DOMINATING SETS FOR ANALYTIC AND HARMONIC FUNCTIONS AND COMPLETENESS OF WEIGHTED BERGMAN SPACES

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ABSTRACT

A set $E \subset \Omega$ is holomorphically dominating for Ω if $\sup_{z \in E} |f(z)| = \sup_{z \in \Omega} |f(z)|$ for all holomorphic functions f on Ω. As follows from a result of Stray, this property is equivalent to the inaccessibility of the Aleksandrov compactification point $*$ (of Ω) from $\Omega \setminus \bar{E}$. Moreover, it is equivalent to a large number of other statements (old and new) of holomorphic, harmonic and topological nature, including that a certain weighted Bergman space with $p = \infty$ is a Banach space. We extend this to the cases of harmonic functions in $\mathbf{R}^n$ and holomorphic functions in $\mathbf{C}^n$. We also present some results on when weighted Bergman spaces are (quasi)-Banach spaces, the case $p = \infty$ being characterised by the result mentioned above.

1. Introduction

In this note, we try to understand better the completeness properties of weighted Bergman spaces. More specifically, let $0 < p \leq \infty$, $\Omega \subset \mathbf{C}$ be a domain, μ be a positive Borel measure on Ω, and the *weighted Bergman space* $A^p_\mu(\Omega)$ be the set of those functions $f \in L^p(\mu)$ that are holomorphic in Ω. In general $\| \cdot \|_{L^p(\mu)}$ is a quasi-seminorm on $A^p_\mu(\Omega)$. It is not known under which conditions the space $A^p_\mu(\Omega)$ is complete with respect to $\| \cdot \|_{L^p(\mu)}$. Completeness means that, if $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $A^p_\mu(\Omega)$, then there exists $f \in \mathrm{Hol}(\Omega)$ such that $f_j \to f$ with respect to the $L^p(\mu)$-distance. Alternatively, if $f_j \in \mathscr{A}^p_\mu(\Omega)$ converges to $f \in L^p_\mu(\Omega)$, then there exists $g \in \mathrm{Hol}(\Omega)$ that is equivalent to f (in $L^p_\mu(\Omega)$). There are (almost) always many non-holomorphic functions in $\overline{\mathscr{A}^p_\mu(\Omega)}$. Of course, completeness results are known for several special classes of measures. For instance, completeness holds if μ is the Lebesgue measure but not if μ is the restriction of the Lebesgue measure to some compact subset of Ω with positive Lebesgue measure.

The general problem of completeness might be very difficult. A complete solution is known only for the case $p = \infty$, in which the measure μ plays a role only through its support. In this note we show that the completeness (strictly speaking, when the space is a Banach space) for the case $p = \infty$ is equivalent to several old and new

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conditions. We extend this to the case of harmonic functions in $\mathbf{R}^n$ and holomorphic functions in $\mathbf{C}^n$ and give some partial results for the case $p < \infty$.

2. The main result

In this section we discuss several conditions equivalent to the completeness of $A^\infty_\mu(\Omega)$. We start with some notation that will be needed.

Let $H_E(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \|f\|_E := \sup_{z\in E}|f| < \infty\}$ and $h_E(\Omega) = \{u \in \mathrm{Harm}(\Omega) : \|u\|_E < \infty\}$, $\mathrm{Hol}(\Omega)$ ($\mathrm{Harm}(\Omega)$) being the set of holomorphic (real-valued harmonic) functions on Ω. We do not assume that $\|\cdot\|_E$ is a norm; in general it is only a seminorm. If $E = \varnothing$, then we make the convention that $\|f\|_E = 0$ for all functions f.

The *holomorphically* (*harmonically*) *convex hull* of E in Ω is the set $\{z \in \Omega : |f(z)| \leq \|f\|_E$ for all $f \in \mathrm{Hol}(\Omega)$ $(\mathrm{Harm}(\Omega))\}$. A set E is *holomorphically* (*harmonically*) *dominating* for Ω if $\|f\|_\Omega = \|f\|_E$ for all $f \in \mathrm{Hol}(\Omega)$ $(\mathrm{Harm}(\Omega))$.

A domain is a non-empty open connected set. A set is *locally connected* at a point z if every neighbourhood of z contains a connected neighbourhood of z; it is *locally connected* if it is locally connected at every point. A *path* is a continuous mapping from an interval.

We also let $D(a,r) = \{z \in \mathbf{C} : |z-a| < r\}$, $\mathbf{D} = D(0,1)$ and T_z be the point evaluation functional at z, defined by $T_z f = f(z)$.

We let $G(\cdot, w; \Omega)$ be the Green function for the domain Ω with pole at w, if it exists. For $\Omega \subset \mathbf{C}$ the Green function exists if and only if the logarithmic capacity of $\mathbf{C} \setminus \Omega$ is positive, in particular if $\mathbf{C} \setminus \Omega$ contains an interior point. If $\Omega \subset \mathbf{R}^n$, $n \geq 3$, then the Green function always exists.

We are now ready to state our main result.

Theorem 2.1. *Let $\Omega \subset \mathbf{C}$ be a domain and let $E \subset \Omega$. Let $\Omega \cup \{*\}$ be the one-point compactification of Ω. Then the following are equivalent:*

(A1) *The space $H_E(\Omega)$ is a Banach space (with respect to the norm $\|\cdot\|_E$).*
(A2) *The functional T_w is bounded on $H_E(\Omega)$ for all $w \in \Omega$.*
(A3) *The functional T_w on $H_E(\Omega)$ has norm 1 for all $w \in \Omega$.*
(A4) *The holomorphically convex hull of E in Ω is Ω.*
(A5) *The set E is holomorphically dominating for Ω.*
(A6) *There does not exist an unbounded function in $H_E(\Omega)$.*
(A7) *The holomorphic boundary maximum principle is valid for the set $G = \Omega \setminus \bar{E}$ with the boundary taken in Ω, that is, for $f \in \mathrm{Hol}(G)$ we have*

$$\sup_{z\in\partial G\cap\Omega}\ \limsup_{G\ni w\to z}|f(w)| = \sup_{z\in G}|f(z)|.$$

(A8) *The function $z \mapsto (z-w)^{-1}$ is not in $\overline{H_E(\Omega)}$ for any $w \in \Omega$.*
(A9) *The function $z \mapsto (z-w)^{-1}$ is not in $\overline{H_E(\Omega)}$ for any $w \in \overline{\Omega}$.*
(SH) *The subharmonic boundary maximum principle is valid for the set $G = \Omega \setminus \bar{E}$ with the boundary taken in Ω, that is, for any subharmonic function $u : G \to \mathbf{R}$*

we have

$$\sup_{z\in\partial G\cap\Omega}\ \limsup_{G\ni w\to z} u(w) = \sup_{z\in G} u(z).$$

(T1) *There does not exist a path* $\gamma : [0,\infty) \to \Omega \setminus \bar{E}$ *such that* $\gamma(t) \to *$, *as* $t \to \infty$.
(T2) *There does not exist a non-empty open set* $G \subset \Omega \setminus \bar{E}$ *such that* $G \cup \{*\}$ *is path-connected.*
(T3) *There does not exist a non-empty open set* $G \subset \Omega \setminus \bar{E}$ *such that* $G \cup \{*\}$ *is connected and locally connected.*
(H1) *The space* $h_E(\Omega)$ *is a Banach space with respect to the norm* $\|\cdot\|_E$, *and either* $\Omega \setminus \bar{E} = \varnothing$ *or there exists* $w \in \Omega \setminus \bar{E}$ *such that* $h_E(\Omega)$ *is a Banach space also with respect to the norm* $\|\cdot\|_{E\cup\{w\}}$.
(H2) *The functional* T_w *is bounded on* $h_E(\Omega)$ *for all* $w \in \Omega$.
(H3) *The functional* T_w *on* $h_E(\Omega)$ *has norm* 1 *for all* $w \in \Omega$.
(H4) *The harmonically convex hull of* E *in* Ω *is* Ω.
(H5) *The set* E *is harmonically dominating for* Ω.
(H6) *There does not exist an unbounded function in* $h_E(\Omega)$.
(H7) *The harmonic boundary maximum principle is valid for the set* $G = \Omega \setminus \bar{E}$ *with the boundary taken in* Ω, *that is, for any* $u \in \mathrm{Harm}(G)$ *we have*

$$\sup_{z\in\partial G\cap\Omega}\ \limsup_{G\ni w\to z} u(w) = \sup_{z\in G} u(z).$$

(H8) *There does not exist a non-constant* $u \in \mathrm{Harm}(G)$, *where* $G = \Omega \setminus \bar{E}$, *such that*

$$\lim_{G\ni z\to w} u(z) = 0, \quad w \in \partial G_{\mathrm{reg}} \cap \Omega$$

and

$$\limsup_{G\ni z\to w} |u(z)| < \infty, \quad w \in \partial G_{\mathrm{irr}} \cap \Omega,$$

where $\partial G_{\mathrm{reg}}$ *denotes the set of regular boundary points of* G (*for the Dirichlet problem*) *and* $\partial G_{\mathrm{irr}} = \partial G \setminus \partial G_{\mathrm{reg}}$.
(H9) *The function* $G(\cdot, w; \Omega \setminus \bar{D})$ *is not in* $\overline{h_E(\Omega)}$ *for any closed disc* $\bar{D} \subset \Omega$ *and any* $w \in \Omega \setminus \bar{D}$.

