# PROPERTIES OF THE LOCAL FUNCTIONAL CALCULUS

By

Teresa Bermúdez*
Departamento de Análisis Matemático, Universidad de La Laguna, Tenerife, Spain

Manuel González
Departamento de Matemáticas, Universidad de Cantabria, Santander, Spain

and

Antonio Martinón
Departamento de Análisis Matemático, Universidad de La Laguna, Tenerife, Spain

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## Abstract

We develop the local functional calculus for continuous operators with the single-valued extension property (SVEP), obtaining a spectral mapping theorem for the local spectrum and a stability theorem for the SVEP. As an application of this calculus, we obtain local resolvent equations.

## 1. Introduction

The *holomorphic functional calculus* was developed by Dunford and Taylor in the 1940s (see [7]; [16]). Given a continuous operator $T$ acting on a complex Banach space $X$, this calculus associates an operator $f(T) \in L(X)$ to each holomorphic function $f$ defined on a neighbourhood of the spectrum of $T$. Gindler [9] extended it by associating a closed operator to each meromorphic function of a specific class. This *meromorphic functional calculus* was also studied in [10].

Many efforts have been devoted to extending the holomorphic functional calculus in other directions. For example, assume that $T \in L(X)$ has the single-valued extension property (SVEP in short) and let $f$ be a holomorphic function on a neighbourhood of a fixed compact subset $K$ of the complex plane. For $x \in X$ such that the local spectrum of $T$ at $x$, denoted by $\sigma(x, T)$, is contained in $K$, Apostol [1] defines the vector $f[T]x$ as

$$f[T]x := \frac{1}{2\pi i} \int_\Gamma f(\lambda)\hat{x}_T(\lambda)\, d\lambda,$$

where $\Gamma$ is an admissible contour and $\hat{x}_T$ is the local resolvent function. These vectors $f[T]x$ were analysed in [14] and [15].

In [3] a *local functional calculus* was defined as follows. Given an operator

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*Corresponding author, e-mail: tbermude@ull.es

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$T \in L(X)$ with the SVEP and an arbitrary holomorphic function $f : \triangle(f) \subset \mathbb{C} \to \mathbb{C}$, we set

$$D(f[T]) := \{x \in X : \ \sigma(x, T) \subset \triangle(f)\} \text{ and } f[T]x := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\hat{x}_T(\lambda)\, d\lambda.$$

In this way we obtain a linear operator $f[T] : D(f[T]) \subset X \to X$.

The purpose of this paper is to develop this local functional calculus and to give some applications to the local spectral theory.

In Section 2 we include the definitions and notation that we use and some basic properties of the local functional calculus. In Section 3 we give sufficient conditions for continuity and closability, since the operator $f[T]$ is neither continuous nor closed in general. We study the relation between $\sigma(x, T)$ and $\sigma(f[T]x, T)$ and prove a local spectral mapping theorem for a certain class of functions and operators, which asserts that $\sigma(x, f[T]) = f(\sigma(x, T))$, for all $x$ in the domain of $f[T]$. As an application of this result, we obtain a stability theorem of the SVEP by the local functional calculus. In fact, we improve one implication of [6, theorem 1.5] using a different argument. Moreover, we show that in some cases the operator $f\{T\}$ of the meromorphic functional calculus is the minimal closed extension of $f[T]$. Finally, we apply a representation of the local resolvent function using the local functional calculus to obtain local resolvent equations.

## 2. Preliminaries

Given a continuous linear operator $T \in L(X)$, a complex number $\lambda$ belongs to the *resolvent set* $\rho(T)$ of $T$ if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote by $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the *spectrum* of $T$. The *resolvent map* $R(., T) : \rho(T) \to L(X)$ is holomorphic.

Moreover, given an arbitrary linear operator $A : D(A) \subset X \to X$ and $x \in X$, we say that a complex number $\lambda$ belongs to the *local resolvent set* of $A$ at $x$, denoted by $\rho(x, A)$, if there exists a holomorphic function $w : U \subset \mathbb{C} \to D(A)$, defined on a neighbourhood $U$ of $\lambda$, that satisfies

$$(\mu - A)w(\mu) = x, \tag{1}$$

for every $\mu \in U$. The *local spectrum* of $A$ at $x$ is $\sigma(x, A) := \mathbb{C}\backslash\rho(x, A)$. Since $w$ is not necessarily unique, a complementary property is needed to prevent ambiguity.

