AMITSUR'S CONJECTURE ON POLYNOMIAL RINGS IN n COMMUTING INDETERMINATES

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Abstract

For each natural number n we construct an algebra over a countable field such that the polynomial ring in n commuting indeterminates over this algebra is Jacobson radical but not a nil-ring. This answers a question of Amitsur.

1. Introduction

A theorem of Amitsur [2] says that the Jacobson radical of the polynomial ring over an associative ring R, $J(R[x])$, is equal to $I[x]$ for some nil ideal I of R. In [3] Amitsur constructed, for a given ring R, the descending chain of its radicals

$$J(R) \supseteq K(R) \supseteq J_1(R) \supseteq K_1(R) \supseteq \ldots \supseteq J_\infty(R) = K_\infty(R),$$

where $J(R)$ and $K(R)$ denote the Jacobson and upper nil radical of R, and, for every n, $J_n(R) = J(R[x_1, \ldots, x_n]) \cap R$, $K_n(R) = K(R[x_1, \ldots, x_n]) \cap R$, where $R[x_1, \ldots, x_n]$ is the polynomial ring over R in n commuting indeterminates. The question of whether there exists a ring R for which this is a strictly descending chain is open. The aim of this paper is to show that for each n there is a ring R_n such that the inclusion $J_n(R_n) \supset K_n(R_n)$ is strict. Observe that the ring $R = \oplus_{n=i,\ldots,\infty} R_i$ is such that $R = J(R) = K(R) = \ldots = J_i(R)$ and $J_n(R) \neq K_n(R)$ for $n \geq i$.

It is known [1] that if A is an algebra over an uncountable field then

$$J(A) = K(A) = J_1(A) = K_1(A) = \ldots = J_\infty(A) = K_\infty(A).$$

If A is a nil algebra over a countable field then $A[x]$ need not be nil (cf. [11]), even if $A[x]$ is Jacobson radical (cf. [12]). Thus $K_1(A) \neq J_1(A)$. Recall that the Köthe conjecture [6] is true if $K(R) = J_1(R)$ for every ring R [7]. Some interesting results and questions connected with this subject can be found in [3; 4; 9; 10].

In this paper we construct, for each countable field K and each i, an algebra R over K such that $J_i(R) \neq K_i(R)$. The most general idea of our example resembles that of the famous Golod–Shafarevich example (cf. [5; 8; 13; 14]). We use techniques similar to those in [11] and [12]. In Sections 1, 2 and 4 we generalise results from sections 2, 3 and 4 of [11]. As in [12], we use Krempa's well-known result that the polynomial ring over a ring R is Jacobson radical if and only if for any n the ring of $n \times n$ matrices over R is nil.

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The question of whether, for all rings R and for each n, $J_n(R) = K_{n-1}(R)$ remains open [3; 4].

In what follows, K is a field, z is a natural number and A is the algebra of polynomials in $z+2$ non-commuting indeterminates $a_1, \ldots, a_{z+2}$ over K. Let $\overline{A}$ be the subalgebra of A consisting of the polynomials with zero constant term.

We denote by M the set of monomials in $a_1, \ldots, a_{z+2}$ over K, and, for each $n \geq 0$, M_n denotes the set of monomials of degree n. Thus $M_0 = 1$, and for $n \geq 1$ the elements in M_n are of the form $x_1 \ldots x_n$, where $x_i \in \{a_1, \ldots, a_{z+2}\}$. The K-subspace of A spanned by M_n will be denoted by H_n, and elements of H_n will be called homogeneous polynomials of degree n. Every polynomial $f \in A$ can be uniquely represented in the form $f = f_0 + f_1 + \ldots + f_d$, where $f_i \in H_i$, $f_d \neq 0$. The f_i are called homogeneous components of f, and $d = degf$ is the degree of f.

Given a natural number n and a set $F \subseteq A$, let $B_n(F)$ denote the right ideal of A generated by the set $\bigcup_{k=0,1,2,\ldots} M_{nk}F$, i.e. $B_n(F) = \sum_{k=0}^{\infty} M_{nk}FA$.

2. Linear mappings

Let $m_1, m_2, \ldots$ be an increasing sequence of natural numbers such that each m_i divides m_{i+1}, and let S_i be a finite subset of H_{m_i}. In [11, section 2] we introduced mappings $R_i : H_{m_i} \to H_{m_i}$ in the case when A has three generators (i.e. $z = 1$). We will use the same definition (recalled below) when z is an arbitrary natural number.

Given a subset V_i of H_{m_i}, let $c_{V_i} : H_{m_i} \to H_{m_i}$ denote the K-linear mapping such that $ker(c_{V_i}) = span_K(V_i)$.

Let $F_i = \{f_{i,1}, \ldots, f_{i,r_i}\} \subseteq H_{m_i}$.

We shall define K-linear mappings $R_i : H_{m_i} \to H_{m_i}$ inductively for $i = 1, 2, \ldots$.