Remarks. Since the conditions (A7), (SH), (T1)–(T3), (H7) and H8 are saying something about $\bar{E} \cap \Omega$, we can replace E by $\bar{E} \cap \Omega$ (or by any set $F \subset \Omega$ with $\bar{F} \cap \Omega = \bar{E} \cap \Omega$) in any of the other statements and obtain fifteen more equivalent statements. Another equivalent statement is to change (A8) to hold just for $w \in \Omega \setminus \bar{E}$ (the statement is trivial for $w \in \bar{E}$, since $\|(z-w)^{-1}\|_E = \infty$). It is also easy to see that the following two statements are trivial reformulations of (T1).

(a) Every path $\gamma : [0,\infty) \to \Omega \setminus \bar{E}$ has a limit value at ∞ within Ω, i.e.

$$\Omega \cap \bigcap_{t\geq 0} \overline{\gamma([t,\infty))} \neq \varnothing.$$

(b) The path-connected component of $(\Omega \setminus \bar{E}) \cup \{*\}$ containing $*$ is $\{*\}$.

In the statement of (H9) we can replace $G(\cdot, w; \Omega \setminus \bar{D})$ with $G(\cdot, w; \Omega)$ if Ω has a Green function or with $\log|z - w|$ if E is bounded. The proof is similar.

The equivalence (T1) $\Leftrightarrow$ (SH) is Theorem 3.3 below; see the comments following it for the history behind it. The equivalence (T1) $\Leftrightarrow$ (H8) for the case when $\Omega = \mathbf{R}^n$, $n \geq 2$, is theorem 7.4 in Gardiner [10]; it was first proved in Gardiner [9], theorem 2. The equivalence (A4) $\Leftrightarrow$ (T1) follows from proposition 1.1 in Stray [18].

No doubt, most of the equivalences in Theorem 2.1 are known to the experts in the field. They do not, however, seem to have been collected together. Moreover, the equivalence to the statements about Banach spaces, (A1)–(A3) and (H1)–(H3), may be new.

If (H1) is true, then the two norms $\|\cdot\|_E$ and $\|\cdot\|_{E \cup \{w\}}$ are equivalent. In fact, since all the point evaluations are bounded, all norms of the form $\|\cdot\|_{E \cup \{w\}}$, $w \in \Omega$, are equivalent to $\|\cdot\|_E$. Moreover, it is always true (also in the case when there are unbounded point evaluations) that $h_E(\Omega) = h_{E \cup \{w\}}(\Omega)$ for all $w \in \Omega$.

We would have liked to replace (H1) with the following condition:

(H1″) *The space $h_E(\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_E$.*

We have not been able to do this, but we still believe that it is possible and therefore make the following conjecture.

Conjecture 2.2. The statement (H1″) is equivalent to the statements in Theorem 2.1.

From the proof below we see that, if (H1″) is true while all the other (equivalent!) statements are false, then $F = \bar{E} \cap \Omega$ cannot contain any interior point and $*$ is accessible from any point in $\Omega \setminus F$ (with a path in $\Omega \setminus F$). Furthermore, adding a point to F or removing a point not in F from Ω makes the space into a non-Banach space. Similarly, increasing Ω or decreasing F (keeping F or Ω, respectively, fixed) also makes the space into a non-Banach space. Thus a counter-example would be extremely sensitive to changes in the domain and F. From theorem 6 in Gauthier [11] it also follows that F cannot be a discrete set.

Before proving Theorem 2.1 we will state and prove the following related theorem about a fixed point $w \in \Omega$.

Theorem 2.3. *Let $\Omega \subset \mathbf{C}$ be a domain; let $E \subset \Omega$; and fix $w \in \Omega$. Let $\Omega \cup \{*\}$ be the one-point compactification of Ω. Then the following are equivalent:*

(A2′) *The functional T_w is bounded on $H_E(\Omega)$.*

(A3′) *The functional T_w on $H_E(\Omega)$ has norm* 1.

(A4′) *The point w is in the holomorphically convex hull of E in Ω.*

(A7′) *Either $w \in \bar{E}$ or for any holomorphic function f on $G = \Omega \setminus \bar{E}$ we have*

$$|f(w)| \leq \sup_{z \in \partial G \cap \Omega} \limsup_{G \ni \zeta \to z} |f(\zeta)|.$$

(A8′) *The function $z \mapsto (z - w)^{-1}$ is not in $\overline{H_E(\Omega)}$.*

(SH′) *Either $w \in \bar{E}$ or for any subharmonic function u on $G = \Omega \setminus \bar{E}$ we have*

$$u(w) \leq \sup_{z \in \partial G \cap \Omega} \limsup_{G \ni \zeta \to z} u(\zeta).$$

(T1′) *There does not exist a path* $\gamma : [0,1] \to (\Omega \setminus \bar{E}) \cup \{*\}$ *with* $\gamma(0) = w$ *and* $\gamma(1) = *$.
(T2′) *There does not exist an open set* $G \subset \Omega \setminus \bar{E}$ *such that* $G \cup \{*\}$ *is path-connected and* $w \in G$.
(T3′) *There does not exist an open set* $G \subset \Omega \setminus \bar{E}$ *such that* $G \cup \{*\}$ *is connected and locally connected, and* $w \in G$.
(H2′) *The functional* T_w *is bounded on* $h_E(\Omega)$.
(H3′) *The functional* T_w *on* $h_E(\Omega)$ *has norm* 1.
(H4′) *The point* w *is in the harmonically convex hull of* E *in* Ω.
(H7′) *Either* $w \in \bar{E}$ *or for any harmonic function* u *on* $G = \Omega \setminus \bar{E}$ *we have*

$$u(w) \le \sup_{z \in \partial G \cap \Omega} \limsup_{G \ni \zeta \to z} u(\zeta).$$

Corollary 2.4. *Let* $\Omega \subset \mathbf{C}$ *be a domain and let* $E \subset \Omega$. *Let* $\Omega \cup \{*\}$ *be the one-point compactification of* Ω. *Then the holomorphically convex hull of* E *in* Ω *is the same as the harmonically convex hull and consists of all* $z \in \Omega$ *such that there is no path* $\gamma : [0,1] \to (\Omega \setminus \bar{E}) \cup \{*\}$ *with* $\gamma(0) = z$ *and* $\gamma(1) = *$.

Remarks. Apart from mentioning the ‘harmonically convex hull’, this is proposition 1.1 in Stray [18], which thus provides an alternative to our proof below of the equivalence (A4′) ⇔ (T1′). Stray uses pole pushing rather than applying the less elementary Arakelyan’s theorem. Stray’s proof is similar to his own earlier proof for $\Omega = \mathbf{D}$, theorem 2 in [17], and the earlier proof by Brown and Shields for Ω bounded, theorem 1 in [3]. Strictly speaking, Stray [17; 18] and Brown–Shields [3] only study the case when E is relatively closed in Ω, but the general case follows quite trivially from this.

One could believe that the statement ‘the function $G(\,\cdot\,, w; \Omega \setminus \bar{D})$ is not in $\overline{h_E(\Omega)}$ for any closed disc $\bar{D} \subset \Omega \setminus w$’ would also be equivalent to the statements in Theorem 2.3. It is true that it implies the statements in Theorem 2.3 (this follows from Theorem 3.2), but the converse is false: consider Ω, D and w fixed and let E be a level set for $G(\,\cdot\,, w; \Omega \setminus \bar{D})$. Nevertheless, (H9) is equivalent to the other statements in Theorem 2.1.

3. Proofs of Theorems 2.1 and 2.3

We will start by quoting a few results from the literature that will be needed.

Theorem 3.1 (Arakelyan’s theorem). *Let* $\Omega \subset \mathbf{C}$ *be a domain and let* $F \subset \Omega$ *be relatively closed. Let* $\Omega \cup \{*\}$ *be the one-point compactification of* Ω. *Then the following are equivalent:*
(a) *For every* $f \in C(F) \cap \mathrm{Hol}(\mathrm{int}\, F)$ *and every* $\varepsilon > 0$ *there exists* $g \in \mathrm{Hol}(\Omega)$ *such that* $\|f - g\|_F < \varepsilon$.
(b) *The set* $(\Omega \cup \{*\}) \setminus F$ *is connected and locally connected.*

This result was proved by Arakelyan [1], theorem 1. It can also be found in Gaier [8], theorem 3, p. 142. We will not need the full power of Arakelyan’s theorem.

Theorem 3.2. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a domain and let $F \subset \Omega$ be relatively closed. Let $\Omega \cup \{*\}$ be the one-point compactification of Ω. If $(\Omega \cup \{*\}) \setminus F$ is connected and locally connected, then, for each function u harmonic in some neighbourhood of F and each $\varepsilon > 0$, there exists $v \in \operatorname{Harm}(\Omega)$ such that $\|u - v\|_F < \varepsilon$.*

This result was proved by Gauthier–Goldstein–Ow, theorem 3 in [12] for $n = 2$ and theorem 1 in [13] for $n \geq 3$. It can also be found as corollary 3.8 in Gardiner [10].

