } \overline{co}(A) \text{ is compact.} \tag{2.6}$$

Then F has a fixed point in Ω.

Remark 2.1. If E is metrisable then (2.5) holds, and if E is quasi-complete then (2.6) holds.

Remark 2.2. In some cases (see [3]) it is possible to drop assumption (2.3) in Theorem 2.2 and Theorem 2.3. (We will discuss this later in this paper; see Theorems 2.7–2.10.)

Our first result is a non-linear alternative of Leray–Schauder type for $\mathcal{U}_c^k$-admissible maps that satisfy a condition of (2.1) type.

Theorem 2.4. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, $U^c C$ an open convex subset of E, and $0 \in U$. Suppose that $F \in \mathcal{U}_c^k(\overline{U}, C)$ satisfies the following two properties:*

$$A \subseteq \overline{U},\ A \subseteq \overline{co}(\{0\} \cup F(A)) \text{ implies that } \overline{A} \text{ is compact} \tag{2.7}$$

and

$$x \notin \lambda F(x) \text{ for } x \in \partial U \text{ and } \lambda \in (0,1). \tag{2.8}$$

Then F has a fixed point in $\overline{U}$.

PROOF. Let μ be the Minkowski functional on $\overline{U}$ and let $r: E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in E. \tag{2.9}$$

Let $G = rF$, and notice that $G: \overline{U} \to \overline{U}$. In addition since $r \in \mathscr{C} \subseteq \mathcal{U} \subseteq \mathcal{U}_c^k$ we have that $G \in \mathcal{U}_c^k(\overline{U}, \overline{U})$. Next we claim that

$$\text{if } A \subseteq \overline{U} \text{ and } A \subseteq \overline{co}(\{0\} \cup G(A)) \text{ then } \overline{A} \text{ is compact.} \tag{2.10}$$

To see this notice that, if $A \subseteq \overline{U}$ and $A \subseteq \overline{co}(\{0\} \cup rF(A))$ — note that (2.9) implies that $r(B) \subseteq co(B \cup \{0\})$ for any subset B of E — then

$$A \subseteq \overline{co}(\{0\} \cup co(F(A) \cup \{0\})) = \overline{co}(co(F(A) \cup \{0\})) = \overline{co}(F(A) \cup \{0\}).$$

Now (2.7) implies that $\overline{A}$ is compact, so (2.10) holds. Apply Theorem 2.1, and we deduce that there exists $x \in \overline{U}$ with $x \in G(x) = rF(x)$. Thus $x = r(y)$ for some $y \in F(x)$. Now, either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$, so $x = y \in F(x)$ and we are finished. If $y \notin \overline{U}$ then $r(y) = \frac{y}{\mu(y)}$ with $\mu(y) > 1$. Then $x = \lambda y$ (i.e. $x \in \lambda F(x)$) with $0 < \lambda = \frac{1}{\mu(y)} < 1$. This contradicts (2.8). ■

Theorem 2.5. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, $U^c C$ an open convex subset of E, and $0 \in U$. Suppose that $F \in \mathcal{U}_c^k(\overline{U}, C)$ satisfies (2.8) and the following two properties:*

$$A \subseteq \overline{U},\ A \subseteq co(\{0\} \cup F(A)) \text{ implies that } \overline{A} \text{ is compact} \tag{2.11}$$

and

$$F(\overline{A}) \subseteq \overline{F(A)} \text{ for any relatively compact subset } A \text{ of } \overline{U}. \tag{2.12}$$

Then F has a fixed point in $\overline{U}$.