Put $R_1 = id$. Suppose that $i \geq 1$ and we have already defined R_j for $j \leq i$. Given $w = x_1 \ldots x_{m_{i+1}} \in M_{m_{i+1}}$, define

$$R_{i+1}(w) = c_{R_i(F_i)}(R_i(x_1 \ldots x_{m_i})) \prod_{j=2}^{m_{i+1}m_i{}^{-1}} R_i(x_{(j-1)m_i+1} \ldots x_{jm_i})$$

where $c_{R_i(F_i)}$ denotes $c_{\{R_i(f_{i,1}), \ldots, R_i(f_{i,r_i})\}}$.

Since increasing the number of generators of the algebra A does not affect the proofs in section 2 of [11], we have the following.

Theorem 1 (cf. [11, theorem 4]). *If* $w \in H_{m_{l+1}} \cap \sum_{i=1}^{l} B_{m_{i+1}}(F_i)$, *then* $R_{l+1}(w) = 0$.

Proof. See [11] proof of theorem 4. ∎

Lemma 2. *Define* $s_1 = m_1$, *and* $s_{i+1} = s_i(m_{i+1}m_i{}^{-1} - 1)$ *for* $i \geq 1$. *If* $m_{i+1} > m_i z 2^{i+4}$ *for all* $i \geq 1$, *then* $m_{i+1}(1 - (8z)^{-1}) < s_{i+1} \leq m_{i+1}$.

Proof. Easy induction arguments give $s_i \leq m_i$. For every $i \geq 1$, $s_{i+1} = s_i m_{i+1} m_i^{-1} (1 - m_i m_{i+1}^{-1}) \geq s_i m_{i+1} m_i{}^{-1}(1 - e_i)$, where $e_i = 2^{-(i+4)} z^{-1}$. One easily checks that $s_{i+1} \geq m_{i+1} \prod_{k=1}^{i}(1 - e_k)$. Hence $s_{i+1} \geq m_{i+1}(1 - \sum_{k=1}^{i} e_k) > m_{i+1}(1 - \sum_{k=1}^{\infty} e_k) \geq m_{i+1}(1 - (8z)^{-1})$. The result follows.

Each polynomial $w \in H_n$ can be uniquely represented in the form $\sum \alpha_p p$, where $\alpha_p \in K$ and $p \in M_n$. Given a monomial $p = x_1 \dots x_n \in M_n$ and a permutation $\sigma \in S_n$, define $p^\sigma = x_{\sigma(1)} \dots x_{\sigma(n)}$ and $w^\sigma = \sum \alpha_p p^\sigma$. ■

Theorem 3 (cf. [11, theorem 6]). *Suppose that m_i, s_i, R_i are as in Lemma 2 and Theorem 1. For every integer $i \geq 0$ there are permutations $\sigma_i \in S_{m_i}$, $\pi_i \in S_{m_i}$ and a K-linear map $h_i : H_{m_i - s_i} \to H_{m_i - s_i}$ such that if $u \in H_{s_i}$ and $v \in H_{m_i - s_i}$ then*

$$(R_i((uv)^{\sigma_i}))^{\pi_i} = uh_i(v).$$

PROOF. See [11], proof of theorem 6. ■

3. Free algebras

Here we will generalise results from section 3 of [11].

Given $w \in M$ and an element $x \in \{a_1, \dots, a_{z+2}\}$, we denote by $d_x w$ the x-degree of w. Obviously the degree of w, which we denote by dw, is equal to $\sum_{i=1}^{z+2} d_{a_i} w$.

Note that for each integer $n \geq 0$ the set $D = \{w \in M : dw = n\}$ is a free basis of the right A-module DA.

Given integers $n_1, \dots, n_{z+2}$, define $w(n_1, \dots, n_{z+2}) = 0$ if some of n_i is < 0, and $w(n_1, \dots, n_{z+2}) = \sum \{w \in M : d_{a_i} w = n_i$ for $i = 1, \dots, z+2\}$ if all $n_i \geq 0$.

We will need the following straightforward lemma.

Lemma 4. (a) *For each $n \geq 0$ the set $S = \{w(n_1, \dots, n_{z+2}) : n_1, \dots, n_{z+2} \geq 0, n_1 + \dots + n_{z+2} = n\}$ is a free basis of the right A-module SA.*

(b) *For arbitrary integers $n_1, \dots, n_{z+2}$, and $r \leq n_1 + \dots + n_{z+2}$, we have*

$$w(n_1, \dots, n_{z+2}) = \sum \{w(r_1, \dots, r_{z+2}) w(n_1 - r_1, \dots, n_{z+2} - r_{z+2}) : r_1 + \dots + r_{z+2} = r\}.$$

In what follows, $\prec$ will denote the lexicographical ordering of triples $(n_1, \dots, n_{z+2})$ of integers (i.e. $(n_1, \dots, n_{z+2}) \prec (n'_1, \dots, n'_{z+2})$ if and only if $n_1 < n'_1$, or $n_1 = n'_1$ and $n_2 < n'_2$, or $n_1 = n'_1$ and $n_2 = n'_2$ and $n_3 < n'_3$ etc.).