Theorem 3.3. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a domain, let $F \subset \Omega$ be relatively closed and let $G = \Omega \setminus F$. Let $\Omega \cup \{*\}$ be the one-point compactification of Ω. Then the following are equivalent:*

(a) *There does not exist a path $\gamma : [0, \infty) \to G$ such that $\gamma(t) \to *$, as $t \to \infty$.*

(b) *For every subharmonic function u on G,*

$$\sup_{z \in \partial G \cap \Omega} \limsup_{G \ni w \to z} u(w) = \sup_{z \in G} u(z).$$

The implication (a) $\Rightarrow$ (b) is a special case of the theorem on p. 268 in Fuglede [7]. An elementary proof was given by Chen–Gauthier [4]. The converse was first observed by Gauthier–Grothmann–Hengartner [14], p. 479. See also theorem 7.3 in Gardiner [10].

We are now ready to prove Theorems 2.1 and 2.3, starting with the latter. In the proofs below we keep in mind generalisations to several variables, see Section 4.

Proof of Theorem 2.3. We start with proving equivalence between the topological statements (T1′)–(T3′), since this can be done without involving any of the other statements.

$\neg$ (T1′) $\Rightarrow$ $\neg$ (T3′) Let $\gamma : [0, \infty) \to \Omega \setminus \bar{E}$ be a path such that $\gamma(0) = w$ and $\gamma(t) \to *$, as $t \to \infty$. Let $G = \bigcup_{t \geq 0} D\big(\gamma(t), \min\{\frac{1}{2}\operatorname{dist}(\gamma(t), E \cup \partial\Omega), 1/t\}\big)$, which is clearly an open set containing w such that $G \cup \{*\}$ is path-connected. Since G is open, $G \cup \{*\}$ is locally connected at every point in G. It remains to prove that $G \cup \{*\}$ is locally connected at $*$. Let $U \neq \mathbf{C} \cup \{*\}$ be a neighbourhood of $*$ (in $G \cup \{*\}$). Then there exists a non-empty compact set $K \subset \Omega$ such that $G \setminus K \subset U$. Since $\gamma(t) \to *$, there exists t_0 such that $\gamma([t_0, \infty)) \subset G \setminus K$ and moreover $\operatorname{dist}(\gamma([t_0, \infty)), K) = \delta > 0$. Let $t_1 \geq t_0$ be so large that $1/t_1 < \delta$. Let $V = \bigcup_{t \geq t_0} D(\gamma(t), \operatorname{dist}(\gamma(t), \mathbf{C} \setminus (U \setminus \{*\}))) \cup \{*\} \subset U$, which is clearly a path-connected open set in $G \cup \{*\}$. Moreover, $\bigcup_{t \geq t_1} D\big(\gamma(t), \min\{\frac{1}{2}\operatorname{dist}(\gamma(t), E \cup \partial\Omega), 1/t\}\big) \subset V$, so V is a neighbourhood of $*$ in $G \cup \{*\}$.

$\neg$ (T3′) $\Rightarrow$ $\neg$ (T2′) If $G \cup \{*\}$ is connected and locally connected, then G is path-connected, since it is open. Furthermore, by the lemma on p. 138 in Gaier [8], $*$ can be path-connected to some point in G.

$\neg$ (T2′) $\Rightarrow$ $\neg$ (T1′) If $G \cup \{*\}$ is path-connected and $w \in G$, then there is a path as in (T1′).

We are now ready to turn our attention to the statements (A2′)–(A4′), (A7′) and (A8′) involving analytic functions and (SH′) involving subharmonic functions.

(A2′) ⇒ (A3′) Since $1 \in H_E(\Omega)$, we have $\| T_w \|_{H_E(\Omega)^*} \geq 1$. Furthermore,

$$\| T_w \|^2_{H_E(\Omega)^*} = \sup_{\substack{\| f \|_E \leq 1 \\ f \in H_E(\Omega)}} |f(w)|^2 = \sup_{\substack{\| f^2 \|_E \leq 1 \\ f \in H_E(\Omega)}} |f^2(w)| \leq \sup_{\substack{\| g \|_E \leq 1 \\ g \in H_E(\Omega)}} |g(w)| = \| T_w \|_{H_E(\Omega)^*}.$$

It follows that if T_w is bounded then $\| T_w \|_{H_E(\Omega)^*} \leq 1$.

(A3′) ⇒ (A2′) Trivial.

(A3′) ⇔ (A4′) This follows from the definition of holomorphically convex hull. Just notice that since $1 \in H_E(\Omega)$ the norm cannot be less than 1.

(T1′) ⇒ (SH′) Assume that $w \notin \bar{E}$ and let $\widetilde{G}$ be the component of $G = \Omega \setminus \bar{E}$ containing w. Thus by Theorem 3.3 we have

$$u(w) \leq \sup_{z \in \partial \widetilde{G} \cap \Omega} \limsup_{\widetilde{G} \ni \zeta \to z} u(\zeta).$$

The conclusion follows since $\partial \widetilde{G} \subset \partial G$.

(SH′) ⇒ (A7′) This follows directly, letting $u = |f|$.

(A7′) ⇒ (A4′) Let $f \in H_E(\Omega)$. If $w \in \bar{E}$, continuity shows that $|f(w)| \leq \| f \|_E$, otherwise we have $|f(w)| \leq \| f \|_{\partial \bar{E} \cap \Omega} \leq \| f \|_E$, again using continuity.

¬ (T3′) ⇒ ¬ (A2′) The set $(G \cup \{*\}) \setminus \{w\}$ is connected and locally connected. Let $M \in \mathbf{R}$ be arbitrary and

$$f(z) = \begin{cases} \frac{1}{2}, & \text{if } z \in \Omega \setminus G, \\ M, & \text{if } z = w. \end{cases}$$

By Arakelyan's theorem, Theorem 3.1, there is $g \in \mathrm{Hol}(\Omega)$ such that $|f(z) - g(z)| < \frac{1}{2}$ for $z \in (\Omega \setminus G) \cup \{w\}$. It follows that $\| g \|_E \leq 1$. Hence $g \in h_E(\Omega)$ and $\| T_w \|_{H_E(\Omega)^*} \geq |g(w)| > M - \frac{1}{2}$. Since M was arbitrary, T_w is unbounded.

¬ (T3′) ⇒ ¬ (A8′) The function $z \mapsto (z - w)^{-1}$ is in $\overline{H_E(\Omega)}$, by Arakelyan's theorem, Theorem 3.1.

(T1′) ⇒ (A8′) Assume that $f(z) = (z - w)^{-1} \in \overline{H_E(\Omega)}$ and let $f_j \in H_E(\Omega)$ be such that $\| f_j - f \|_E \to 0$, as $j \to \infty$. By continuity, f is bounded on $\bar{E} \cap \Omega$, and therefore $w \notin \bar{E} \cap \Omega$. Hence w belongs to a component G of $\Omega \setminus \bar{E}$ from which there is no path to $*$. Let

$$F = \{ z \in \Omega : \text{there is no path } \gamma : [0,1] \to (\Omega \setminus \bar{E}) \cup \{*\} \text{ with } \gamma(0) = z,\ \gamma(1) = * \},$$

which, by the already established equivalence (T1′) ⇔ (A4′), is exactly the holomorphically convex hull of E in Ω. Let $g(z) = \lim_{j \to \infty} f_j(z)$, whenever it exists. Since $f_j \to g$ uniformly on F, as $j \to \infty$, g is continuous on F and $g \in \mathrm{Hol}(G)$. The functions f and g coincide on $\partial G \cap \Omega \subset \bar{E} \cap \Omega$, by continuity. Let $h(z) = g(z)(z - w) - 1$, an analytic function on G with boundary values 0 on $\partial G \cap \Omega$ and $h(w) = -1$. But this contradicts Theorem 3.3. Hence $f \notin \overline{H_E(\Omega)}$.

The implications (SH′) ⇒ (H7′) ⇒ (H4′) ⇔ (H3′) ⇒ (H2′) ⇒ (T3′) are proved in a similar way as (SH′) ⇒ (A7′) ⇒ (A4′) ⇔ (A3′) ⇒ (A2′) ⇒ (T3′), just replacing Theorem 3.1 by Theorem 3.2 in the proof of ¬ (T3′) ⇒ ¬ (A2′). ∎

Proof of Theorem 2.1. The equivalence of (A2)–(A4), (A7), (A8), (SH), (T1)–(T3), (H2)–(H4) and (H7) follows directly from Theorem 2.3.

(A3) $\Rightarrow$ (A1) First of all it is clear that $\|\cdot\|_E$ is a norm, since otherwise we would have some $f \in H_E(\Omega)$ and $w \in \Omega$ with $T_w f \neq 0 = \|f\|_E$, which would imply that $\|T_w\|_{H_E(\Omega)^*} = \infty$. Secondly, take a Cauchy sequence $\{f_j\}_{j=1}^{\infty}$ in $H_E(\Omega)$. By (A3), $\{f_j(z)\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathbf{C}$ for every $z \in \Omega$. Hence, $g(z) = \lim_{j\to\infty} f_j(z)$ exists for all $z \in \Omega$, and $f_j \to g$ in the norm $\|\cdot\|_E$. By (A3), $f_j \to g$ uniformly, as $j \to \infty$, and hence g is holomorphic, and $g \in H_E(\Omega)$.

