QUASI-WEAKLY COMPACT OPERATORS IN NORMED SPACES

By T. Alvarez*
Departamento de Matemáticas, Universidad de Oviedo, Spain

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Abstract

We study the quasi-weakly compact operators between normed spaces, which are described in terms of their first and second conjugates. Using a result on factorisation of an arbitrary linear operator due to T. Alvarez, R.W. Cross and M. Gonzalez (Factorization of unbounded thin and cothin operators, *Quaestiones Mathematicae* 22 (1999), 519–29), we obtain that quasi-weakly compact operators admit factorisations through quasi-reflexive Banach spaces. As an application, we derive the connection between quasi-weak compactness of the operator and that of this conjugate, and we show examples and special cases. A condition is given for a quasi-weakly compact operator to be strictly singular or strictly cosingular.

1. Introduction and preliminaries

In this paper we shall consider a linear operator T defined on a subspace $D(T)$ of a normed space X with range $R(T)$ contained in a normed space Y. If T is continuous we write $T \in B(X, Y)$, and $B[X, Y]$ denotes the class of all $T \in B(X, Y)$ with $D(T) = X$. For $X = Y$ we will use $B(X)$ (resp. $B[X]$) instead of $B(X, X)$ (resp. $B[X, X]$). For a subspace E of X, $E^{\circ} = \{x' \in X' : x'(x) = 0, x \in E\}, J_E^X$ denotes the natural injection of E into X, with the quotient map from X onto X/E being denoted by Q_E^X (J_E and Q_E when X is understood). The conjugate T' of T is the conjugate of $TJ_{D(T)}$ in the usual sense (see e.g. [9, II.2.2]). Let Q denote the quotient map of Y'' onto $Y''/D(T')^{\circ}$; then, with usual identification $QY'' = D(T')'$, we have that $T'' : D(T'') \subset D(T)'' \to D(T')'$. Also we write J_X for the injection of X into its completion $X^{\tilde{}}$, and we shall use J to denote the natural embedding of a given normed space into its second dual.

If T is continuous then $T^{\tilde{}}$ is the continuous extension to $D(T)^{\tilde{}}$ of T regarded as an element of $B(X^{\tilde{}}, Y^{\tilde{}})$. With the natural identification of the isometric spaces X' and $(X^{\tilde{}})'$, we have $T' = (T^{\tilde{}})'$.

The operator T is said to be strictly singular if, whenever the restriction of T to a subspace Z of $D(T)$ has a continuous inverse, Z is finite-dimensional, and strictly cosingular if there is no closed infinite-codimensional subspace N of Y for which $(Q_N T)'$ has a continuous inverse [10].

Recall that a subspace W of X' is called total on X if, for any $x \in X$, $x'(x) = 0$, $x' \in W$ implies that $x = 0$; W is norm-determining for X if there is an $\varepsilon > 0$ (depending only on W) such that $\varepsilon \|x\| \leq sup\{|x'(x)| : x' \in W, \|x'\| \leq 1\}$ (see e.g. [14, pp 157, 172]).

Let M be a closed, $\sigma(Y', Y)$-dense subspace of Y'. Then M induces a weak

* E-mail: seco@pinon.ccu.uniovi.es

topology $\sigma(Y, M)$ on Y. In [15], for Y complete, $\sigma(Y, M)$ is called an ultraweak topology on Y, where an extension of the Eberlein–Smulian theorem is obtained under certain conditions on M.

In this paper we study some other aspects of the ultraweak topologies on Y in relation to linear operators $T: D(T) \subset X \to Y$ where X and Y are normed spaces.

The following lemma is elementary and helps to explain Definition 2 and the next results.

Lemma 1. *Let M be a $\sigma(Y', Y^{\tilde{}})$-dense subspace of Y'. Then M is total in Y and $\sigma(Y, M)$ is a Hausdorff topology in Y. If M° is finite-dimensional then M is norm-determining for Y.*

Proof. M is $\sigma(Y', Y^{\tilde{}})$-dense in Y' if and only if M is total on $Y^{\tilde{}}$ or, equivalently, $JY^{\tilde{}} \cap M^\circ = \{0\}$ [14, p. 164], and since JY can be regarded as a subspace of $JY^{\tilde{}}$ we have that if M is $\sigma(Y', Y^{\tilde{}})$-dense in Y' then M is total on Y and consequently $\sigma(Y, M)$ is Hausdorff. Furthermore, if M° is finite-dimensional, then $JY^{\tilde{}} \oplus M^\circ$ is closed, and thus, by a result of Dixmier [7], M is norm-determining for $Y^{\tilde{}}$ and also for Y. ∎

Definition 2. Let M be a closed, $\sigma(Y', Y^{\tilde{}})$-dense subspace of Y' such that M° is finite-dimensional. We say that T is M°-*weakly compact* if it maps bounded subsets in $D(T)$ to relatively $\sigma(Y, M)$-compact subsets in Y, and T is called *quasi-weakly compact* if it is M°-weakly compact for some M as above.

The corresponding classes of operators will be abbreviated to $M^\circ\text{-}WC(X, Y)$ and $QWC(X, Y)$ respectively.