Lemma 5. *If $(n_1, \dots, n_{z+2}) \succ (n'_1, \dots, n'_{z+2})$, then for arbitrary integers $m_1, \dots, m_{z+2}$, $(m_1 - n_1, \dots, m_{z+2} - n_{z+2}) \prec (m_1 - n'_1, \dots, m_{z+2} - n'_{z+2})$.*

Lemma 6 (cf. [11, lemma 9]). *Let $f : H_p \to A$, $g : H_q \to A$ and $h : H_{p+q} \to A$ be K-linear mappings such that for all $u \in M_p$, $v \in M_q$, $h(uv) = f(u)g(v)$. If $h(w(n_1, \dots, n_{z+2})) = 0$ for all $n_1 + \dots + n_{z+2} = p + q$, then either $f(w(p_1, \dots, p_{z+2})) = 0$ for all $p_1 + \dots + p_{z+2} = p$ or $g(w(q_1, \dots, q_{z+2})) = 0$ for all $q_1 + \dots + q_{z+2} = q$.*

PROOF. Analogous to proof of lemma 9 from [11]. ■

Theorem 7. *Let p, r be integers such that $p + r > 80z(z+2)$, $r > 4zp$, and let $f : H_p \to A$ and $g : H_{r+p} \to A$ be K-linear mappings such that for $u \in M_r$, $v \in$*

M_p, we have $g(uv) = uf(v)$. If the dimension of the K-subspace of A spanned by $\{g(w(n_1, \ldots, n_{z+2})) : n_1 + \ldots + n_{z+2} = p + r\}$ is $< (p+r)^{z+1}(40z(z+2))^{-z-1}$, then $f(w(p_1, \ldots, p_{z+2})) = 0$, for all $p_1, \ldots, p_{z+2}$ such that $p_1 + \ldots + p_{z+2} = p$. Consequently for all $n_1 + \ldots + n_{z+2} = p + r$ we have $g(w(n_1, \ldots, n_{z+2})) = 0$.

Proof. The result holds if some $p_i < 0$. Hence it suffices to show that each $f(w(p_1, \ldots, p_{z+2}))$, where $p_1 + \ldots + p_{z+2} = p$, is a linear combination of $f(w(q_1, \ldots, q_{z+2}))$ such that $q_1 + \ldots + q_{z+2} = p$ and $(q_1, \ldots, q_{z+2}) \prec (p_1, \ldots, p_{z+2})$. Let $S = \{(n_1, \ldots, n_{z+2}) : n_1 + \ldots + n_{z+2} = p + r,\ (p+r)(z+2)^{-1} < n_i < (p+r)(z+2)^{-1}(1 + (20z)^{-1})$ for $i = 1, \ldots, z+1\}$. First we shall prove that $cardS \geq (p+r)^{z+1}(40z(z+2))^{-z-1}$. Observe that there are at least $(p+r)(20z(z+2))^{-1} - 2$ natural numbers lying between $(p+r)(z+2)^{-1}$ and $(p+r)(z+2)^{-1}(1 + (20z)^{-1})$, so we can choose $((p+r)(20z(z+2))^{-1} - 2)^{z+1}$ distinct sequences $(n_1, \ldots, n_{z+1})$ of natural numbers such that $(p+r)(z+2)^{-1} < n_i < (p+r)(z+2)^{-1}(1 + (20z)^{-1})$ for $i = 1, \ldots, z+1$. For each such sequence we can choose a natural number n_{z+2} such that $n_1 + \ldots + n_{z+2} = p + r$ and $n_{z+2} \geq (p+r)(z+2)^{-1}(1 - 10^{-1})$. Since $p + r > 80z(z+2)$, we get that $cardS \geq ((p+r)(20z(z+2))^{-1} - 2)^{z+1} \geq (p+r)^{z+1}((40z)(z+2))^{-z-1}$.

Hence the assumption of the theorem implies that $\sum_{(n_1, \ldots, n_{z+2}) \in S} l_{n_1, \ldots, n_{z+2}} g(w(n_1, \ldots, n_{z+2})) = 0$ for some $l_{n_1, \ldots, n_{z+2}} \in K$ not all of which are zeros. Let $(j_1, \ldots, j_{z+2})$ be the maximal element in S, with respect to $\prec$, such that $l_{j_1, \ldots, j_{z+2}} \neq 0$. Then $g(w(j_1, \ldots, j_{z+2})) = \sum k_{n_1, \ldots, n_{z+2}} g(w(n_1, \ldots, n_{z+2}))$ for some $k_{n_1, \ldots, n_{z+2}} \in K$, where the sum runs over all $(n_1, \ldots, n_{z+2}) \in S$ such that $(n_1, \ldots, n_{z+2}) \prec (j_1, \ldots, j_{z+2})$.