$\neg$ (A3) $\Rightarrow$ $\neg$ (A1) Let w be such that $\|T_w\|_{H_E(\Omega)^*} > 1$ and find $f \in H_E(\Omega)$ with $f(w) = 1$ and $\|f\|_E < 1$. Let further $g = 1/(1-f)$. Then $\left\|g - \sum_{j=0}^{N} f^j\right\|_E \to 0$, as $N \to \infty$, so $g \in \overline{H_E(\Omega)}$. Assume that there exists $h \in H_E(\Omega)$ such that $\left\|h - \sum_{j=0}^{N} f^j\right\|_E \to 0$, as $N \to \infty$. Then g and h represent the same equivalence class with respect to $\|\cdot\|_E$ and $h(z) = g(z)$ for $z \in E$. It follows that $k(z) := (1-f(z))h(z)-1 = 0$ for $z \in E$. Moreover, k is analytic in Ω. Since $\|k\|_E = 0$ and $k(w) = -1$, the seminorm $\|\cdot\|_E$ is not a norm.

(A4) $\Leftrightarrow$ (A5) Follows just by checking the definitions.

(A5) $\Rightarrow$ (A6) For $f \in H_E(\Omega)$ we have $\|f\|_\Omega = \|f\|_E < \infty$.

$\neg$ (A5) $\Rightarrow$ $\neg$ (A6) Let $f \in H_E(\Omega)$ be non-constant and such that $\|f\|_E < \|f\|_\Omega$. (That f be non-constant is an extra requirement only when $E = \varnothing$.) Without loss of generality we can assume that $\|f\|_E \leq 1 < \|f\|_\Omega$. If $\|f\|_\Omega = \infty$ we are finished; otherwise there exists a sequence $\{z_j\}_{j=1}^{\infty}$ in Ω such that $\lim_{j\to\infty} |f(z_j)| = \|f\|_\Omega$. Since $\|f\|_\Omega < \infty$, there exists a convergent subsequence $\{f(z_{j_k})\}_{k=1}^{\infty}$ of the sequence $\{f(z_j)\}_{j=1}^{\infty}$; let $w = \lim_{k\to\infty} f(z_{j_k})$. Since $|w| = \|f\|_\Omega$ and f is non-constant, the maximum principle shows that $w \notin f(\Omega)$. Therefore $1/(w - f(z))$ is an unbounded holomorphic function on Ω that is bounded on E.

(A8) $\Rightarrow$ (A9) Let $f_w(z) = (z-w)^{-1}$. We already know that $f_w \notin \overline{H_E(\Omega)}$ if $w \in \Omega$. Consider $w \in \partial\Omega$. Assume that $f_w \in \overline{H_E(\Omega)}$. Then $\|f_w\|_E < \infty$, and, since f_w is analytic in Ω, we deduce that $f_w \in H_E(\Omega)$. We have already proved that (A8) $\Rightarrow$ (A6), but f_w is unbounded in Ω, a contradiction.

(A9) $\Rightarrow$ (A8) Trivial.

Finally, we will study the harmonic statements (H1)–(H9).

(H3) $\Rightarrow$ (H1) The proof that $h_E(\Omega)$ is a Banach space with respect to $\|\cdot\|_E$ is similar to the proof of (A3) $\Rightarrow$ (A1). Thus we are finished if $\Omega \setminus \bar{E} = \varnothing$. Otherwise, let $w \in \Omega \setminus \bar{E}$ be arbitrary. Then T_w is bounded on $h_E(\Omega)$ with respect to $\|\cdot\|_E$, and hence $\|\cdot\|_E$ and $\|\cdot\|_{E\cup\{w\}}$ are equivalent norms on $h_E(\Omega)$.

$\neg$ (T1) $\Rightarrow$ $\neg$ (H1) Assume first $\neg$ (T1) and (H1$''$) (i.e. that $h_E(\Omega)$ is a Banach space with respect to $\|\cdot\|_E$). Let $F = \bar{E} \cap \Omega$ and let Ω' be the union of the components of $\Omega \setminus F$ that contain a path to $*$. By $\neg$ (T1), $\Omega' \neq \varnothing$. Let $\bar{D} \subset \Omega'$ be a closed disc and let $w \in \Omega' \setminus \bar{D}$. By Theorem 3.2 we find that $G(\cdot, w; \Omega \setminus \bar{D}) \in \overline{h_E(\Omega)}$ ($\Omega' \cup \{*\}$ is not necessarily locally connected, but there is a suitable subdomain that is). By (H1$''$) there exists $u \in h_E(\Omega)$ such that $u(z) = G(z, w; \Omega \setminus \bar{D})$ for all $z \in E$, and by continuity for all $z \in F$. For each component of $\Omega \setminus (F \cup \Omega')$ we can apply Theorem 3.3 and obtain that $G(z, w; \Omega \setminus \bar{D}) - u(z) = 0$ for $z \in \Omega \setminus \Omega'$. This implies that $\Omega \setminus \Omega'$ does not contain any interior point and hence that $F = \Omega \setminus \Omega'$ and $\operatorname{int} F = \varnothing$. Pick an arbitrary point $w \in \Omega \setminus F = \Omega'$. By $\neg$ (T1$'$) $\Rightarrow$ $\neg$ (H2$'$)

of Theorem 2.3, there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of functions in $h_E(\Omega)$ such that $\|f_j\|_E \to 0$, as $j \to \infty$, and $f_j(w) = 1$, $j = 1, 2, \ldots$. If $h_E(\Omega)$ were a Banach space also with respect to the norm $\|\cdot\|_{E \cup \{w\}}$, then there would be a limit element $g \in h_E(\Omega)$ such that $\|f_j - g\|_{E \cup \{w\}} \to 0$, as $j \to \infty$. But this would require that $g(z) = 0$ for $z \in E$ and that $g(w) = 1$, and thus g would have been a non-zero function with $\|g\|_E = 0$, violating the assumption that $\|\cdot\|_E$ was a norm on $h_E(\Omega)$.

(H4) $\Leftrightarrow$ (H5) $\Rightarrow$ (H6) This is similar to (A4) $\Leftrightarrow$ (A5) $\Rightarrow$ (A6).

$\neg$ (T3) $\Rightarrow$ $\neg$ (H6) Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of distinct points in G such that $z_j \to *$, as $j \to \infty$. Let

$$u(z) = \begin{cases} 1, & \text{if } z \in E, \\ j, & \text{if } z = z_j. \end{cases}$$

By Theorem 3.2 we can find v harmonic in Ω such that $|u(z) - v(z)| < 1$ for $z \in E \cup \{z_j : j = 1, 2, \ldots\}$. It follows that $v \in h_E(\Omega)$ and that v is unbounded.

(T1) $\Leftrightarrow$ (H8) For the case when $\Omega = \mathbf{C}$, this is theorem 7.4 in Gardiner [10]; the proof is similar in the general case, and we refrain from copying it here.

$\neg$ (T3) $\Rightarrow$ $\neg$ (H9) Let $\bar{D} \subset \Omega \setminus \bar{E}$ be a closed disc and $w \in (\Omega \setminus \bar{E}) \setminus \bar{D}$. Then the function $G(\cdot, w; \Omega \setminus \bar{D})$ is in $\overline{h_E(\Omega)}$, by Theorem 3.2.

(T1) $\Rightarrow$ (H9) Assume (T1) and $\neg$ (H9). Let the closed disc $\bar{D} \subset \Omega$ and $w \in \Omega \setminus \bar{D}$ be such that $G(\cdot, w; \Omega \setminus \bar{D}) \in \overline{h_E(\Omega)}$, where we assume that $G(\cdot, w; \Omega \setminus \bar{D})$ is extended in an arbitrary way to $\bar{D}$, if $E \cap \bar{D} \neq \varnothing$ (the result does not depend on the extension), that is, we assume that there exists $u_j \in h_E(\Omega)$ such that $\|u_j - G(\cdot, w; \Omega \setminus \bar{D})\|_E \to 0$, as $j \to \infty$. We have already proved that (T1) $\Rightarrow$ (H1), so $h_E(\Omega)$ is complete and there exists $u \in h_E(\Omega)$ such that $u_j \to u$ in $h_E(\Omega)$, as $j \to \infty$. That means that $u(z) = G(z, w; \Omega \setminus \bar{D})$ for $z \in E$. Let $F = \bar{E} \cap (\Omega \setminus \bar{D})$ and $v = G(\cdot, w; \Omega \setminus \bar{D}) - u$. By continuity, $v(z) = 0$ for $z \in F$. Moreover, $w \notin F$. The set $(\Omega \setminus \bar{D}) \setminus F$ can be partitioned into three open sets: the component Ω' of $(\Omega \setminus \bar{D}) \setminus F$ containing w; the union Ω'' of the components of $(\Omega \setminus \bar{D}) \setminus F$ not containing w and not containing a path to $*$; the union Ω''' of the components of $(\Omega \setminus \bar{D}) \setminus F$ not containing w but containing a path to $*$. By (T1), Ω''' is empty and F is non-empty. For each component of Ω'' we can apply Theorem 3.3 and obtain that $v(z) = 0$ for $z \in F \cup \Omega''$. Since $\lim_{z \to w} v(z) = \infty$, we can find a disc $D' \subset \Omega'$ so that $\inf_{z \in D' \setminus \{w\}} v(z) > 0$. By (T1) and Theorem 3.3 (applied to $-v$), $v(z) \geq 0$ for all $z \in \Omega \setminus (\bar{D} \cup \{w\})$. But, since F is non-empty, v has a local minimum at an interior point, which shows that $v \equiv 0$, a contradiction. ■

4. Generalisations

The topological statements (T1)–(T3) are equivalent quite generally, and so are (T1′)–(T3′). It is not the object of this paper to describe exact conditions under which they are equivalent.