In the case where X, Y are Banach spaces, $M = Y'$ and $T \in B[X, Y]$ our definition of M°-weakly compact coincides with the standard notion of *weakly compact operator*. In this case the following properties are equivalent:

(a) T is weakly compact.
(b) $R(T'') \subset JY$.
(c) T' is $\sigma(Y', Y) - \sigma(X', X)$-continuous.
(d) There is a reflexive space Z and operators $U \in B[Z, Y]$, $V \in B[X, Z]$ such that $T = UV$.
(e) T' is weakly compact.

See [8, ths 2 and 8, and lemma 7].

In Section 2 we shall obtain statements analogous to (a), (b) and (c) for quasi-weakly compact operators. The properties corresponding to (a), (b) and (c) are:

(a′) T is M°-weakly compact.
(b′) $R(T'') \subset Q(JY \oplus M^\circ)$.
(c′) T': $D(T')$, $\sigma(D(T'), JY \oplus M^\circ)) \to (D(T)', \sigma(D(T'), D(T'')))$ is continuous.

We will establish that (a′) ⇒ (b′) ⇔ (c′) and that all three conditions are equivalent under the assumption that T is continuous.

In Section 3 we give a result of factorisation for quasi-weakly compact operators, showing that if $T \in QWC(X, Y)$ then $T_\circ := J_Y T$ factors through a quasi-reflexive

Banach space. As an application of this factorisation, we derive the connection between the quasi-weak compactness of T and that of T'.

Section 4 describes examples and special cases. We will investigate the relationship between quasi-reflexive spaces and range-closed quasi-weakly compact operators. The remaining portion of this section gives some conditions under which a quasi-weakly compact operator is strictly singular or strictly cosingular.

2. First and second conjugates of a quasi-weakly compact operator

Proposition 3. *If $T \in QWC(X, Y)$ then $T \in B(X, Y)$.*

PROOF. Suppose that there is a closed, $\sigma(Y', Y^{\tilde{}})$-dense subspace M of Y' with M° finite-dimensional such that T is M°-weakly compact. Then if B is a bounded subset in $D(T)$ it is TB relatively $\sigma(Y, M)$-compact. Since $\sigma(Y, M)$ is Hausdorff and for $y' \in M$ the set $\{y \in Y : |y'(y)| < 1\}$ is a neighbourhood of 0_Y in $\sigma(Y, M)$, it follows that $sup\{|y'(Tx)| : x \in B\} < \infty$. Considering this fact, and that M is closed and norm-determining for Y, we conclude, from the generalisation of the uniform boundedness principle [14, p. 172], that TB is a bounded subset in Y. ∎

The following lemma will be used extensively.

Lemma 4. *Let M be a $\sigma(Y', Y)$-dense subspace of Y' and K be a $\sigma(Y'', M)$-compact subset in Y''. Then we have*

(a) *$K + M^\circ$ is $\sigma(Y'', M)$-compact;*

(b) *$K + M^\circ$ is $\sigma(Y'', M)$-closed;*

A subset F in Y is $\sigma(Y, M)$-compact if and only if JF is $\sigma(Y'', M)$-compact in Y''.

PROOF. Let (u_α) be a net in $K + M^\circ$, $u_\alpha = k_\alpha + v_\alpha$, with $k_\alpha \in K$, $v_\alpha \in M^\circ$.

(a) As K is $\sigma(Y'', M)$-compact, a subnet (k_β) of (k_α) converges in $\sigma(Y'', M)$ to some $k \in K$. Therefore the subnet (u_β) of (u_α) converges to k with respect to $\sigma(Y'', M)$, so $K + M^\circ$ is $\sigma(Y'', M)$-compact.

(b) Let $u_\alpha \to u \in Y''$ in $\sigma(Y'', M)$. Then $u_\alpha(y') \to u(y')$ for all $y' \in M$. By (a), there exists a subnet (u_β) of (u_α) such that $u_\beta(y') = k_\beta(y') \to k(y') = u(y')$, for all $y' \in M$. Hence $(u - k)(y') = 0$, $y' \in M$, and consequently $u = k + (u - k) \in K + M^\circ$, so that $K + M^\circ$ is $\sigma(Y'', M)$-closed.

The remaining part of the lemma is clear. ∎

Lemma 5 ([12, ch. 9, th. 9.13]). *Let $y'' \in Y''$. Then $y'' \in JY$ if and only if y'' is $\sigma(Y', Y)$-continuous.*

Theorem 6. *Let M be a closed, $\sigma(Y', Y^{\tilde{}})$-dense subspace of Y' such that M° is finite-dimensional. Consider the following properties:*

(a) *$T \in M^\circ$-$WC(X, Y)$;*

(b) *$R(T'') \subset Q(JY \oplus M^\circ)$;*

(c) *$T' : (D(T'), \sigma(D(T'), JY \oplus M^\circ)) \to (D(T)', \sigma(D(T'), D(T'')))$ is continuous.*

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). *If T is continuous then all three statements are equivalent.*