From Lemma 4(b) we get that if $n_1 + \ldots + n_{z+2} = p + r$ then $g(w(n_1, \ldots, n_{z+2})) = \sum_{r_1 + \ldots + r_{z+2} = r} w(r_1, \ldots, r_{z+2}) f(w(n_1 - r_1, \ldots, n_{z+2} - r_{z+2}))$, so $g(w(j_1, \ldots, j_{z+2})) = \sum_{r_1 + \ldots + r_{z+2} = r} w(r_1, \ldots, r_{z+2}) f(w(j_1 - r_1, \ldots, j_{z+2} - r_{z+2}))$ and $g(w(j_1, \ldots, j_{z+2})) = \sum k_{n_1, \ldots, n_{z+2}} \sum_{r_1 + \ldots + r_{z+2} = r} w(r_1, \ldots, r_{z+2}) f(w(n_1 - r_1, \ldots, n_{z+2} - r_{z+2}))$ where the sum runs over $(n_1, \ldots, n_{z+2}) \in S$ such that $(n_1, \ldots, n_{z+2}) \prec (j_1, \ldots, j_{z+2})$. These and Lemma 4(a) imply that, for each $(r_1, \ldots, r_{z+2})$ satisfying $r_1 + \ldots + r_{z+2} = r$, we have $f(w(j_1 - r_1, \ldots, j_{z+1} - r_{z+2})) = \sum k_{n_1, \ldots, n_{z+2}} f(w(n_1 - r_1, \ldots, n_{z+2} - r_{z+2}))$ where the sum runs over $(n_1, \ldots, n_{z+2}) \in S$, $(n_1, \ldots, n_{z+2}) \prec (j_1, \ldots, j_{z+2})$.

The definition of S and the assumption $r > 4zp$ imply that $j_i > p$ for $i = 1, \ldots, z+2$. Hence for arbitrary $p_1, \ldots, p_{z+2}$, such that $p_1 + \ldots + p_{z+2} = p$, the integers $r_1 = j_1 - p_1, \ldots, r_{z+2} = j_{z+2} - p_{z+2}$ are positive and $r_1 + \ldots + r_{z+2} = r$. Thus

$$\begin{aligned} f(w(p_1, \ldots, p_{z+2})) &= f(w(j_1 - r_1, \ldots, j_{z+2} - r_{z+2})) \\ &= \sum_{(n_1, \ldots, n_{z+2}) \prec (j_1, \ldots, j_{z+2})} k_{n_1, \ldots, n_{z+2}} f(w(n_1 - r_1, \ldots, n_{z+2} - r_{z+2})). \end{aligned}$$

Clearly $(n_1 - r_1, \ldots, n_{z+2} - r_{z+2}) \prec (j_1 - r_1, \ldots, j_{z+2} - r_{z+2}) = (p_1, \ldots, p_{z+2})$, so the result holds. ■

Now, from Lemma 4(b) we get immediately that if $n_1 + \ldots + n_{z+2} = p + r$ then $g(w(n_1, \ldots, n_{z+2})) = 0$.

Theorem 8. *Let $m_1, m_2, \ldots$ be natural numbers such that, for each i, m_i divides m_{i+1}, $m_{i+1} > m_i z 2^{i+4}$ and $m_1 > 80z(z+2)$.*

Let, for $i > 0$, $F_i = \{f_{i,1}, \ldots, f_{i,r_i}\} \subseteq H_{m_i}$, with $r_i < {m_i}^{z+1}(40z(z+2))^{-z-1}$, and let R_i be defined as in Section 1. For every $i > 0$ there are $n_1, \ldots, n_{z+2}$ such that $R_i(w(n_1, \ldots, n_{z+2})) \neq 0$.

PROOF. Suppose the contrary. Let i be the minimal number such that $R_i(w(n_1, \ldots, n_{z+2})) = 0$ for all $n_1 + \ldots + n_{z+2} = m_i$. Clearly $i > 1$. By the definition of R_i we have that if $w = x_1 \ldots x_{m_i} \in M_{m_i}$ then $R_i(w) = c_{R_{i-1}(F_{i-1})}(R_{i-1}(x_1 \ldots x_{m_{i-1}})) \prod_{j=2}^{m_i m_{i-1}^{-1}} R_{i-1}(x_{(j-1)m_{i-1}+1} \ldots x_{jm_{i-1}})$. Applying Lemma 6 several times, we get that $R_{i-1}(w(n_1, \ldots, n_{z+2})) = 0$ for all $n_1 + \ldots + n_{z+2} = m_{i-1}$ or $c_{R_{i-1}(F_{i-1})}(R_{i-1}(w(n_1, \ldots, n_{z+2}))) = 0$ for all $n_1 + \ldots + n_{z+2} = m_{i-1}$. Since i is minimal, the former is impossible. Thus, suppose the latter holds. Then $R_{i-1}(w(n_1, \ldots, n_{z+2})) \in span_K\{R_{i-1}(f_{i-1,1}), \ldots, R_{i-1}(f_{i-1,r_{i-1}})\}$ $(= ker c_{R_{i-1}(F_{i-1})})$.