4.1. Several real variables

In several real variables, i.e. assuming that $\Omega \subset \mathbf{R}^n$, $n \geq 3$, and considering the harmonic conditions (H1)–(H9), we can simplify the last condition (H9) as follows:

(H9″) *The function* $x \mapsto |x - y|^{2-n}$ *is not in* $\overline{h_E(\Omega)}$ *for any* $y \in \Omega$.

With this simplification we can use the same proofs as in two real variables and obtain the equivalence between (SH), (T1)–(T3), (H1)–(H9) and (H9″).

As in the plane case, we would have liked to replace (H1) with (H1″). A possible counter-example has to satisfy the same remarked properties as in the plane case, and we therefore make the same conjecture for $n \geq 3$ as for $n = 2$.

The proofs of the harmonic parts of Theorem 2.3 given above for two real variables also carry over verbatim to several real variables to prove the equivalence between (SH′), (T1′)–(T3′), (H2′)–(H4′) and (H7′).

4.2. Several complex variables

In several complex variables, i.e. assuming that $\Omega \subset \mathbf{C}^n$, $n \geq 2$, and considering the holomorphic conditions (A1)–(A7), (A2′)–(A4′) and (A7′) (perhaps (A8), (A9) and (A8′) should be replaced by similar statements, involving the fundamental solution for the $\bar{\partial}$ problem in Ω), the equivalence between (A1)–(A6) is true and so is the equivalence between (A2′)–(A4′) with the same proofs as for one complex variable. Moreover, (T1) $\Rightarrow$ (A7) $\Rightarrow$ (A4) and (T1′) $\Rightarrow$ (A7′) $\Rightarrow$ (A4′), with the only change in the proofs being that we prove the implication (T1′) $\Rightarrow$ (A7′) directly, in a similar way to the proof of (T1′) $\Rightarrow$ (SH′).

On the other hand, (A4) $\not\Rightarrow$(A7) $\not\Rightarrow$(T1) and (A4′) $\not\Rightarrow$(A7′) $\not\Rightarrow$(T1′) in several complex variables.

(A7) $\not\Rightarrow$(T1) and (A7′) $\not\Rightarrow$(T1′) Let Ω be the polydisc $D(0,2)^n$, $n \geq 2$, $E = \{(z_1,\ldots,z_n) \in D(0,2)^n : |z_1| > 1\}$, and $w = (0,\ldots,0)$. We thus have $G = \Omega \setminus \bar{E} = \{(z_1,\ldots,z_n) \in D(0,2)^n : |z_1| < 1\}$. Let $f \in \mathrm{Hol}(G)$. Then for every fixed $(z_2,\ldots,z_n) \in D(0,2)^{n-1}$ the maximum principle for one complex variable gives us that

$$\begin{aligned}\sup_{\zeta_1\in\mathbf{D}} |f(\zeta_1,z_2,\ldots,z_n)| &= \sup_{|z_1|=1}\ \limsup_{\mathbf{D}\ni\zeta_1\to z_1} |f(\zeta_1,z_2,\ldots,z_n)|\\ &\leq \sup_{|z_1|=1}\ \limsup_{G\ni\zeta\to(z_1,\ldots,z_n)} |f(\zeta)|.\end{aligned}$$

It follows directly that

$$|f(w)| \leq \sup_{z\in G}|f(z)| \leq \sup_{z\in\partial G\cap\Omega}\ \limsup_{G\ni\zeta\to z}|f(\zeta)|.$$

To complete the proof we notice that the inequality

$$\sup_{z\in G}|f(z)| \geq \sup_{z\in\partial G\cap\Omega}\ \limsup_{G\ni\zeta\to z}|f(\zeta)|$$

is obvious.

(A4) $\not\Rightarrow$(A7) and (A4′) $\not\Rightarrow$(A7′) Let

$$\begin{aligned}&\Omega = \{(z_1,\ldots,z_n) \in D(0,4)^n : (z_2,\ldots,z_n) \in \mathbf{D}^{n-1} \text{ or } |z_1| < 1 \text{ or } |z_1| > 2\},\\ &E = \{(z_1,\ldots,z_n) \in \Omega : z_1 \notin \mathbf{D}\},\\ &w = (0,\ldots,0,2),\end{aligned}$$

and $h(z_1,\ldots,z_n) = z_n$. Thus $G = \Omega \setminus \bar{E} = \mathbf{D} \times D(0,4)^{n-1}$ and $\partial G \cap \Omega = (\partial\mathbf{D}) \times \mathbf{D}^{n-1}$.

It is easy to see that

$$\sup_{z\in\partial G\cap\Omega} |h(z)| = 1 < 2 = h(w),$$

and thus that (A7) and (A7′) are false.

It remains to prove that (A4) is true, and this is essentially done as in example 2.1 in Fornæss–Stensønes [6] (which studies domains of holomorphy). Let $f \in \mathrm{Hol}(\Omega)$. Let for $(z_1,\dots,z_n) \in \widetilde{\Omega} := D(0,3) \times D(0,4)^{n-1}$,

$$g(z_1,\dots,z_n) = \frac{1}{2\pi i}\int_{|\zeta|=3} \frac{f(\zeta, z_2,\dots,z_n)}{\zeta - z_1}\, d\zeta$$

(the circle $|\zeta| = 3$ is assumed to be positively oriented), a continuous function that is holomorphic in each variable separately, and hence $g \in \mathrm{Hol}(\widetilde{\Omega})$ (by definition or by Hartogs's theorem, depending on how you define holomorphic). Moreover, $f = g$ on $D(0,3) \times \mathbf{D}^{n-1}$. Since $\Omega \cap \widetilde{\Omega}$ is connected, the uniqueness theorem for holomorphic functions of one complex variable (fix z_1 and vary the other variables one at a time) shows that $f = g$ on $\Omega \cap \widetilde{\Omega}$. We can thus define

$$\tilde{f}(z) = \begin{cases} f(z), & \text{if } z \in \Omega, \\ g(z), & \text{if } z \in \widetilde{\Omega}, \end{cases}$$

a holomorphic extension of f to $D(0,4)^n$. By the maximum principle of one complex variable we have

$$|f(z_1,\dots,z_n)| \le \sup_{|\zeta|=3} |f(\zeta, z_2,\dots,z_n)| \le \|f\|_E \quad \text{for } (z_1,\dots,z_n) \in \Omega \setminus E.$$

Hence (A4), and in particular (A4′), is true.

Remark. Notice that in the example for (A7) $\not\Rightarrow$(T1), E, G and Ω are all pseudoconvex. Pseudoconvexity seems to have something to do with our problem but does not give the whole picture. As an example of what it can give, we have the following proposition.

Proposition 4.1. *Let $\Omega \subset \mathbf{C}^n$ be a domain and $E \subset \Omega$. If every function that is holomorphic in a neighbourhood of E has a holomorphic extension to Ω, then the holomorphically convex hull of E in Ω is Ω.*

PROOF. Assume that the holomorphically convex hull of E in Ω is *not* Ω. Then, there is f holomorphic on Ω and $w \in \Omega$ with $|f(w)| > \|f\|_E$. That means that the function $z \mapsto 1/(f(z) - f(w))$ is holomorphic in a neighbourhood of E but not on all of Ω. ■

5. Completeness of weighted Bergman spaces

In this section we discuss some partial results concerning the completeness of $A^p_\mu(\Omega)$, whose interest, in the light of Theorem 2.1, lies in the case $p < \infty$.

Throughout the section we assume that $0 < p \le \infty$, that $\Omega \subset \mathbf{C}$ is a domain and that μ is a positive Borel measure on Ω.

We let $L^p_\mu(\Omega)$ denote the usual Lebesgue space (quasi)-normed by $\|f\|_{L^p_\mu(\Omega)} := (\int_\Omega |f|^p \, d\mu)^{1/p}$, $0 < p < \infty$, and $\|f\|_{L^\infty_\mu(\Omega)} = \inf\{C \ge 0 : \mu(\{z \in \Omega : |f(z)| > C\}) = 0\}$. A quasi-norm does not satisfy the usual triangle inequality, only a quasi-triangle inequality of the type $\|f + g\| \le C(\|f\| + \|g\|)$ for some fixed constant $C > 1$. For $p \ge 1$ we have a norm, for $0 < p < 1$ only a quasi-norm.

Definition 5.1. The *weighted Bergman space* $\mathscr{A}^p_\mu(\Omega)$ is defined by

$$\mathscr{A}^p_\mu(\Omega) = \{f \in \mathrm{Hol}(\Omega) : \|f\|_{L^p_\mu(\Omega)} < \infty\}.$$

The starting point of our study was to try to find out when the Bergman space $\mathscr{A}^p_\mu(\Omega)$ is complete. We can directly observe that $\mathscr{A}^\infty_\mu(\Omega) = H_{\operatorname{supp}\mu \cap \Omega}(\Omega)$. Thus we have already studied when $\mathscr{A}^\infty_\mu(\Omega)$ is a Banach space in Theorem 2.1. So far, the picture is very incomplete for $p < \infty$. Of course, we also need to know when $\|\cdot\|_{L^p_\mu(\Omega)}$ is a norm on $\mathscr{A}^p_\mu(\Omega)$.