PROOF. Assume that (a) holds. Then by Proposition 3 T is continuous and, hence, we have that $T' \in B[Y', D(T)']$, $D(T'') = D(T)''$ [9, II.2.8], T'' is $\sigma(D(T)'', D(T)') - \sigma(Y'', Y')$-continuous with $T''J = JT$, $B_{D(T)''}$ is $\sigma(D(T)'', D(T)')$-compact (Alaoglu theorem) and $JB_{D(T)}$ is $\sigma(D(T)'', D(T)')$-dense in $B_{D(T)''}$ (Goldstine theorem). Since T is M°-weakly compact, if A denotes the $\sigma(Y, M)$-closure of $TB_{D(T)}$, then A is $\sigma(Y, M)$-compact, and $JA + M^\circ$ is $\sigma(Y'', M)$-closed, by Lemma 4. Now,

$T''B_{D(T)''} \subset (JTB_{D(T)})^{-\sigma(Y'', Y')} \subset (JTB_{D(T)})^{-\sigma(Y'', M)} \subset (JA)^{-\sigma(Y'', M)} \subset (JA + M^\circ)^{-\sigma(Y'', M)} = JA + M^\circ \subset JY \oplus M^\circ$ as required.

(b) $\Leftrightarrow$ (c) [2, prop. 4.5].

It remains only to show that (b) $\Rightarrow$ (a) if T is continuous. Assume that T is continuous and (b) is true. Let (x_α) be a net in $B_{D(T)}$. Then (Jx_α) is a net in $B_{D(T)''}$, which is $\sigma(D(T)'', D(T)')$-compact, and so (x_α) has a subnet (x_β) such that $Jx_\beta \to x''$ in $\sigma(D(T)'', D(T)')$ for some $x'' \in B_{D(T)''}$. Therefore $T''Jx_\beta \to T''x''$ in $\sigma(Y'', Y')$, and, since $R(T'') \subset JY \oplus M^\circ$, there exists $y \in Y$ such that if $y' \in M$ we have that $(T''x'')(y') = y'(y)$ and $y'(Tx_\beta) = (JTx_\beta)(y') = (T''Jx_\beta)(y') \to (T''x'')(y') = y'(y)$, that is, $Tx_\beta \to y$ with respect to $\sigma(Y, M)$. ∎

In general (a) is not equivalent to (b), as the following example shows.

Example 7. Let X be a reflexive space and S be the restriction of the identity operator to a dense subspace of codimension one. Put $Y := R(S)$, select a point $x_\circ \in X \backslash D(S)$ and define T by $Tx = Sx$, $x \in D(S)$, $Tx_\circ = 0$. Then it is clear that T satisfies (b), but T is not continuous, so that T is not quasi-weakly compact.

Proposition 8. *If* $T \in QWC(X, Y)$*, then* $T \in M^\circ$-$WC(X, Y)$ *for some closed,* $\sigma(Y', Y^{\tilde{}})$*-dense subspace* M *of* Y' *with* M° *finite-dimensional and contained in* $R(T'')$.

PROOF. Let $T \in Z^\circ$-$WC(X, Y)$ with $dim Z^\circ = n < \infty$. Let $F_1 := Z^\circ \bigcap R(T'')$ and $Z^\circ = F_1 \oplus F_2$ where F_2 is a complementary subspace of F_1 in Z°. If $F_2 = \{0\}$, then $F_2 = Z^\circ \subset R(T'')$ as required. Therefore we can assume that $F_2 \neq \{0\}$ and let $\{u_1, u_2, \ldots, u_m\}$ be a basis of F_2 $(1 \le m \le n)$. For each u_j, $j \in \{1, 2, \ldots, m\}$, there is $v_j \in R(T'')$ such that $v_j = Jy_j + w_j + \lambda u_j \neq 0$, where $y_j \in Y$, $w_j \in F_1$ and $\lambda_j \in K \backslash \{0\}$, since T is Z°-weakly compact. Now, if F_3 denotes the subspace spanned by $v_1, v_2, \ldots, v_m$, then it is easy to verify that the subspace $F := F_1 \oplus F_3 \subset R(T'')$ is finite-dimensional and T is $({}_\circ F)^\circ$-weakly compact. ∎

Lemma 9. *Let* $T \in M^\circ$-$WC(X, Y)$ *and* H *be a subspace of* Y *such that* $R(T'') \subset JH \oplus M^\circ$*. Then the subset* $G(H) := \{h \in H : Jh + m^\circ \in T''B_{D(T)''}$ *for some* $m^\circ \in M^\circ\}$ *is* $\sigma(Y, M)$*-compact in* Y*.*

PROOF. Let (h_α) be a net in $G(H)$. Then there exists a net (m_α°) in M° such that $(Jh_\alpha + m_\alpha^\circ)$ is a net in $T''B_{D(T)''}$ that is $\sigma(Y'', Y')$-compact, and so a subnet $(Jh_\beta + m_\beta^\circ)$ of $(Jh_\alpha + m_\alpha^\circ)$ converges to some $u \in T''B_{D(T)''} \subset JH \oplus M^\circ$ in $\sigma(Y, M)$. Hence, if $u = Jh + m^\circ$, $h \in H$, $m^\circ \in M^\circ$, it follows that $h_\beta \to h$ with respect to $\sigma(Y, M)$. ∎

Theorem 10. *Let* $T \in QWC(X, Y)$*. Then there is a closed,* $\sigma(Y', Y^{\tilde{}})$*-dense subspace*

M of Y' such that T is M°-weakly compact with M° finite-dimensional, $M^\circ \subset R(T'')$ and the $\sigma(Y,M)$-closure of $TB_{D(T)}$ is bounded in the norm topology on Y.