Proof. Let μ, r, G be as in Theorem 2.4. Essentially the same reasoning as in Theorem 2.4 guarantees that $G \in \mathscr{U}_c^k(\overline{U}, \overline{U})$, and,

$$\text{if } A \subseteq \overline{U} \text{ and } A \subseteq co(\{0\} \cup G(A)), \text{ then } \overline{A} \text{ is compact.} \tag{2.13}$$

Next we claim that

$$G(\overline{A}) \subseteq \overline{G(A)} \text{ for any relatively compact subset } A \text{ of } \overline{U}. \tag{2.14}$$

This is immediate since F satisfies (2.12) and r is continuous, i.e. if A is a relatively compact subset of $\overline{U}$ then

$$G(\overline{A}) = r(F(\overline{A})) \subseteq r(\overline{F(A)}) \subseteq \overline{r(F(A))} = \overline{G(A)}.$$

Apply Theorem 2.2, and we deduce that there exists $x \in \overline{U}$ with $x \in G(x) = rF(x)$. Thus $x = r(y)$ for some $y \in F(x)$, and, as in Theorem 2.4, we have $y \in \overline{U}$. As a result we have $x = r(y) = y \in F(x)$. ■

Essentially the same reasoning as in Theorem 2.5 (except that here we use Theorem 2.3) establishes the following result.

Theorem 2.6. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, $U \subseteq C$ an open convex subset of E, and $0 \in U$. Suppose that $F \in \mathscr{U}_c^k(\overline{U}, C)$ maps compact sets into relatively compact sets. Assume that* (2.8) *and* (2.12) *are satisfied and suppose that the following properties hold:*

$$\begin{cases} A \subseteq \overline{U},\ A \subseteq co(\{x_0\} \cup F(A)),\ \textit{with } \overline{A} = \overline{C} \\ \textit{and } C \subseteq A \textit{ countable, implies that } \overline{A} \textit{ is compact;} \end{cases} \tag{2.15}$$

$$\begin{cases} \textit{for any relatively compact subset } A \textit{ of } \overline{U} \textit{ there} \\ \textit{exists a countable set } B \subseteq A \textit{ with } \overline{B} = \overline{A} \end{cases} \tag{2.16}$$

and

$$\textit{if } A \textit{ is a compact subset of } \overline{U} \textit{ then } \overline{co}(A) \textit{ is compact.} \tag{2.17}$$

Then F has a fixed point in $\overline{U}$.

We can remove condition (2.12) for certain subclasses of $\mathscr{U}_c^k(\overline{U}, C)$. Given a class $\mathscr{A}$ of maps (with $\mathscr{C} \subseteq \mathscr{A}$), we let $\mathscr{A}(X, Y)$ denote the set of maps $F : X \to 2^Y$ belonging to $\mathscr{A}$, and $\mathscr{A}_c$ the set of finite compositions of maps in $\mathscr{A}$. We say that $F \in \mathscr{A}_c^k(X, Y)$ if $F : X \to 2^Y$ is such that for any compact convex subset K of X there exists a closed map $G \in \mathscr{A}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. We also assume throughout that $\mathscr{A}_c^k$ is closed under compositions.

In [3] the following fixed-point result was established.

Theorem 2.7. *Let Ω be a closed convex subset of a locally convex Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose that $F \in \mathscr{A}_c^k(\Omega, \Omega)$ satisfies* (2.2) *and the* *following condition:*

$$\begin{cases} \textit{for any relatively compact convex subset } A \textit{ of } \Omega \textit{ and any} \\ \textit{closed map } G \in \mathscr{A}_c(\overline{A}, \Omega) \textit{ with } G(x) \subseteq F(x) \textit{ for } x \in \overline{A}, \\ G^\star \in \mathscr{U}_c^k(\overline{A}, \overline{A}) \textit{ if } G^\star(x) \neq \emptyset \textit{ for } x \in \overline{A}; \\ \textit{here } G^\star(x) = G(x) \cap \overline{A} \textit{ for } x \in \overline{A}. \end{cases} \tag{2.18}$$

Then F has a fixed point in Ω.