Proposition 5.2. *The following are equivalent:*

(a) $\|\cdot\|_{L^p_\mu(\Omega)}$ *is a (quasi)-norm on* $\mathscr{A}^p_\mu(\Omega)$*;*

(b) $\operatorname{supp}\mu \cap \Omega$ *contains a limit point in* Ω.

If, moreover, $\mu(K) < \infty$ *for all compact* $K \subset \Omega$*, then*

(c) $\mathscr{A}^p_\mu(\Omega)/\sim \; \ne L^p_\mu(\operatorname{supp}\mu \cap \Omega)$*, where* $f \sim g$ *if* $\|f - g\|_{L^p_\mu(\Omega)} = 0$*,*

is also equivalent.

Proof. (b) $\Rightarrow$ (a) Let $f \in \mathscr{A}^p_\mu(\Omega)$ and assume that $\|f\|_{L^p_\mu(\Omega)} = 0$. By continuity, $f(z) = 0$ for $z \in \operatorname{supp}\mu \cap \Omega$. By (b) that means that $f(z_j) = 0$ for a sequence of points $\{z_j\}_{j=1}^\infty$ with a limit point in Ω. The uniqueness theorem implies that $f \equiv 0$.

$\neg$ (b) $\Rightarrow$ $\neg$ (a) Let $w \in \Omega \setminus \operatorname{supp}\mu$ and let $E = (\operatorname{supp}\mu \cap \Omega) \cup \{w\}$. Then $E \subset \Omega$ does not contain any limit point in Ω. By theorem 15.13 in Rudin [15], we can find a function f holomorphic in Ω such that $f(w) = 1$ and $f(z) = 0$ for $z \in \operatorname{supp}\mu \cap \Omega$, i.e. $f \not\equiv 0 = \|f\|_{L^p_\mu(\Omega)}$.

$\neg$ (b) $\Rightarrow$ $\neg$ (c) Let $f \in L^p_\mu(\operatorname{supp}\mu \cap \Omega)$ be arbitrary. By theorem 15.13 in Rudin [15], we can find a function g holomorphic in Ω such that $f(z) = g(z)$ for $z \in \operatorname{supp}\mu \cap \Omega$. Hence $f \sim g$ and $g \in \mathscr{A}^p_\mu(\Omega)$.

(b) $\Rightarrow$ (c) Let $\{z_j\}_{j=1}^\infty$ and $\{r_j\}_{j=1}^\infty$ be sequences such that $D(z_j, 2r_j) \subset \Omega$ are pairwise disjoint, $z_j \in \operatorname{supp}\mu$, and $z_0 = \lim_{j\to\infty} z_j \in \Omega$. Let further

$$f(z) = \begin{cases} 1, & \text{if } z \in \bigcup_{k=1}^\infty D(z_{2k}, r_{2k}), \\ 0, & \text{otherwise.} \end{cases}$$

Then $f \in L^p_\mu(\operatorname{supp}\mu \cap \Omega)$. Assume that there exists $g \in \mathscr{A}^p_\mu(\Omega)$ such that $f \sim g$. Thus, by continuity, $1 = \lim_{k\to\infty} g(z_{2k}) = g(z_0) = \lim_{k\to\infty} g(z_{2k-1}) = 0$, a contradiction. ■

Lemma 5.3. *Let* $0 < p \le q \le \infty$ *and assume that* $\mu(\Omega) < \infty$ *and that* $\mathscr{A}^p_\mu(\Omega)$ *is a* *(quasi)-Banach space. Then $\mathscr{A}^q_\mu(\Omega)$ is also a (quasi)-Banach space. Moreover, there does not exist a path $\gamma : [0,\infty) \to \Omega \setminus \operatorname{supp}\mu$ such that $\gamma(t) \to *$, as $t \to \infty$.*

PROOF. Let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $\mathscr{A}^q_\mu(\Omega)$. Since $\mu(\Omega) < \infty$, Hölder's inequality shows that $\{f_j\}_{j=1}^\infty$ is also a Cauchy sequence in $\mathscr{A}^p_\mu(\Omega)$, and by completeness of $\mathscr{A}^p_\mu(\Omega)$ there exists $g \in \mathscr{A}^p_\mu(\Omega)$ such that $f_j \to g$ in $L^p_\mu(\Omega)$, as $j \to \infty$. Moreover, $L^q_\mu(\Omega)$ is complete, so there exists $h \in L^q_\mu(\Omega)$ such that $f_j \to h$ in $L^q_\mu(\Omega)$, as $j \to \infty$. Again, since $\mu(\Omega) < \infty$, we find that $f_j \to h$ in $L^p_\mu(\Omega)$, as $j \to \infty$. Hence $g = h$ μ-a.e. in Ω and $f_j \to g$ in $L^q_\mu(\Omega)$, as $j \to \infty$. Since the Cauchy sequence was arbitrary, we have proved that $\mathscr{A}^q_\mu(\Omega)$ is complete. The condition for having a norm is, by Proposition 5.2, the same for p and q, and we are thus finished with the first part.

By the first part, $H_{\operatorname{supp}\mu\cap\Omega}(\Omega) = \mathscr{A}^\infty_\mu(\Omega)$ is a Banach space. We can therefore apply the implication (A1) $\Rightarrow$ (T1) of Theorem 2.1 to obtain the second part. ■

The radial case can serve as a model case.

Proposition 5.4. *Let $d\mu(re^{i\theta}) = d\nu(r)\,d\theta$ and assume that $\mu(K) < \infty$ for all compact $K \subset \mathbf{D}$. Then the following are equivalent:*

(a) *$\mathscr{A}^p_\mu(\mathbf{D})$ is a (quasi)-Banach space;*
(b) *$\partial\mathbf{D} \subset \operatorname{supp}\mu|_{\mathbf{D}}$;*
(c) *$1 \in \operatorname{supp}\nu|_{[0,1)}$;*
(d) *T_z is a bounded functional on $\mathscr{A}^p_\mu(\mathbf{D})$ for all $z \in \mathbf{D}$;*
(e) *$z \mapsto \|T_z\|_{\mathscr{A}^p_\mu(\mathbf{D})^*}$ is uniformly bounded on compact subsets of $\mathbf{D}$.*

Remark. Since (b) and (c) are independent of p, the other statements are also independent of p.

PROOF. (a) $\Rightarrow$ (b) If $\mu(\mathbf{D}) = \infty$, then $\operatorname{supp}\mu|_{\mathbf{D}} \cap \partial\mathbf{D} \neq \varnothing$, by assumption. On the other hand, if $\mu(\mathbf{D}) < \infty$, then $\operatorname{supp}\mu|_{\mathbf{D}} \cap \partial\mathbf{D} \neq \varnothing$, by Lemma 5.3. Since $\operatorname{supp}\mu$ is radially symmetric, (b) follows.

(b) $\Rightarrow$ (c) Follows directly from the fact that

$$\operatorname{supp}\mu|_{\mathbf{D}} = \{re^{i\theta} : r \in \operatorname{supp}\nu|_{[0,1)},\ 0 \le \theta < 2\pi\}.$$

(c) $\Rightarrow$ (d) Let $z = re^{i\theta} \in \mathbf{D}$ and $f \in \mathscr{A}^p_\mu(\mathbf{D})$. Using the Poisson integral formula, we get

$$\begin{aligned}|f(z)| &\le \frac{1}{2\pi\nu((\sqrt{r},1))}\int_{\sqrt{r}}^1\int_0^{2\pi}\left|\frac{1-(r/\rho)^2}{1-2(r/\rho)\cos(\theta-t)+(r/\rho)^2}\right||f(\rho e^{it})|\,dt\,d\nu(\rho)\\ &\le \frac{1-r^2}{2\pi\nu((\sqrt{r},1))(1-\sqrt{r})^2}\int_{\sqrt{r}}^1\int_0^{2\pi}|f(\rho e^{it})|\,dt\,d\nu(\rho)\\ &\le \frac{1-r^2}{2\pi\nu((\sqrt{r},1))(1-\sqrt{r})^2}\|f\|_{L^p_\mu(\mathbf{D})}.\end{aligned}$$

By (c), $\nu((\sqrt{r},1)) > 0$, and thus T_z is bounded on $\mathscr{A}^p_\mu(\mathbf{D})$.

(d) $\Rightarrow$ (e) Let $K \subset \mathbf{D}$ be compact and $r < 1$ be such that $K \subset D(0,r)$. Let $f \in \mathscr{A}^p_\mu(\mathbf{D})$ with $\|f\|_{L^p_\mu(\mathbf{D})} \le 1$. By the maximum principle and symmetry, $\|f\|_K \le \|f\|_{\partial D(0,r)} \le \|T_r\|_{\mathscr{A}^p_\mu(\mathbf{D})^*}$. Hence $\|T_z\|_{\mathscr{A}^p_\mu(\mathbf{D})^*} \le \|T_r\|_{\mathscr{A}^p_\mu(\mathbf{D})^*} < \infty$ for $z \in K$.