PROOF. First we prove the following property:

$$\text{For any } S \in Z^\circ\text{-}WC(X,Y) \text{ with } Z^\circ \subset R(S''), \quad R(S'') \subset J((R(S))^-) \oplus Z^\circ. \tag{1}$$

Let $S \in Z^\circ$-$WC(X,Y)$; then by Theorem 6 it follows that $R(S'') \subset JY \oplus Z^\circ$. If $u = Jy + z^\circ \in R(S'') \subset N(S')^\circ$, $y \in Y, z^\circ \in Z^\circ$, then for $y' \in N(S')$ we have that $0 = u(y') = Jy(y') + z^\circ(y') = y'(y) + z^\circ(y') = y'(y)$ because $Z^\circ \subset R(S'')$ by hypothesis. Thus, $y \in_\circ ((R(S)^-)^\circ) = (R(S))^-$ and consequently $R(S'') \subset J((R(S))^-) \oplus Z^\circ$.

If T is quasi-weakly compact, so is the restriction, say T_1, of T to $B_{D(T)}$. By Definition 2, Proposition 8 and the above property (1), it follows that there is a closed, $\sigma(Y',Y^\tilde{})$-dense subspace M of Y' such that T_1 is M°-weakly compact, $M^\circ \subset R(T_1'')$ and $R(T_1'') \subset J((R(T_1))^-) \oplus M^\circ = J((TB_{D(T)})^-) \oplus M^\circ$.

Now, for $H := (R(T_1))^-$, the set $G(H)$ defined in the previous lemma is $\sigma(Y,M)$-compact and hence $\sigma(Y,M)$-closed, and it is clear that $TB_{D(T)}$ is contained in $G(H)$ and this is trivially contained in H. Consequently,

$$(TB_{D(T)})^{-\sigma(Y,M)} \subset (G(H))^{-\sigma(Y,M)} = G(H) \subset H,$$

and so the result follows. ■

Theorem 11. *Let $T \in B(X,Y)$. Then T is quasi-weakly compact if and only if $T^\tilde{}$ is quasi-weakly compact.*

PROOF. The implication ($\Rightarrow$) is immediate from Theorem 6 (a) $\Leftrightarrow$ (b) upon observing that $D(T^\tilde{}) = (D(T))^\tilde{}$, $T'' = (T^\tilde{})''$, and that we can consider JY as a subspace of $JY^\tilde{}$. The converse follows from (a) $\Leftrightarrow$ (c) in Theorem 6. ■

3. Factorisation of a quasi-weakly compact operator

First we introduce some notations that will be used in the main result. We shall denote by D_T the normed space $(D(T), \|.\|_T)$ where $\|x\|_T := \|x\| + \|Tx\|, x \in D(T)$. The operator G_T (or simply G if there is no danger of confusion) denotes the natural injection from D_T into $D(T)$; TG is thus a continuous operator.

We shall say that T factors through a quasi-reflexive Banach space Z if there are linear operators $V : D(T) \subset X \to Z$, $U : Z \to Y$ such that $T = UV$.

Also, for every $S \in B[X,Y]$ with X, Y complete, we shall consider the bar functor of Yang [16], S^-, defined by $S^-(x'' + JX) := S''x'' + JY$, $x'' \in X''$.

The following theorem describes the Davies, Figiel, Johnson and Pelczynski factorisation of unbounded operators acting between normed spaces.

Theorem 12 [3, th. 2.1]. *Let $T : D(T) \subset X \to Y$ be given. Then corresponding to each $1 \le p \le \infty$ there exists a Banach space Z_p and a factorisation $R : D(T) \subset X \to Z_p$, $J_p : Z_p \to Y\tilde{}$, $T_\circ = J_p R$ in which J_p is a continuous injective operator with $D(J_p) = Z_p$, RG is continuous, and $(TG)\tilde{} = J_p(RG)\tilde{}$. Moreover, if $1 < p < \infty$ then J_p coincides with the tauberian injection in the factorisation result of Davies, Figiel, Johnson and Pelczynski of $(TG)\tilde{}$ corresponding to p.*

Theorem 13. *If T is quasi-weakly compact then $T_\circ$ factors through a quasi-reflexive Banach space. The converse is true if T is continuous.*