Remark 2.3. If, for example, $\mathscr{A}(X, Y)$ denotes the class of upper semicontinuous maps $H : X \to 2^Y$ with non-empty compact convex values, then clearly (2.18) is satisfied. In fact in this case $G^\star \in \mathscr{A}_c^k(\overline{A}, \overline{A}) \subseteq \mathscr{U}_c^k(\overline{A}, \overline{A})$.

Next we obtain a Leray–Schauder alternative for the subclass $\mathscr{A}$. (We will state and prove the result with Fr here. The result with rF we will leave to the reader.)

Theorem 2.8. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, $U^c C$ an open convex subset of E, and $0 \in U$. Suppose that $F \in \mathscr{A}_c^k(\overline{U}, C)$ satisfies* (2.8) *and the following conditions:*

$$A \subseteq C,\ A \subseteq co(\{0\} \cup F(co(\{0\} \cup A) \cap \overline{U}))\ \textit{implies that } \overline{A} \textit{ is compact} \tag{2.19}$$

and

$$\begin{cases} \textit{for any relatively compact convex subset } A \textit{ of } C \textit{ and any} \\ \textit{closed map } G \in \mathscr{A}_c(\overline{A}, C) \textit{ with } G(x) \subseteq Fr(x) \textit{ for } x \in \overline{A}, \\ G^\star \in \mathscr{U}_c^k(\overline{A}, \overline{A}) \textit{ if } G^\star(x) \neq \emptyset \textit{ for } x \in \overline{A}; \\ \textit{here } G^\star(x) = G(x) \cap \overline{A} \textit{ for } x \in \overline{A}, \textit{ and } r \\ \textit{is defined in } (2.9). \end{cases} \tag{2.20}$$

Then F has a fixed point in $\overline{U}$.

PROOF. Let r be as in (2.9) and let $H = Fr$. Notice since $\mathscr{A}_c^k$ is closed under compositions that $H \in \mathscr{A}_c^k(C, C)$. Next we claim that

$$\text{if } A \subseteq C \text{ and } A \subseteq co(\{0\} \cup H(A)) \text{ then } \overline{A} \text{ is compact.} \tag{2.21}$$

To see this, notice that if $A \subseteq C$ and $A \subseteq co(\{0\} \cup Fr(A))$ then, since $r(A) \subseteq co(A \cup \{0\})$, we have

$$A \subseteq co(\{0\} \cup F(co(\{0\} \cup A) \cap \overline{U})).$$

Now (2.19) implies that $\overline{A}$ is compact, so (2.21) holds. Apply Theorem 2.7, and we deduce that there exists $x \in C$ with $x \in Fr(x)$. If we let $z = r(x) \in \overline{U}$, then $z \in rF(z)$. Thus $z = r(y)$ for some $y \in F(z)$. Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$, so $z = r(y) = y \in F(z)$, and we are finished. If $y \notin \overline{U}$ then $r(y) = \lambda y$ with $\lambda = \frac{1}{\mu(y)} \in (0, 1)$. Thus $z = r(y) = \lambda y \in \lambda F(z)$ with $\lambda \in (0, 1)$. This contradicts (2.8). ■

Similarly we can obtain the analogue of Theorem 2.6 for maps in the subclass $\mathscr{A}$. First we recall the following fixed-point result from [3].

Theorem 2.9. *Let Ω be a closed convex subset of a locally convex Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose that $F \in \mathscr{A}_c^k(\Omega, \Omega)$ maps compact sets into relatively compact sets and assume that* (2.4), (2.5), (2.6) *and* (2.18) *hold. Then F has a fixed point in Ω.*

Essentially the same reasoning as in Theorem 2.8 establishes the following result.