Consequently we get that the dimension of the K-subspace of A spanned by $\{R_{i-1}(w(n_1, \ldots, n_{z+2})) : n_1 + \ldots + n_{z+2} = m_{i-1}\}$ is $\leq r_{i-1} < {m_{i-1}}^{z+1}(40z(z+2))^{-z-1}$. For $w \in M_{m_{i-1}}$ let $\overline{R}_{i-1}(w) = (R_{i-1}(w^{\sigma_{i-1}}))^{\pi_{i-1}}$ where π_{i-1}, σ_{i-1} are those of Theorem 3. Observe that $(w(n_1, \ldots, n_{z+2}))^{\sigma} = w(n_1, \ldots, n_{z+2})$. Consequently $\overline{R}_{i-1}(w(n_1, \ldots, n_{z+2})) = (R_{i-1}(w(n_1, \ldots, n_{z+2})))^{\pi_{i-1}}$. Consequently the dimension of the K-subspace of A spanned by $\{\overline{R}_{i-1}(w(n_1, \ldots, n_{z+2})) : n_1 + \ldots + n_{z+2} = m_{i-1}\}$ is $< {m_{i-1}}^{z+1}(40z(z+2))^{-z-1}$. By Lemma 2 and Theorem 3, $r = s_{i-1}, p = m_{i-1} - s_{i-1}$, $f = h_{i-1}$ and $g = \overline{R}_{i-1}$ satisfy the assumptions of Theorem 7. Hence $\overline{R}_{i-1}(w(n_1, \ldots, n_{z+2})) = 0$ and $R_{i-1}(w(n_1, \ldots, n_{z+2})) = 0$ for all $n_1 + \ldots + n_{z+2} = m_{i-1}$. We are finished. ∎

4. Algebras T and T_1

In what follows, K is a field and T is the algebra of polynomials in non-commuting indeterminates $b_1, \ldots, b_{z+2}$ over K.

We denote by $\overline{T}$ the subalgebra of polynomials with zero constant term in T, and by $\overline{M}$ the set of monomials in $b_1, \ldots, b_{z+2}$. Let Q be the additive monoid generated by $1/2$. Define the semigroup homomorphism $\xi : \overline{M} \to Q$ as follows: $\xi(b_i) = 1$ for $i = 1, \ldots, z+1$, $\xi(1) = 0$ and $\xi(b_{z+2}) = 1/2$.

Set $\overline{M}_0 = \{1\}$, and for each $q \in Q$ denote by $\overline{M}_q$ the set of monomials w such that $\xi(w) = q$. The K-subspace of T spanned by $\overline{M}_q$ will be denoted by $\overline{H}_q$. Thus $\overline{H}_q = span_K \overline{M}_q$.

Given $F \subseteq T$, we denote by $M^l(F)$ the set of all matrices $l \times l$ over F.

Let $T[y_1, \ldots, y_{z-1}]$ be the polynomial ring in $z-1$ commuting indeterminates $y_1, \ldots, y_{z-1}$ over T, and let $\overline{K}$ or $K[y_1, \ldots, y_{z-1}]$ denote the polynomial ring over K in $z-1$ commuting indeterminates $y_1, \ldots, y_{z-1}$.

Define $K_p = span_K\{w_1 \ldots w_j : j < p \text{ and } w_i \in \{y_1, \ldots, y_{z-1}\} \text{ for all } i \leq j\}$.

Lemma 9. *Let $f = \sum_{i=1}^{t} f_i$ where $f_i \in M^l(K_p \overline{H}_{i/2})$ for each i. Suppose that either n or $n + 1/2$ is a natural number $> t$. Then there exists a set $G(f, n) \subseteq \overline{H}_n$ with $card G(f, n) < l^2(2np)^z t(z+2)^t$ such that, for every natural number $m > 2n$,*

$$f^m \in M^l\left(\left(G(f,n) \sum_{i=1}^{t} \overline{M}_{i/2} \overline{K} f^{m-2n} + \overline{M}_{n-1/2}\{b_1, \ldots, b_{z+1}\}\right) T\overline{K}\right).$$

PROOF. Given $1 \leq j \leq t$ and $1 \leq k \leq 2n-j$, we set $w_{j,k} = \sum_{(i_1,\ldots,i_k):i_1+\ldots+i_k=2n-j} f_{i_1}\ldots f_{i_k}$. Note that $w_{j,k} \in \overline{H}_{n-j/2}$. Let $G_1(f,n) = \bigcup_{1\leq j\leq t}\bigcup_{1\leq k\leq 2n-j} w_{j,k}\overline{M}_{j/2}$. Clearly $G_1(f,n) \subseteq M^l(\overline{H}_n K_{2np})$. Observe that, for all natural j, $card\overline{M}_{j/2} \leq (z+2)^j$. Note that for each j the number of polynomials $w_{j,k}$ is $< 2n$. Thus we get $cardG_1(f,n) < 2ntsup_{1\leq j\leq t}(card(\overline{M}_{j/2})) \leq 2nt(z+2)^t$. Note that, since $y_1,\ldots,y_{z-1}$ are commuting indeterminates, we have $dim_K K_{2np} \leq (2np)^{z-1}$. Thus there is a set $G(f,n) \subseteq \overline{H}_n$ such that $G_1(f,n) \subseteq M^l(G(f,n)K_{2np})$ and $cardG(f,n) \leq l^2(2np)^{z-1}cardG_1(f,n) \leq l^2(2np)^z t(z+2)^t$.