Proof. Case I: $T \in B[X, Y]$, X and Y complete. Assume that $T \in QWC(X, Y)$. Then $T \in M^\circ$-weakly compact for some closed subspace M of Y' such that $(M)^{-\sigma(Y',Y)} = Y'$ and M° is finite-dimensional. Then $(TB_X)^{-\sigma(Y,M)}$ is $\sigma(Y, M)$-compact and, by Lemma 4, is $J((TB_X)^{-\sigma(Y,M)}) + M^\circ$, $\sigma(Y'', M)$-closed. Now, with reference to [6, lemma 1], put $W = TB_X$, $F = \{y \in Y : |||y||| < \infty\}$ and let j denote the identity embedding of F into Y; j is continuous and $jB_F \subset 2^n W + 2^{-n} B_Y$. Hence

$$JjB_F = j''JB_F \subset 2^n JW + 2^{-n} JB_Y \subset 2^n JW + 2^{-n}(JB_Y)^{-\sigma(Y'',Y')} = 2^n JW + 2^{-n} B_{Y''}.$$

Since j'' is $\sigma(F'', F') - \sigma(Y'', Y')$-continuous and $B_{Y''}$ is $\sigma(Y'', Y')$-compact, if D denotes the $\sigma(Y, M)$-closure of W, we have

$$\begin{aligned} j''B_{F''} &= j''((JB_F)^{-\sigma(F'',F')}) \subset (j''JB_F)^{-\sigma(Y'',Y')} \subset (2^n JW + 2^{-n} B_{Y''})^{-\sigma(Y'',Y')} \\ &= (2^n JW)^{-\sigma(Y'',Y')} + 2^{-n} B_{Y''} \subset (2^n JD)^{-\sigma(Y'',M)} + 2^{-n} B_{Y''} \\ &\subset 2^n \{(JD + M^\circ)^{-\sigma(Y'',Y')}\} + 2^{-n} B_{Y''} \subset JY \oplus M^\circ + 2^{-n} B_{Y''}, n \in N. \end{aligned}$$

Consequently $j''B_{F''} \subset JY \oplus M^\circ$, and $B_{F''} \subset (j'')^{-1} JY \oplus (j'')^{-1} M^\circ = JF \oplus (j'')^{-1} M^\circ$ by virtue of [6, lemma 1]. Let $C := j'M$; then C is closed in F' since $C = R(j'J_M)$ is closed if and only if $R((J_M)'j'')$ is closed [9, IV.1.2] or, equivalently, $N((J_M)') + R(j'') = M^\circ + R(j'')$ is closed [9, IV.2.9], which is trivially true since M° is finite-dimensional and $R(j'')$ is closed. Moreover, the subspace C is total in F' because $JY \cap M^\circ = \{0\}$ implies that $JF \cap C^\circ = \{0\}$ and also $dim C^\circ = dim (j'')^{-1} M^\circ < \infty$. Therefore $R(I''_F) \subset JF \oplus C^\circ$, and thus F is a quasi-reflexive Banach space. Now, we obtain $T = j(j^{-1}T)$ with $j \in B[F, Y]$, $j^{-1}T \in B[X, F]$ because $TB_X \subset B_F$ by [6, Lemma 1].

Suppose that there is a quasi-reflexive space Z and $U \in B[Z, Y]$, $V \in B[X, Z]$ such that $T = UV$. Then $T^- = U^- V^- \in B[X''/JX, Y''/JY]$ with $dim R(T^-) < \infty$. Hence there exists a closed subspace Γ of Y'' such that $JY \cap \Gamma = \{0\}$, $R(T'') \subset JY \oplus \Gamma$ and $dim \Gamma < \infty$. It is clear that $M := {}_\circ\Gamma$ is a closed, $\sigma(Y', Y)$-dense subspace of Y' with $R(T'') \subset JY \oplus M^\circ$, and, applying Theorem 6, we conclude that T is quasi-weakly compact.

Case II: T an arbitrary linear operator and X, Y normed spaces. The result desired follows from the observation that if T is continuous then so is G^{-1}, and we deduce from Definition 2, Theorem 11, Case I and Theorem 6 that $T \in QWC \Leftrightarrow TG \in QWC \Leftrightarrow (TG)\tilde{} \in QWC \Leftrightarrow (TG)\tilde{}$ factors through a quasi-reflexive Banach space. ■

The following example shows that, in general, the converse to the above theorem is not true if T is not required to be continuous.

Example 14. Let $X = L_1[0,1]$, $Y = L_2[0,1]$ and define T by $D(T) = \{f \in L_1[0,1]$: f' exists almost everywhere and $f' \in L_2[0,1]\}$, $Tf = f'$, $f \in D(T)$ where f' is the derivative of f. Then it is obvious that T is not continuous, and so by Proposition 3 T is not quasi-weakly compact but T factors through Y, which is a quasi-reflexive Banach space.

As an application of the previous theorem we derive the relationship between a quasi-weakly compact and its conjugate.

Theorem 15. *If T is quasi-weakly compact then so is T'. The converse is true if T is continuous.*

PROOF. Assume that $T \in QWC(X,Y)$. Then by the above theorem and Theorem 6 we deduce that $T_\circ$ and $(TG)^{\sim}$ factors through a quasi-reflexive Banach space, and the same is true for $(TG)' = ((TG)^{\sim})'$ since a Banach space is quasi-reflexive if and only if its dual space is quasi-reflexive. But $T' = (G^{-1})'(TG)'$ [11, lemma 2.4] with $(G^{-1})' \in B[D'_T, D(T)']$, and so by Theorem 13 we conclude that T' is quasi-weakly compact.