Theorem 2.10. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, $U^c C$ an open convex subset of E, and $0 \in U$. Suppose that $F \in \mathscr{A}_c^k(\overline{U}, C)$ maps compact sets into relatively compact sets and assume that the following properties hold:*

$$\begin{cases} A \subseteq C,\ A \subseteq co(\{x_0\} \cup F(co(\{0\} \cup A)),\ \text{with } \overline{A} = \overline{Q} \\ \text{and } Q \subseteq A \text{ countable, implies that } \overline{A} \text{ is compact;} \end{cases} \tag{2.22}$$

$$\begin{cases} \text{for any relatively compact subset } A \text{ of } C \text{ there} \\ \text{exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A} \end{cases} \tag{2.23}$$

and

$$\text{if } A \text{ is a compact subset of } C \text{ then } \overline{co}(A) \text{ is compact.} \tag{2.24}$$

Finally assume that (2.8) *and* (2.20) *hold. Then F has a fixed point in $\overline{U}$.*

Our next result is a Furi–Pera-type fixed-point theorem for compact closed $\mathscr{U}_c^k$-admissible maps.

Theorem 2.11. *Let Q be a closed convex subset of a metrisable locally convex topological vector space E with $0 \in Q$. Suppose that $F \in \mathscr{U}_c^k(Q, E)$ is a closed compact map with the following condition satisfied:*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } x \in \lambda F(x) \text{ and } 0 \leqslant \lambda < 1, \text{ then} \\ \{\lambda_j F(x_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases} \tag{2.25}$$

Then F has a fixed point in Q.

Proof. Let $r : E \to Q$ be a continuous retraction (the existence of r follows from Dugundji's extension theorem).

Remark 2.4. If $0 \in int\, Q$ we may take

$$r(x) = \frac{x}{\max\{1, \mu(x)\}},\ x \in E,$$

where μ is the Minkowski functional on Q. Note that if $int\, Q = \emptyset$ then $\partial Q = Q$.

From Remark 2.4, we may choose (and we do so) the retraction r above so that $r(z) \in \partial Q$ if $z \in E \backslash Q$. Consider

$$B = \{x \in E : x \in Fr(x)\}.$$

Firstly, $B \neq \emptyset$. To see this, notice that $Fr \in \mathscr{U}_c^k(E, E)$ and also notice that Fr is a compact map (r is continuous and F is compact). Theorem 2.1 guarantees that Fr has a fixed point, and therefore $B \neq \emptyset$. In addition, since F is closed, we have that B is closed. In fact B is compact, since

$$B \subseteq Fr(B) \subseteq F(Q).$$

It remains to show that $B \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose that $B \cap Q = \emptyset$. Then, since B is compact and Q is closed, there exists a $\delta > 0$ with $dist(B, Q) > \delta$. Choose $m \in \{1, 2, \ldots\}$ with $1 < \delta m$. Define

$$U_i = \left\{x \in E : d(x, Q) < \frac{1}{i}\right\} \quad \text{for} \quad i \in \{m, m+1, \ldots\};$$

here d is the metric associated with E. Fix $i \in \{m, m+1, \ldots\}$. Since $dist(B, Q) > \delta$, we see that $B \cap \overline{U_i} = \emptyset$. In addition, U_i is open and convex, $0 \in U_i$ and $Fr \in \mathscr{U}_c^k(\overline{U_i}, E)$ is a compact map. Theorem 2.4 guarantees, since $B \cap \overline{U_i} = \emptyset$, that there exists

$$(y_i, \lambda_i) \in \partial U_i \times (0, 1) \quad \text{with} \quad y_i \in \lambda_i Fr(y_i).$$

We can do this for each $i \in \{m, m+1, \ldots\}$. Thus we have

$$\{\lambda_i Fr(y_i)\} \not\subseteq Q \text{ for each } i \in \{m, m+1, \ldots\}. \tag{2.26}$$