Observe now that $f^m = \sum_{1\leq i_1,\ldots,i_m\leq t} f_{i_1}\ldots f_{i_m}$. For each term $f_{i_1}\ldots f_{i_m}$ there exists $1 \leq k < 2n$ such that $i_1+\ldots+i_k < 2n < m$ and $i_1+\ldots+i_{k+1} \geq 2n$ (because $i_1 \leq t < 2n$ and $i_1+\ldots+i_{2n} \geq 2n$). Thus (cf. [11], proof of lemma 1) $f^m = \sum_{k=1}\sum_{i_1+\ldots+i_k<2n\leq i_1+\ldots+i_{k+1}} f_{i_1}\ldots f_{i_{k+1}}\sum_{1\leq i_{k+2},\ldots,i_m\leq t} f_{i_{k+2}}\ldots f_{i_m}$. However, $\sum_{1\leq i_{k+2},\ldots,i_m\leq t} f_{i_{k+2}}\ldots f_{i_m} = \sum_{1\leq j_1,\ldots,j_{m-k-1}\leq t} f_{j_1}\ldots f_{j_{m-k-1}} = f^{m-k-1}$. Consequently we get that $f^m = \sum_{k=1}^{2n-1}\sum_{i_1+\ldots+i_k<2n\leq i_1+\ldots+i_{k+1}} f_{i_1}\ldots f_{i_{k+1}} f^{m-k-1}$. Thus $f^m = \sum_{k=1}^{2n-1}\sum_{j=1}^{min(2n-k,t)}$ $\sum_{i_1+\ldots+i_k=2n-j} f_{i_1}\ldots f_{i_k}(f_j+\ldots+f_t)f^{m-k-1}$. So $f^m = \sum_{j=1}^{t}\sum_{k=1}^{2n-j}\sum_{i_1+\ldots+i_k=2n-j} f_{i_1}\ldots f_{i_k}$ $(f_j+\ldots+f_t)f^{m-k-1} = \sum_{j=1}^{t}\sum_{k=1}^{2n-j} w_{j,k}(f_j+\ldots+f_t)f^{m-k-1}$.

Observe first that, since $1 \leq k \leq 2n-j$, $f^{m-k-1} = f^{m-2n}f^l$ for some $l \geq 0$. Clearly each $w_{j,k}(f_j+\ldots+f_t)$ is a linear combination of elements from $G_1(f,n)M^l(\sum_{i=1}^{t}\overline{M}_{i/2}\overline{K})$ and elements from $M^l(\overline{M}_{n-1/2}\{b_1,\ldots,b_{z+1}\}T\overline{K})$. Since $G_1(f,n) \subseteq M^l(\overline{K}G(f,n))$, we get the result. ■

Let T_1 be the subalgebra of T generated by $b_1,\ldots,b_{z+1}$ and b_{z+2}^2.

Let $S_1 = span_K\{s \in \overline{M} : s \notin T_1\}$ and $S_2 = span_K\{s \in \overline{M} : sT \cap T_1 = 0\}$.

Given a natural number n and a set $F \subseteq T$, let $D_n(F)$ denote the right ideal of R generated by the set $\bigcup_{k=0,1,2,\ldots}\overline{M}_{nk}F$. Similarly, given a natural number n and a set $\overline{F} \subseteq T_1$, let $C_n(\overline{F})$ denote the right ideal of T_1 generated by the set $\bigcup_{k=0,1,2,\ldots}(\overline{M}_{nk} \cap T_1)\overline{F}$. Thus $D_n(F) = \sum_{k=0,1,2,\ldots}\overline{M}_{nk}FT$ and $C_n(\overline{F}) = \sum_{k=0,1,2,\ldots}(\overline{M}_{nk} \cap T_1)\overline{F}T_1$.

Given a matrix S with entries in $\overline{K}T$, we denote by $[S]$ the set of coefficients of entries of S. Thus $[S] \subseteq T$.

Theorem 10. *Let f be as in Lemma 9. Let r, w be natural numbers with $w > r > 2t$. There exists a set $F(f,r) \subseteq \overline{H}_r$ with $cardF(f,r) < l^2(2rp)^z(z+2)^{4t}$ such that the two-sided ideal of T generated by $[f^{10w}]$ is contained in $D_w(F(f,r)) + S_2$.*

PROOF. Let $F(f,r) = \bigcup_{0\leq i\leq t}\overline{M}_{i/2}G(f,r-i/2)$ where $G(f,n)$ is defined as in Lemma 9. Observe that, for all i, $cardM_{i/2} \leq (z+2)^i$. Hence $cardF(f,r) < (t+1)sup_{1\leq i\leq t}$ $(cardM_{i/2}G(f,r-i/2)) \leq l^2(t+1)t(z+2)^{2t}(2rp)^z \leq l^2(2rp)^z(z+2)^{4t}$.

Since S_2 is the right ideal of T, it suffices to show that $u[f^{10w}] \subseteq D_w(F(f,r)) + S_2$ for all monomials u. Observe that either $u \in S_2$ or $u = u''u'$ for some $u'', u' \in \overline{M}$ such that $\xi(u'')$ is a natural number divsible by w and $\xi(u') \leq w$. Observe that if, for some $v \in \overline{M}$, $\xi(v)$ is neither a natural number nor 0, then $vb_i \in S_2$ for $i = 1,\ldots,z+1$. Hence $u\overline{M}_{2w-\xi(u')-1/2}\{b_1,\ldots,b_{z+1}\} \subseteq S_2$. Applying Lemma 9 to f, $n = 2w-\xi(u')$ and