Conversely, suppose that T is continuous and T' is quasi-weakly compact. Then by previous argument $T'' = (T^{\sim})'' \in QWC[(X^{\sim})'', (Y^{\sim})'']$ and hence $(T^{\sim})''J = JT^{\sim}$ is quasi-weakly compact or, equivalently, factors through a quasi-reflexive Banach space, and the same holds for $T^{\sim}$. Now, Theorem 11 ensures that T is quasi-weakly compact.

The operator T considered in Example 14 is not quasi-weakly compact, but since $D(T') = \{0\}$ we have that T' is obviously a quasi-weakly compact operator. ■

4. Examples of quasi-weakly compact operators

A normed space X is called paracomplete (or operator range) if it is the range $R(\alpha_X)$ of a continuous injective operator α_X defined on a Banach space X_s. This happens if and only if X admits a new norm, stronger than the initial one, that makes it complete [4] or, equivalently, the algebra $B[X]$ is paracomplete [1, th. 3.3]. The inverse of α_X is denoted by β_X.

Not all linear subspaces of a Banach space are operator ranges; for example, the null space of a discontinuous linear functional is not an operator range. We observe that many normed spaces that appear in applications are paracomplete, like the space $C[0,1]$ with the norm of $L_2[0,1]$ or some Sobolev spaces with suitable L_2-norms.

We define a paracomplete space X to be pre-quasi-reflexive if its pre-image X_s is quasi-reflexive. The following result is known for the class QR of all quasi-reflexive Banach spaces.

Theorem 16 [13, th. 2.1]. *Let E be a closed subspace of a quasi-reflexive Banach space Z. Then E and Z/E are quasi-reflexive Banach spaces. Moreover, the class QR has the three-space property, that is, if E is a closed subspace of a Banach space Z such that E and Z/E are quasi-reflexive spaces, then Z is quasi-reflexive.*

Next, we generalise this theorem for the class $P-QR$ of all pre-quasi-reflexive spaces.

Proposition 17. *Let E be a closed subspace of a pre-quasi-reflexive space X. Then E and X/E are pre-quasi-reflexive spaces. The class $P-QR$ has the three-space property.*

Proof. Let E be a closed subspace of a paracomplete space X.

(a) Assume that X is pre-quasi-reflexive. Since β_X is open it follows that $\beta_X E$ is a closed subspace of the quasi-reflexive space X_s. By the previous theorem $\beta_X E$ and $X_s/\beta_X E$ are quasi-reflexive spaces. It is easily seen that $\beta_X E = E_s$ and $X_s/\beta_X E = (X/E)_s$.

(b) Suppose that X/E and E are pre-quasi-reflexive spaces. Then E_s and $(X/E)_s$ are quasi-reflexive Banach spaces, and so, by Theorem 16, $X_s \in QR$, that is, X is pre-quasi-reflexive. ∎

Proposition 18. *Let $T \in B(X, Y)$. If either $(D(T))\tilde{}$ or $Y\tilde{}$ is quasi-reflexive then $T \in QWC(X, Y)$.*

Proof. It follows immediately by combining Theorem 11 with Theorem 13. ∎

Proposition 19. *If $D(T)$ is Banach and Y is pre-quasi-reflexive then*

$$B[D(T), Y] = QWC[D(T), Y].$$

Proof. Suppose that T is continuous with $D(T)$ complete and Y pre-quasi-reflexive. Then T, and therefore $\beta_Y T$, is closed. By the closed graph theorem, $\beta_Y T$ is continuous, and, since α_Y is quasi-weakly compact by Proposition 18, it follows that $T = \alpha_Y(\beta_Y T)$ is quasi-weakly compact. ∎

The following example shows that the completeness on $D(T)$ cannot be weakened.

Example 20. Consider the map $T \in B[l_2, l_1]$ defined by $T(\alpha_n) = (\alpha_n/n)$, $(\alpha_n) \in l_2$. Then $R(T)$ is dense in l_1, and $R(T)_s = l_2$ is quasi-reflexive. However, the identity map on $R(T)$, $I_{R(T)}$, is not quasi-weakly compact since, by Theorem 11, if $I_{R(T)} \in QWC[R(T)]$, then the identity on l_1 that coincides with $(I_{R(T)})\tilde{}$ is also quasi-weakly compact. Consequently l_1 is quasi-reflexive, which thus is not true.

We give some applications of Theorems 6 and 13 to quasi-reflexive Banach spaces and range-closed quasi-weakly compact operators acting between Banach spaces.

Proposition 21. *Let X, Y be Banach spaces and $T \in B[X, Y]$ with closed range. Then*

T is quasi-weakly compact if and only if $R(T)$ is quasi-reflexive. Moreover, a Banach space is quasi-reflexive if and only if it is the image of a range-closed quasi-weakly compact $S \in B[X, Y]$ with X and Y Banach spaces.