We now look at

$$D = \{x \in E : x \in \lambda Fr(x) \text{ for some } \lambda \in [0, 1]\}.$$

Notice that $D \neq \emptyset$ is closed and in fact compact (so sequentially compact), since $F \in \mathscr{U}_c^k(Q, E)$ is a compact map. This, together with

$$d(y_j, Q) = \frac{1}{j} \quad \text{and} \quad |\lambda_j| \leqslant 1 \quad \text{for} \quad j \in \{m, m+1, \ldots\},$$

implies that we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0, 1] \quad \text{and} \quad y_j \to y^\star \in \partial Q.$$

In addition we have $y_j \in \lambda_j Fr(y_j)$ with F closed, and so $y^\star \in \lambda^\star Fr(y^\star)$. Note that $\lambda^\star \neq 1$ since $B \cap Q = \emptyset$. Hence $0 \leqslant \lambda^\star < 1$. However, (2.25), with

$$x_j = r(y_j) \in \partial Q \quad \text{and} \quad x = y^\star = r(y^\star),$$

implies that $\{\lambda_j Fr(y_j)\} \subseteq Q$ for j sufficiently large. This contradicts (2.26). Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x) = F(x)$. ∎

We can improve Theorem 2.11 if E is a Hilbert space.

Theorem 2.12. *Let Q be a closed convex subset of a Hilbert space E with $0 \in Q$. Suppose that $F \in \mathscr{U}_c^k(Q, E)$ is a closed condensing map with $F(Q)$ a bounded subset of E, and assume that* (2.25) *holds. Then F has a fixed point in Q.*

Proof. Define $r : E \to Q$ by $r(x) = P_Q(x)$, i.e. r is the nearest point projection. Note that r is non-expansive. Let

$$B = \{x \in E : x \in Fr(x)\}.$$

Now $Fr \in \mathscr{U}_c^k(E, E)$ is a condensing map with $Fr(E)$ a bounded subset of E. Theorem 2.1 guarantees that Fr has a fixed point, so $B \neq \emptyset$. Also B is closed and compact (if $\alpha(r(B)) \neq 0$ then

$$\alpha(B) \leqslant \alpha(Fr(B))) < \alpha(r(B)) \leqslant \alpha(B),$$

a contradiction, so we have $\alpha(r(B)) = 0$, and thus $\alpha(B) = 0$; here α is the Kuratowski measure of non-compactness). Suppose that $B \cap Q = \emptyset$. Let m, U_i be as in Theorem 2.11, and essentially the same reasoning as in Theorem 2.11 (here Fr is a condensing map) guarantees that

$$\{\lambda_i Fr(y_i)\} \nsubseteq Q \text{ for each } i \in \{m, m+1, \dots\}. \tag{2.27}$$

Let D be as in Theorem 2.11, and notice that D is closed and compact since $D \subseteq \overline{co}(Fr(D) \cup \{0\})$. As a result we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0,1] \quad \text{and} \quad y_j \to y^\star \in \partial Q.$$

This implies that $y^\star \in \lambda^\star Fr(y^\star)$ with $0 \leqslant \lambda^\star < 1$. This, together with (2.25), implies that $\{\lambda_j Fr(y_j)\} \subseteq Q$ for j sufficiently large. This contradicts (2.27), so $B \cap Q \neq \emptyset$. ∎

3. Fixed-point results for *PK*-admissible maps

We begin this section by presenting a very general Leray–Schauder alternative for PK maps.

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space, C a closed convex subset of E, U^c C an open convex subset of E, and $0 \in U$. Suppose that $\overline{U}$ is paracompact and $F \in PK(\overline{U}, C)$ satisfies*

$$x \notin \lambda F(x) \textit{ for } x \in \partial U \textit{ and } \lambda \in (0,1). \tag{3.1}$$