$m = 10w$, we get

$$u[f^{10w}] \subseteq uG(f, 2w - \xi(u')) \left(\sum_{i=1}^{t} \overline{M}_{i/2}\overline{K} \right) [f^{6w+2\xi(u')}] + S_2.$$

For each $h \in G(f, 2w - \xi(u'))$, $\xi(u'h) = 2w$, so w divides $\xi(uh)$. Thus it suffices to show that $[gf^{6w+2\xi(u')}] \subseteq F(f, r) + S_2$ for each monomial g such that $\xi(g) \leq t/2$. Observe first that, since r is a natural number, we have $g\overline{M}_{r-\xi(g)-1/2}\{b_1, \ldots, b_{z+1}\} \subseteq S_2$. Hence, applying Lemma 9 to f and $n = r - \xi(g)$, $m = 6w + 2\xi(u')$, we get

$$[gf^{6w+2\xi(u')}] \subseteq gG(f, r - \xi(g)) \left(\sum_{i=1}^{t} \overline{M}_{i/2}\overline{K} \right) [f^{6w+2\xi(u)-2r+2\xi(g)}] + S_2.$$

Since $gG(f, r - \xi(g)) \subseteq F(f, r)$, we obtain our claim. ■

Corollary 11. *Let f, r, w be as in Theorem 10. There exists a set $\overline{F}(f, r) \subseteq \overline{H}_r \cap T_1$ with $card\overline{F}(f, r) < l^2(2rp)^z(z + 2)^{4t}$ such that the two-sided ideal of T generated by $[f^{10w}]$ is contained in $C_w(\overline{F}(f, r)) + S_1$.*

PROOF. Let $F(f, r)$ be as in Theorem 10. It suffices to show that there is a set $\overline{F}(f, r) \subseteq \overline{H}_r \cap T_1$ with $card\overline{F}(f, r) \leq cardF(f, r)$ such that $D_w(F(f, r)) \subseteq C_w(\overline{F}(f, r)) + S_1$. Each $h \in F(f, r)$ can be uniquely represented in the form $h = h' + h''$, where $h' \in \overline{H}_r \cap T_1$ and $h'' \in S_1$. The set of all such h' will be denoted by $\overline{F}(f, r)$. Thus, $card\overline{F}(f, r) \leq cardF(f, r)$. Each element of $D_w(F(f, r))$ is a linear combination of elements ghg' where $g' \in M$, $h \in F(f, r)$ and $g \in \overline{M}_{wk}$ for some $k \in \{0, 1, \ldots\}$. Since k, w, r are natural numbers, it follows that if $g \in S_1$ or $g' \in S_1$ then $gF(f, r)g' \subseteq S_1$. Suppose now that $g, g' \in T_1$. Observe that if $h'' \in S_1$ then $gh''g' \in S_1$ and $gh'g' \in C_w(\overline{F}(f, r))$. This proves the corollary. ■

From Theorem 10 and Corollary 11 we get the following.

Corollary 12. *Let $f_i \in M^{l_i}(K_{p_i}(\sum_{i=1}^{t_i} H_{i/2}))$ for $i = 1, 2 \ldots$ and for some natural numbers l_i, p_i, t_i, and let m_i, $i = 1, 2 \ldots$, be an increasing sequence of natural numbers such that $m_i > 2t_i$. There exist subsets $F_i \subseteq \overline{H}_{m_i} \cap T_1$ with $cardF_i < l_i^2(2m_ip_i)^z(z + 2)^{4t_i}$ such that the ideal of T generated by $[f_i^{10m_{i+1}}]$, $i = 1, 2 \ldots$, is contained in $\sum_{i=1}^{\infty} C_{m_{i+1}}(F_i) + S_1$.*

5. The main result

Let $\alpha : A \to T_1$ be the homomorphism defined by $\alpha(a_i) = b_i$ for $i \leq z + 1$ and $\alpha(a_{z+2}) = b_{z+2}^2$. Observe that $\alpha : A \to T_1$ is an isomorphism.

Corollary 13. *For all n, $\alpha(M_n) = \overline{M}_n \cap T_1$ and $\alpha^{-1}(C_n(F)) = B_n(\alpha^{-1}(F))$.*

Let $w(n_1,\dots,n_{z+2})$ be defined as in Section 2. Given integers $n_1,\dots,n_{z+2}$, define $p(n_1,\dots,n_{z+2})=0$ if some of n_i is <0, and $p(n_1,\dots,n_{z+2})=\sum\{v\in\overline{M}: d_{b_i}v=n_i$ for $i=1,\dots,z+2\}$ if all $n_i\geq 0$, where d_xv is the x-degree of v for $x\in\{b_1,\dots,b_{n+2}\}$.

Lemma 14. *For all $n_1,\dots,n_{z+2}$, we have $p(n_1,\dots,n_{z+2})\in S_1$ if n_{z+2} is odd, and $p(n_1,\dots,n_{z+2})-\alpha(w(n_1,\dots,n_{z+1},n_{z+2}/2))\in S_1$ if n_{z+2} is even.*

Proof. By induction on $n_1+\dots+n_{z+2}$ and Lemma 4(b). ■

Now we will prove the main result.