PROOF. Suppose that $T \in B[X, Y]$ where X and Y are Banach spaces and $R(T)$ is closed. Since $T = J_{R(T)}T$, if $R(T)$ is quasi-reflexive, then T factors through a quasi-reflexive Banach space, and so Theorem 13 ensures that T is quasi-weakly compact. Conversely, let $T \in M^{\circ}\text{-}WC[X, Y]$ for some M closed, $\sigma(Y', Y)$-dense subspace of Y' with $dim M^{\circ} < \infty$ and $R(T)$ closed. Then $(J_{R(T)})'' R(T)'' = R(T'') \subset JY \oplus M^{\circ}$ by Theorem 6. Moreover, it follows immediately from the equality $(J_{R(T)})'' JR(T) = JY \bigcap (J_{R(T)})'' R(T)''$ that $R(T)'' \subset JR(T) \oplus ((J_{R(T)})'')^{-1} M^{\circ}$, and, continuing as in the final part of the proof of Case I of Theorem 13, we deduce that $R(T)$ is a quasi-reflexive space.

If Z is a quasi-reflexive Banach space then by Theorem 13 the identity map on Z is obviously a quasi-weakly compact operator. The other implication follows from the above.

If X, Y are Banach spaces and $T \in B[X, Y]$ does not have closed range, then, in general, the closure of $R(T)$ is not quasi-reflexive when T is quasi-weakly compact. Indeed, it suffices to consider the inclusion map $i : l_2 \to c_o$, $i \in QWC[l_2, c_o]$. But $R(i)$ is dense in c_o, which is not quasi-reflexive. ■

We may relax the completeness condition in the above proposition by substituting an alternative assumption. Thus we have the following.

Proposition 22. *Let the paracomplete space X be pre-quasi-reflexive. If there is a continuous linear operator mapping X onto a Banach space Y, then Y is quasi-reflexive.*

PROOF. This follows trivially from Proposition 21, since, if $T \in B[X, Y]$ with $R(T) = Y$, $T\alpha_X$ is a continuous operator from a quasi-reflexive space X_s onto Y, and so, by Theorem 13, we obtain that $T\alpha_X$ is quasi-weakly compact. ■

Proposition 23. *Let Y be a paracomplete space. Then Y is the image of a quasi-weakly compact operator defined on a Banach space and having range in $Y^{\tilde{}}$ if and only if there exists a pre-quasi-reflexive space containing Y and contained in $Y^{\tilde{}}$.*

PROOF. Suppose that $T \in QW[X, Y^{\tilde{}}]$ with X complete and $R(T) = Y$. From Theorem 13 it follows that there are $V \in B[X, Z]$, $U \in B[Z, Y^{\tilde{}}]$ such that $T = UV$ and Z is a quasi-reflexive Banach space. Since $Z/N(U)$ is quasi-reflexive and $\hat{U} \in B[Z/N(U), Y^{\tilde{}}]$ we have that $R(U) = R(\hat{U})$ is pre-quasi-reflexive and that $R(U)_s = Z/N(U)$ and $Y \subset R(U) \subset Y^{\tilde{}}$.

Conversely, assume that $Y \subset Z \subset Y^{\tilde{}}$ is such that Z is a pre-quasi-reflexive space. Consider the subspace $X := \beta_Z Y$ of Z_s with the norm $|||.||| = ||.||_1 + ||\beta_Y \alpha_Z .||_2$ where $||.||_1$ and $||.||_2$ are the norms on Y_s and Z_s respectively. It is easy to prove that X is complete and that the injective map, J_X, from X into Z_s is continuous. Since Z_s is quasi-reflexive, we deduce from Theorem 13 that $j\alpha_Z J_X \in QWC[X, Y^{\tilde{}}]$ (where j is the inclusion from Z into $Y^{\tilde{}}$) with $R(j\alpha_Z J_X) = Y$. ■

Recall that T is an F_+-operator [5] if there is a closed finite-codimensional subspace E of $D(T)$ for which $(TJ_E)^{-1}$ exists and is continuous.

Proposition 24. *Let T be a quasi-weakly compact F_+-operator. Then $(D(T))\tilde{}$ is quasi-reflexive.*

Proof. Let E be a closed finite-codimensional subspace of $D(T)$ such that $(TJ_E)^{-1}$ exists and is continuous. We write $I_E = (TJ_E)^{-1}(TJ_E)$. If T is quasi-weakly compact then so is $(TJ_E)'$ by Theorem 15 and, consequently, $(I_E)'$ is quasi-weakly compact. Now, by Theorem 15, Theorem 11 and Theorem 6 we deduce that the completion of E is quasi-reflexive. But $E\tilde{}$ is a closed finite-codimensional subspace of $D(T))\tilde{}$ and hence, by the three-space property, it follows that $(D(T))\tilde{}$ is quasi-reflexive as required. ■

Proposition 25. *Let $T \in QWC(X, Y)$ such that either $D(T))\tilde{}$ or $Y\tilde{}$ contains no closed infinite-dimensional quasi-reflexive subspaces. Then T is strictly singular.*

Proof. (a) Suppose that the completion of $D(T)$ contains no isomorphic copies of a quasi-reflexive Banach space.