(e) $\Rightarrow$ (a) First of all it is clear that $\|\cdot\|_{L^p_\mu(\mathbf{D})}$ is a norm, since otherwise we would have some $f \in \mathscr{A}^p_\mu(\mathbf{D})$ and $z \in \mathbf{D}$ with $\|f\|_{L^p_\mu(\mathbf{D})} = 0 \neq T_z f$, which would imply that $\|T_z\|_{\mathscr{A}^p_\mu(\mathbf{D})^*} = \infty$. Secondly, take a Cauchy sequence $\{f_j\}_{j=1}^\infty$ in $\mathscr{A}^p_\mu(\mathbf{D})$. By (e), $\{f_j(z)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{C}$ for every $z \in \mathbf{D}$. Hence $g(z) = \lim_{j\to\infty} f_j(z)$ exists for all $z \in \mathbf{D}$. By (e), $f_j \to g$ uniformly on compact subsets of $\mathbf{D}$, as $j \to \infty$, and therefore $g \in \mathrm{Hol}(\mathbf{D})$. Moreover, $L^p_\mu(\mathbf{D})$ is a complete space, so there exists $h \in L^p_\mu(\mathbf{D})$ such that $f_j \to h$ in $L^p_\mu(\mathbf{D})$, as $j \to \infty$. It follows that $f_j(z) \to h(z)$, as $j \to \infty$, for μ-a.e. $z \in \mathbf{D}$. Hence $g = h$ μ-a.e. in $\mathbf{D}$. Therefore $g \in \mathscr{A}^p_\mu(\mathbf{D})$ and $\mathscr{A}^p_\mu(\mathbf{D})$ is complete. ∎

The following result provides a general criterion for a weighted Bergman space to be a (quasi)-Banach space.

Proposition 5.5. *Let $w = d\mu/dm$, the Radon–Nikodym derivative with respect to the Lebesgue measure m. Let also $K \subset \Omega$ be compact. If $\log^- w \in L^1_{\mathrm{loc}}(\Omega \setminus K)$, then $\mathscr{A}^p_\mu(\Omega)$ is a (quasi)-Banach space.*

Remarks. The functions $\log^+$ and $\log^-$ are defined by $\log^+ x = \max\{\log x, 0\}$ and $\log^- x = \max\{-\log x, 0\}$.

Notice also that the condition is independent of p.

Proof. We notice that $w > 0$ a.e. on $\Omega \setminus K$ and hence $\Omega \setminus K \subset \operatorname{supp}\mu$. For $p = \infty$ the result follows from Theorem 2.1. Assume, for the rest of the proof, that $0 < p < \infty$. We will prove that (c) in Proposition 5.8 below holds, from which the result follows.

Let $K' \subset \Omega$ be compact and such that $K \subset \operatorname{int} K'$. Let $r = \frac{1}{2}\operatorname{dist}(\partial K', K \cup \partial\Omega)$ and let $G = \{z \in \mathbf{C} : \operatorname{dist}(z, \partial K') < r\}$. By assumption, $\log^- w \in L^1(G)$. Further, let $z \in \partial K'$ and $D = D(z,r)$. Let also $f \in \mathscr{A}^p_\mu(\Omega)$. We get, using the subharmonicity of $\log|f|^p$ and Jensen's inequality,

$$\begin{aligned}
|f(z)|^p &= \exp(\log|f(z)|^p) \\
&\le \exp\left(\frac{1}{m(D)}\int_D \log|f(\zeta)|^p\,dm(\zeta)\right) \\
&= \exp\left(\frac{1}{m(D)}\int_D \log(|f(\zeta)|^p w(\zeta))\,dm(\zeta)\right)\exp\left(\frac{1}{m(D)}\int_D \log\frac{1}{w(\zeta)}\,dm(\zeta)\right) \\
&\le \frac{1}{m(D)}\int_D |f(\zeta)|^p w(\zeta)\,dm(\zeta)\exp\left(\frac{1}{m(D)}\int_G \log^+\frac{1}{w(\zeta)}\,dm(\zeta)\right) \\
&\le \frac{1}{m(D)}\|f\|^p_{L^p_\mu(\Omega)}\exp\left(\frac{1}{m(D)}\|\log^- w\|_{L^1(G)}\right).
\end{aligned}$$

This shows that T_z is uniformly bounded for $z \in \partial K'$.

By the maximum principle,

$$\sup_{z\in K'} |T_z f| \le \sup_{z\in\partial K'} |T_z f| \quad \text{for all } f \in \mathscr{A}^p_\mu(\Omega).$$

Hence

$$\sup_{z\in K'} \| T_z \|_{\mathscr{A}^p_\mu(\Omega)^*} \le \sup_{z\in\partial K'} \| T_z \|_{\mathscr{A}^p_\mu(\Omega)^*} < \infty. \quad \blacksquare$$

Theorem 5.6. *Let $\Omega \subset \mathbf{C}$ be a bounded domain and $0 < p < \infty$. Then there exists a measure μ absolutely continuous with respect to Lebesgue measure, with $\Omega \subset \operatorname{supp} \mu$, and such that $\overline{\mathscr{A}^p_\mu(\Omega)} = L^p_\mu(\Omega)$, in fact such that the (holomorphic) polynomials are dense in $L^p_\mu(\Omega)$.*

Remarks. This stands out in contrast with the case $p = \infty$ and shows that the problem of determining exactly when a weighted Bergman space is a (quasi)-Banach space is not so easy.

For $p = 2$ this result was proved by Bram; see the proof of theorem 6 in Bram [2] or the proof of theorem 14.21 in Conway [5]. The proof essentially carries over to $p \neq 2$.

SKETCH OF PROOF. By changing exponent in the proof of Bram, one obtains that $\bar{z}$ is in the closure of the holomorphic polynomials, and hence any polynomial in two real variables is in the closure. An appeal to the Stone–Weierstrass theorem (see e.g. Rudin [16], p. 122) shows that all continuous functions are in the closure—they can be approximated by polynomials uniformly—and hence in our (quasi)-norm. Finally, the continuous functions are dense in $L^p_\mu(\Omega)$ (see e.g. theorems 3.13 and 3.14 in Rudin [15]), the proofs therein being valid also for $0 < p < 1$.

In order to have an easier example of non-completeness for a measure with full support we give the following example.

Example 5.7. Let $0 < p < \infty$ and let $\mathbf{D}$ be the domain under consideration. Let also $f(z) = 1/z$. Runge's theorem provides us with polynomials p_j, $j = 1, 2, \ldots$, such that

$$|f(z) - p_j(z)| < \frac{1}{j}, \quad z \in \mathbf{D},\ |\operatorname{Im} z| \ge \frac{1}{j}.$$

Let $M_j = \sup_{z\in\mathbf{D}} |p_j(z)|$, $a_0 = 1$, $a_j = \min\{j/(j+1)^p 2^j, M_j^{-p}, a_{j-1}\}$, $j = 1, 2, \ldots$,

$$w(z) = \begin{cases} a_j, & 1/(j+1) \le |\operatorname{Im} z| < 1/j, \\ 0, & z \in \mathbf{R}, \end{cases}$$

and $d\mu = w\, dm$. Then

$$\|f\|^p_{L^p_\mu(\mathbf{D})} \le \sum_{j=1}^{\infty} (j+1)^p a_j \frac{4}{j} \le \sum_{j=1}^{\infty} \frac{4}{2^j} = 4.$$

Thus

$$\begin{aligned}\|f-p_j\|^p_{L^p_\mu(\mathbf{D})} &= \int_{\{z\in\mathbf{D}:|\mathrm{Im}\,z|\geq 1/j\}} |f-p_j|^p w\,dm + \int_{\{z\in\mathbf{D}:|\mathrm{Im}\,z|<1/j\}} |f-p_j|^p w\,dm \\ &\leq \frac{\pi}{j^p} + 2^p \int_{\{z\in\mathbf{D}:|\mathrm{Im}\,z|<1/j\}} |f|^p w\,dm + 2^p \int_{\{z\in\mathbf{D}:|\mathrm{Im}\,z|<1/j\}} |p_j|^p w\,dm \\ &\leq \frac{\pi}{j^p} + 2^p \|f\|^p_{L^p(\{z\in\mathbf{D}:|\mathrm{Im}\,z|<1/j\})} + \frac{2^p\cdot 4a_j M_j^p}{j} \to 0, \quad \text{as } j\to\infty.\end{aligned}$$

Since obviously $p_j \in \mathscr{A}^p_\mu(\mathbf{D})$, we find that $f \in \overline{\mathscr{A}^p_\mu(\mathbf{D})}$. If there exists $g \in \mathscr{A}^p_\mu(\mathbf{D})$ such that $p_j \to g$ in $\mathscr{A}^p_\mu(\mathbf{D})$, then $f = g$ μ-a.e., and hence, by continuity, $f(z) = g(z)$ for all $z \in \mathbf{D}$, a contradiction. Thus $\mathscr{A}^p_\mu(\mathbf{D})$ is not complete.

We can easily find a C^∞ weight $\widetilde{w} \leq w$ that is zero only on the real axis, and therefore $\mathscr{A}^p_{\tilde\mu}(\mathbf{D})$, where $d\tilde\mu = \widetilde{w}\,dm$, is also non-complete, in the same way as above.

In the rest of this section we present some results on the connection between $\mathscr{A}^p_\mu(\Omega)$ being a (quasi)-Banach space and the boundedness of its point evaluations.