In addition assume that one of the following conditions holds:

$$A \subseteq \overline{U},\ A \subseteq \overline{co}(\{0\} \cup F(A)) \textit{ implies that } \overline{A} \textit{ is compact,} \tag{3.2a}$$

$$A \subseteq \overline{U},\ A \subseteq co(\{0\} \cup F(A)) \textit{ implies that } \overline{A} \textit{ is compact} \tag{3.2b}$$

or

$$\begin{cases} A \subseteq \overline{U},\ A \subseteq co(\{x_0\} \cup F(A)), \textit{ with } \overline{A} = \overline{C} \textit{ and} \\ C \subseteq A \textit{ countable, implies that } \overline{A} \textit{ is compact; in addition} \\ \textit{assume that for any relatively compact subset } A \textit{ of } \overline{U} \textit{ there} \\ \textit{exists a countable set } B \subseteq A \textit{ with } \overline{B} = \overline{A}, \textit{ and also if} \\ D \textit{ is a compact subset of } \overline{U} \textit{ assume that } \overline{co}(D) \textit{ is compact.} \end{cases} \tag{3.2c}$$

Then F has a fixed point in $\overline{U}$.

PROOF. Let $f : \overline{U} \to C$ be the continuous selection of F guaranteed from Theorem 1.1, μ be the Minkowski functional on $\overline{U}$ and $r : E \to \overline{U}$ be

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in E.$$

Let $G = rf$, and notice that $G : \overline{U} \to \overline{U}$ is a continuous map.

Case (i). Suppose that (3.2a) holds.

Then we claim that

$$\text{if } A \subseteq \overline{U} \text{ and } A \subseteq \overline{co}(\{0\} \cup G(A)) \text{ then } \overline{A} \text{ is compact.} \tag{3.3}$$

To see this notice that if $A \subseteq \overline{U}$ and $A \subseteq \overline{co}(\{0\} \cup G(A))$ then (see Theorem 2.4)

$$A \subseteq \overline{co}(\{0\} \cup rF(A)) \subseteq \overline{co}(\{0\} \cup F(A)),$$

and so (3.2a) implies that $\overline{A}$ is compact. Thus (3.3) holds. We may apply Theorem 2.1 (with F replaced by G) to deduce that there exists $x \in \overline{U}$ with $x = G(x)$.

Case (ii). Suppose that (3.2b) holds.

Then as above we can prove that

$$\text{if } A \subseteq \overline{U} \text{ and } A \subseteq co(\{0\} \cup G(A)) \text{ then } \overline{A} \text{ is compact.} \tag{3.4}$$

Apply Theorem 2.2 (with F replaced by $G = rf$; note that $rf(\overline{A}) \subseteq \overline{rf(A)}$ for any relatively compact subset A of $\overline{U}$ since r, f are continuous) to deduce that there exists $x \in \overline{U}$ with $x = G(x)$.

Case (iii). Suppose that (3.2c) holds.

Then as above we can prove that

$$\begin{cases} \text{if } A \subseteq \overline{U},\ A \subseteq co(\{x_0\} \cup F(A)), \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies that } \overline{A} \text{ is compact.} \end{cases} \tag{3.5}$$

Apply Theorem 2.3 to deduce that there exists $x \in \overline{U}$ with $x = G(x)$.

In all cases there exists $x \in \overline{U}$ with $x = G(x)$. Thus $x = r(y)$ with $y = f(x)$. Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$, so $x = y = f(x) \in F(x)$, and we are finished. If $y \notin \overline{U}$ then $r(y) = \frac{y}{\mu(y)}$ with $\mu(y) > 1$. Then $x = \lambda y = \lambda f(x) \in \lambda F(x)$ with $0 < \lambda = \frac{1}{\mu(y)} < 1$. This contradicts (3.1). ■

Our final result is a Furi–Pera result for PK maps.

Theorem 3.2. *Let Q be a closed convex subset of a metrisable locally convex topological vector space E with $0 \in Q$. Suppose that $F \in PK(Q, E)$ is a compact map with the following condition satisfied:*

$$\begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging} \\ \textit{to } (x, \lambda) \textit{ with } x \in \lambda F(x) \textit{ and } 0 \leqslant \lambda < 1, \textit{ then} \\ \{\lambda_j F(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large.} \end{cases} \tag{3.6}$$

Then F has a fixed point in Q.