Case I: $T \in QWC[X, Y]$, X and Y Banach spaces. If T is not strictly singular then there exists a closed infinite-dimensional subspace E of X for which $(TJ_E)^{-1}$ is continuous. Hence $I_E = (TJ_E)^{-1}(TJ_E)$ is quasi-weakly compact and, by Theorem 6, we obtain that E is a quasi-reflexive subspace of X, contradicting the assumption for $D(T)\tilde{}$.

Case II: $T \in QWC(X, Y)$, X and Y normed spaces. Since

$$T\tilde{} \in QWC[D(T)\tilde{}, Y\tilde{}]$$

by Theorem 11, Case I ensures that $T\tilde{}$ is strictly singular and hence T is strictly singular.

(b) Assume that $Y\tilde{}$ contains no closed infinite-dimensional quasi-reflexive subspaces.

Case I: $T \in QWC[X, Y]$, X and Y Banach spaces. If T is not strictly singular then there is a closed infinite-dimensional subspace E of X for which $B_{T(E)} \subset \lambda TB_E$ for some $\lambda > 0$, where $F := T(E)$ is a closed subspace of Y.

If A denotes the $\sigma(Y, Y')$-closure of TB_E, we have $(J_F)''(JB_F)^{-\sigma(Y'', Y')} = (J_F)''B_{F''} \subset \lambda(JA)^{-\sigma(Y'',Y')}$ (2).

Since $T \in QWC[X, Y]$, it follows that $TJ_E \in QWC[E, Y]$, and hence there is a closed, $\sigma(Y', Y)$-dense subspace M of Y' such that TJ_E is M°-weakly compact. Then, by Lemma 4, $J((TB_E)^{-\sigma(Y,M)}) + M^\circ$ is $\sigma(Y'', M)$-closed, and, since the $\sigma(Y'', Y')$-closure of a set is contained in the $\sigma(Y'', M)$-closure of the set, it follows from property (2) that $(J_F)''B_{F''} \subset \lambda(JA)^{-\sigma(Y'',Y')} \subset \lambda J((TB_E)^{-\sigma(Y,M)}) + M^\circ \subset JY \oplus M^\circ$. Now, continuing as in the final part of the proof of Case I of Theorem 13, we conclude that F is a quasi-reflexive subspace of Y.

Case II: T an arbitrary quasi-weakly compact operator, X and Y normed spaces. Since $T \in QWC(X, Y)$, $(TG)\tilde{} \in QWC[(D_T)\tilde{}, Y\tilde{}]$ by Theorem 13. Then by Case I

$(TG)^{\tilde{}}$ is strictly singular, and hence TG is strictly singular, and consequently T is strictly singular. Indeed, suppose that E is a subspace of $D(T)$ for which $\alpha\|e\| \leq \|Te\|$ for $e \in E$ and some $\alpha > 0$. Writing $e = Ge_o$ with $e_o \in D_T$, we have $\alpha\|Ge_o\| \leq \|TGe_o\|$. Therefore $(1/2)\alpha\|e_o\|_T \leq \|TGe_o\|$, showing that E is finite-dimensional. Therefore T is strictly singular as required. ■

The following example shows that the condition *either* $(D(T))^{\tilde{}}$ *or* $Y^{\tilde{}}$ *contains no isomorphic copies of a quasi-reflexive space* in the previous result is not superfluous.

Example 26. Let T be a non-compact operator from l_2 into l_2 (for example, the identity map). Then it is obvious that l_2 contains closed infinite-dimensional quasi-reflexive subspaces, $T \in QWC[l_2]$ but T is not strictly singular.

Proposition 27. *Let* $T \in QWC(X, Y)$ *and suppose that either* $D(T)'$ *or* Y' *contains no closed infinite-dimensional quasi-reflexive subspaces. Then* T *is strictly cosingular.*

PROOF. By Theorem 15, $T' \in QWC[Y', D(T)']$, and so the above proposition ensures that T' is strictly singular, and consequently T is strictly cosingular. Indeed, suppose that T is not strictly cosingular. By definition, there is a closed infinite-codimensional subspace F of Y such that $(Q_F T)' = T' J_{F^\circ}$ has a continuous inverse. Since $dimF^\circ = dim(Y/F)' = \infty$, it follows that T' is not strictly singular.

This proposition does not hold if the assumption on $D(T)'$ or Y' is not required. We first observe that, if T is continuous, then T is strictly singular if T' is strictly cosingular. Assume that T' is strictly cosingular and that there exists an infinite-dimensional subspace E of $D(T)$ such that TJ_E has a continuous inverse. Then $(TJ_E)' = Q_{E^\circ} T'$ is surjective. Since $dimE = dimE' = dimD(T)'/E^\circ = \infty$, it follows that T' is not strictly cosingular. ■

Example 28. Let T be the operator considered in Example 26. Then there is a non-compact operator, say S, from l_2 into l_2 such that $T = S'$, Now, the previous observation ensures that T is not strictly cosingular.

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