Proposition 5.8. *Consider the following statements:*

(a) $\mathscr{A}^p_\mu(\Omega)$ *is a* (*quasi*)*-Banach space;*

(b) T_z *is a bounded functional on* $\mathscr{A}^p_\mu(\Omega)$ *for all* $z \in \Omega$*;*

(c) $z \mapsto \|T_z\|_{\mathscr{A}^p_\mu(\Omega)^*}$ *is uniformly bounded on compact subsets of* Ω.

Then ((a) *and* (b)) $\Leftrightarrow$ (c).

Remark. In the radial case and the case where $p = \infty$ we have already seen that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c), but we have not been able to prove this in general. On the other hand, we do not have a counter-example either.

Proof. (c) $\Rightarrow$ (b) Trivial.

(c) $\Rightarrow$ (a) This is proved exactly as (e) $\Rightarrow$ (a) in Proposition 5.4.

((a) and (b)) $\Rightarrow$ (c) Since $\mathscr{A}^p_\mu(\Omega)$ is a (quasi)-Banach space, it is an F-space in the terminology of Rudin [16], and the Banach–Steinhaus theorem is at our disposal. Let $K \subset \Omega$ be compact. Since $\sup_{z\in K}|T_z f| = \sup_{z\in K}|f(z)| < \infty$ for all $f \in \mathscr{A}^p_\mu(\Omega)$ (in fact for all $f \in \mathrm{Hol}(\Omega)$), the Banach–Steinhaus theorem (combine theorems 2.4 and 2.6 in Rudin [16]) shows that $\sup_{z\in K}\|T_z\|_{\mathscr{A}^p_\mu(\Omega)^*} < \infty$. ∎

Proposition 5.9. *Let* $w \in \Omega$ *and* δ_w *be the Dirac measure at* w. *Then* $\mathscr{A}^p_\mu(\Omega) = \mathscr{A}^p_{\mu+\delta_w}(\Omega)$*, as sets. Moreover,*

(i) *if* T_w *is bounded on* $\mathscr{A}^p_\mu(\Omega)$*, then the* (*quasi*)*-*(*semi*)*norms on* $\mathscr{A}^p_\mu(\Omega)$ *and* $\mathscr{A}^p_{\mu+\delta_w}(\Omega)$ *are equivalent;*

(ii) *if* T_w *is unbounded on* $\mathscr{A}^p_\mu(\Omega)$*, then* $\mathscr{A}^p_{\mu+\delta_w}(\Omega)$ *is not a* (*quasi*)*-Banach space;*

(iii) T_w *is bounded on* $\mathscr{A}^p_{\mu+\delta_w}(\Omega)$ (*with norm* ≤ 1).

PROOF. The first part is clear, since $\|f\|^p_{L^p_{\mu+\delta_w}(\Omega)} = \|f\|^p_{L^p_\mu(\Omega)} + |f(w)|^p$, and $f(w)$ is bounded for every $f \in \mathrm{Hol}(\Omega)$.

(i) For all $f \in \mathscr{A}^p_\mu(\Omega) = \mathscr{A}^p_{\mu+\delta_w}(\Omega)$ we have

$$\|f\|^p_{L^p_\mu(\Omega)} \le \|f\|^p_{L^p_{\mu+\delta_w}(\Omega)} = \|f\|^p_{L^p_\mu(\Omega)} + |f(w)|^p \le (1 + \|T_w\|^p_{\mathscr{A}^p_\mu(\Omega)^*})\|f\|^p_{L^p_\mu(\Omega)}.$$

(ii) Since T_w is unbounded, there exists a sequence $\{f_j\}_{j=1}^\infty$ of functions $f_j \in \mathscr{A}^p_\mu(\Omega)$ with $\|f_j\|_{L^p_\mu(\Omega)} \le 1/j$ and $T_w f_j = f_j(w) = 1$. Let

$$g(z) = \begin{cases} 1, & z = w, \\ 0, & z \ne w. \end{cases}$$

We see directly that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathscr{A}^p_{\mu+\delta_w}(\Omega)$ and that $f_j \to g$ in $L^p_{\mu+\delta_w}(\Omega)$. Assume that $\mathscr{A}^p_{\mu+\delta_w}(\Omega)$ is a (quasi)-Banach space. Then there exists $h \in \mathscr{A}^p_{\mu+\delta_w}(\Omega)$ such that $h = g$ $(\mu + \delta_w)$-a.e. This implies that $h(w) = 1$ and, by continuity, that $h(z) = 0$ for $z \in \operatorname{supp}\mu \cap \Omega \setminus \{w\}$. By Proposition 5.2 we find a sequence of zeros of h with a limit point in Ω (possibly w). The uniqueness theorem shows that $h \equiv 0 \ne h(w)$, a contradiction.

(iii) We have

$$|T_w f|^p \le \|f\|^p_{L^p_\mu(\Omega)} + |f(w)|^p = \|f\|^p_{L^p_{\mu+\delta_w}(\Omega)}$$

for $f \in \mathscr{A}^p_{\mu+\delta_w}(\Omega)$. ■

Proposition 5.10. *Assume that* $\mathscr{A}^p_\mu(\Omega)$ *is a (quasi)-Banach space. Then the set* $M = \{z \in \Omega : T_z \text{ is bounded on } \mathscr{A}^p_\mu(\Omega)\}$ *is relatively closed in* Ω.

PROOF. Take a converging sequence $\{z_j\}_{j=1}^\infty$ of points in M with $z_0 = \lim_{j\to\infty} z_j \in \Omega$. Since $\sup_{j\ge1}|T_{z_j}f| = \sup_{j\ge1}|f(z_j)| < \infty$ for all $f \in \mathscr{A}^p_\mu(\Omega)$ and $\{T_{z_j} : j \ge 1\}$ is a collection of bounded functionals, the Banach–Steinhaus theorem (combine theorems 2.4 and 2.6 in Rudin [16]) shows that $\sup_{j\ge1}\|T_{z_j}\|_{\mathscr{A}^p_\mu(\Omega)^*} < \infty$. Hence $|T_{z_0}f| = \lim_{j\to\infty}|T_{z_j}f| \le \sup_{j\ge1}\|T_{z_j}\|_{\mathscr{A}^p_\mu(\Omega)^*} < \infty$ for all $f \in \mathscr{A}^p_\mu(\Omega)$ with $\|f\|_{L^p_\mu(\Omega)} \le 1$. It follows that T_{z_0} is a bounded functional, and therefore $z_0 \in M$. ■

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REFERENCES

[1] N.U. Arakelyan, Uniform and tangential approximations by analytic functions, *Izvestiya Akademii Nauk Armyanskoĭ SSR. Seriya Matematika* **3** (1968), 273–86 [in Russian]; English transl.: in L.J. Leifman (ed.), *Ten papers on complex analysis*, American Mathematical Society Translations 122, American Mathematical Society, Providence, RI, 1984, pp 85–97.

[2] J. Bram, Subnormal operators, *Duke Mathematical Journal* **22** (1955), 75–94.

[3] L. Brown and A.L. Shields, Approximation by analytic functions uniformly continuous on a set, *Duke Mathematical Journal* **42** (1975), 71–81.
[4] H. Chen and P.M. Gauthier, A maximum principle for subharmonic and plurisubharmonic functions, *Canadian Mathematical Bulletin* **35** (1992), 34–9.
[5] J.B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs 36, American Mathematical Society, Providence, RI, 1991.
[6] J.E. Fornæss and B. Stensønes, *Lectures on counterexamples in several complex variables*, Mathematical Notes 33, Princeton University Press, NJ, 1987.
[7] B. Fuglede, Asymptotic paths for subharmonic functions, *Mathematische Annalen* **213** (1975), 261–74.
[8] D. Gaier, *Lectures on complex approximation*, Birkhäuser, Boston, MA, 1987.
[9] S.J. Gardiner, The Dirichlet problem with non-compact boundary, *Mathematische Zeitschrift* **213** (1993), 163–70.
[10] S.J. Gardiner, *Harmonic approximation*, London Mathematical Society Lecture Note Series 221, Cambridge University Press, 1995.
[11] P.M. Gauthier, Mittag-Leffler theorems on Riemann surfaces and Riemannian manifolds, *Canadian Journal of Mathematics* **50** (1998), 547–62.
[12] P.M. Gauthier, M. Goldstein and W.H. Ow, Uniform approximation on unbounded sets by harmonic functions with logarithmic singularities, *Transactions of the American Mathematical Society* **261** (1980), 169–83.
[13] P.M. Gauthier, M. Goldstein and W.H. Ow, Uniform approximation on closed sets by harmonic functions with Newtonian singularities, *Journal of the London Mathematical Society* **28** (1983), 71–82.
[14] P.M. Gauthier, R. Grothmann and W. Hengartner, Asymptotic maximum principles for subharmonic and plurisubharmonic functions, *Canadian Journal of Mathematics* **40** (1988), 477–86.
[15] W. Rudin, *Real and complex analysis* (3rd edn), McGraw-Hill, New York, 1987.
[16] W. Rudin, *Functional analysis* (2nd edn), McGraw-Hill, New York, 1991.
[17] A. Stray, Characterization of Mergelyan sets, *Proceedings of the American Mathematical Society* **44** (1974), 347–52.
[18] A. Stray, Approximation by analytic functions which are uniformly continuous on a subset of their domain of definition, *American Journal of Mathematics* **99** (1977), 787–800.