Proof. Note that Q is paracompact. We let $f : Q \to E$ be the continuous selection of F guaranteed from Theorem 1.1. Let

$$B = \{x \in E : x = fr(x)\},$$

where r is as described in Theorem 2.11. Firstly, $B \neq \emptyset$. To see this, note that fr is a continuous compact map, and we may apply Theorem 2.1 to deduce that $B \neq \emptyset$. Also B is closed and compact since

$$B \subseteq fr(B) \subseteq f(Q) \subseteq F(Q).$$

Suppose that $B \cap Q = \emptyset$. Then, since B is compact and Q is closed, there exists a $\delta > 0$ with $dist(B, Q) > \delta$. Choose $m \in \{1, 2, \ldots\}$ with $1 < \delta m$. Define

$$U_i = \left\{x \in E : d(x, Q) < \frac{1}{i}\right\} \quad \text{for} \quad i \in \{m, m+1, \ldots\};$$

here d is the metric associated with E. Fix $i \in \{m, m+1, \ldots\}$. Since $dist(B, Q) > \delta$, we see that $B \cap \overline{U_i} = \emptyset$. In addition U_i is open and convex, $0 \in U_i$ and $fr : \overline{U_i} \to E$ is a continuous compact map. Theorem 2.4 (or Theorem 3.1, note that $\overline{U_i}$ is paracompact) guarantees, since $B \cap \overline{U_i} = \emptyset$, that there exists

$$(y_i, \lambda_i) \in \partial U_i \times (0,1) \quad \text{with} \quad y_i = \lambda_i fr(y_i) \in \lambda_i Fr(y_i).$$

We can do this for each $i \in \{m, m+1, \ldots\}$. Thus we have

$$\{\lambda_i Fr(y_i)\} \nsubseteq Q \text{ for each } i \in \{m, m+1, \ldots\}. \tag{3.7}$$

Let

$$D = \{x \in E : x = \lambda fr(x) \text{ for some } \lambda \in [0,1]\}.$$

Notice that $D \neq \varnothing$ is closed and in fact compact (so sequentially compact), so we may assume without loss of generality that

$$\lambda_j \to \lambda^\star \in [0,1] \quad \text{and} \quad y_j \to y^\star \in \partial Q.$$

Also $y^\star = \lambda^\star fr(y^\star)$ with $0 \leqslant \lambda^\star < 1$. However, (3.6), with

$$x_j = r(y_j) \in \partial Q \quad \text{and} \quad x = y^\star = r(y^\star),$$

implies $\{\lambda_j Fr(y_j)\} \subseteq Q$ for j sufficiently large. This contradicts (3.7). Thus $B \cap Q \neq \varnothing$, so there exists $x \in Q$ with $x = fr(x) = f(x) \in F(x)$. ■

Remark 3.1. If E is a Hilbert space in Theorem 3.2 then we can replace $F \in PK(Q, E)$ a compact map with $F \in PK(Q, E)$ a condensing map with $F(Q)$ a bounded subset of E.

REFERENCES

[1] M. Furi and P. Pera, A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals, *Annales Polonici Mathematici* **47** (1987), 331–46.

[2] D. O'Regan, Fixed point theory for closed multifunctions, *Archivum Mathematicum* **34** (1998), 191–7.

[3] D. O'Regan, Fixed point theorems for the $\mathscr{B}^k$-admissible maps of Park, *Applicable Analysis* **79** (2001), 173–85.

[4] S. Park, Ninety years of the Brouwer fixed point theorem, *Vietnam Journal of Mathematics* **27** (1999), 187–222.

[5] S. Park and H. Kim, Admissible classes of multifunctions on generalized convex spaces, *Proceedings of the College of Natural Sciences Seoul National University* **18** (1993), 1–21.