Theorem 15. *For every countable field K and for every natural number z there is an algebra $\tilde{T}$ over K (generated by $z+2$ elements) such that the polynomial algebra $\tilde{T}[y_1,\dots,y_z]$ over $\tilde{T}$ in z commuting indeterminates is Jacobson radical but not nil.*

Proof. Recall first that $\overline{K}=K[y_1,\dots,y_{z-1}]$. Let $\overline{T}$ be the subalgebra of T of polynomials with zero constant term. Since K is countable, elements in $\bigcup_{l=1}^{\infty}M^l(\overline{K}\,\overline{T})$ can be enumerated, i.e. $\bigcup_{l=1}^{\infty}M^l(\overline{K}\,\overline{T})=\{f_1,f_2,\dots\}$. For each f_i fix natural numbers l_i,t_i,p_i such that $f_i\in M^{l_i}(K_{p_i}(\sum_{i=1}^{t_i}\overline{H}_{i/2}))$.

It is clear that there are natural numbers $m_1,m_2,\dots$ satisfying

(i) $m_1>80z(z+2)$ and, for each i, $m_i\geq l_i^2(2p_i)^z(z+2)^{4t_i}(40z(z+2))^{z+1}$;

(ii) for each $i\geq 1$, $m_{i+1}>m_iz2^{i+4}$ and m_i divides m_{i+1}.

Let I be the ideal of $\overline{T}$ generated by $\{[f_i^{10m_{i+1}}]: i=1,2,\dots\}$. Hence, for each l, $M^l((\overline{T}/I)[y_1,\dots,y_{z-1}])$ is nil. Consequently from Krempa's result quoted at the beginning of the paper we see that $(\overline{T}/I)[y_1,\dots,y_z]$ is Jacobson radical.

Suppose that $(\overline{T}/I)[y_1,\dots,y_z]$ is nil. Then there is a number k such that, for all $l\geq k$, $(b_1y_1+\dots+b_zy_z+b_{z+1}+b_{z+2})^l\in I[y_1,\dots,y_z]$.

Consequently, for all $0\leq n_1,\dots,n_z\leq l$, $\sum_{0\leq i\leq l}p(n_1,\dots,n_z,l-(n_1+\dots+n_z+i),i)\in I$.

From Corollary 12, $I\subseteq\sum_{i=1}^{\infty}C_{m_{i+1}}(F_i)+S_1$ for some $F_i\subseteq\overline{H}_{m_i}\cap T_1$ with $cardF_i<l_i^2(2m_ip_i)^z(z+2)^{4t_i}$. Consequently from (i) $cardF_i<m_i{}^{z+1}(40z(z+2))^{-z-1}$. From this and from Lemma 14 we get that for all $n_1,\dots,n_z$

$$\sum_{0\leq j\leq l/2}\alpha(w(n_1,\dots,n_z,l-(n_1+\dots+n_z+2j),j))-q_{n_1,\dots,n_z}\in S_1$$

for some $q_{n_1,\dots,n_z}\in\sum_{i=1}^{\infty}C_{m_{i+1}}(F_i)$. Since the left side of this equation belongs to T_1 and $T_1\cap S_1=0$, it is equal to 0. Now, from Corollary 13,

$$\sum_{0\leq j\leq l/2}w(n_1,\dots,n_z,l-(n_1+\dots+n_z+2j),j)\in\sum_{i=1}^{\infty}B_{m_{i+1}}(E_i)$$

where $E_i=\alpha^{-1}(F_i)$.

Observe now that $w(n_1,\dots,n_z,l-(n_1+\dots+n_z+2j),j)\in H_{l-j}$. Since $\sum_{i=1}^{\infty}B_{m_{i+1}}(E_i)$ is homogeneous, we get that $w(n_1,\dots,n_{z+2})\in\sum_{i=1}^{\infty}B_{m_{i+1}}(E_i)$ for all $n_1+\dots+n_{z+1}+2n_{z+2}=l$. This holds for all $l\geq k$. Moreover, if $n_1+\dots+n_{z+1}+n_{z+2}=m_k$ then $n_1+\dots+n_{z+1}+2n_{z+2}\geq m_k\geq k$. Consequently $w(n_1,\dots,n_{z+2})\in\sum_{i=1}^{\infty}B_{m_{i+1}}(E_i)$ for all

$n_1 + \ldots + n_{z+2} = m_k$. Since $\sum_{i=1}^{\infty} B_{m_{i+1}}(E_i)$ is homogeneous and $E_i \subseteq H_{m_i}$, we get that $w(n_1, \ldots, n_{z+2}) \in \sum_{i=1}^{k} B_{m_{i+1}}(E_i)$ for all $n_1 + \ldots + n_{z+2} = m_k$. Now, by Lemma 4(b), for all $n_1 + \ldots + n_{z+2} = m_{k+1}$, $w(n_1, \ldots, n_{z+2}) \in \sum_{i=1}^{k} B_{m_{i+1}}(E_i)$.

By Theorem 1 we have $R_{k+1}(w(n_1, \ldots, n_{z+2})) = 0$ for all $n_1 + \ldots + n_{z+2} = m_{k+1}$. By Theorem 8 this is impossible. Thus $\tilde{T} = \overline{T}/I$ satisfies the assumptions of our theorem. ■

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