A linear operator $A$ satisfies the SVEP if, for every holomorphic function $h : U \to D(A)$ defined on an open subset $U$ of $\mathbb{C}$, the condition $(\lambda - A)h(\lambda) \equiv 0$ implies that $h \equiv 0$. If $A$ satisfies the SVEP, then for every $x \in X$ there exists a unique holomorphic function $\hat{x}_A$ defined on $\rho(x, A)$ satisfying (1), which is called the *local resolvent function of* $A$ *at* $x$. For example, if the *point spectrum* of $T$, $\sigma_p(T)$, has empty interior, then $T$ has the SVEP. See [6] and [12] for further details.

For every subset $H \subset \mathbb{C}$, $X(A, H) := \{x \in X : \sigma(x, A) \subset H\}$ is a linear manifold of $X$. If $X(A, F)$ is closed for all closed sets $F$, we say that $A$ has *property* $(C)$. If $T \in L(X)$ satisfies property (C) then $T$ has the SVEP [15, theorem 2.3] and [11, proposition 1.2].

It is clear that if $T$ has the SVEP and $x \in N(\lambda - T)\backslash\{0\}$ for certain $\lambda \in \mathbb{C}$, then $\sigma(x, T) = \{\lambda\}$. The converse is not true. For example, take any quasi-nilpotent operator. However, Mbekhta [13, lemma 3.1.5] proved that if $T$ is a *hyponormal* ($\| T^*x\| \leq \| Tx\| \; \forall x \in X$) then

$$\sigma(x, T) = \{\lambda\} \Rightarrow x \in N(\lambda - T)\backslash\{0\}. \tag{2}$$

A set $D$ in the complex plane is called a *Cauchy domain* if it is open, it has a finite number of components, and the boundary of $D$ is composed of a finite number of simple closed rectifiable curves, no two of which intersect.

The *local functional calculus* may be defined for any holomorphic function $f$ on its domain $\triangle(f)$ and for any $T \in L(X)$ satisfying the SVEP [3]. We obtain a linear operator $f[T] : D(f[T]) \subset X \to X$, with domain $D(f[T]) := \{x \in X : \sigma(x, T) \subset \triangle(f)\}$ and $f[T]x$ given by

$$f[T]x = \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)\hat{x}_T(\lambda)\, d\lambda, \tag{3}$$

where $\Gamma$ is the boundary of any Cauchy domain $D$ such that $\sigma(x, T) \subset D \subset \overline{D} \subset \triangle(f)$. It is easy to prove that $D(f[T])$ is a linear subspace invariant under $T$. Moreover, in the case $\sigma(T) \subset \triangle(f)$, we have that $f[T]$ coincides with $f(T)$, the operator of the holomorphic functional calculus.

The following example shows that $f[T]$ is not well defined when $T$ does not satisfy the SVEP.

*Example 2.1.* Let $T$ be the operator on the Hilbert space $\ell_2(\mathbb{N})$ defined by $Te_1 = 0$ and $Te_{n+1} = e_n$ with $n > 1$, where $\{e_n : n \in \mathbb{N}\}$ is the canonical orthonormal basis for $\ell_2(\mathbb{N})$.

Given a non-zero vector $y = (y_1, y_2, \ldots) \in \ell_2(\mathbb{N})$, if we take $x_1(\lambda) = 0$ and $x_{n+1}(\lambda) = -y_n - \lambda y_{n-1} - \ldots - \lambda^{n-2}y_2 - \lambda^{n-1}y_1$, then $\eta(\lambda) := (x_1(\lambda), x_2(\lambda), \ldots)$ and $\omega(\lambda) := (1, \lambda, \lambda^2, \ldots)$ define holomorphic functions from the open unit disc $\mathbb{D}$ into $\ell_2(\mathbb{N})$, such that $(\lambda - T)(\eta(\lambda) + t\omega(\lambda)) \equiv y$ on $\mathbb{D}$, for every fixed $t \in \mathbb{C}$. Hence $T$ does not have the SVEP. Taking $f(\lambda) := 1/\lambda$, we get

$$\frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\eta(\lambda) + t\omega(\lambda))\, d\lambda = \frac{1}{2\pi i}\int_{\Gamma} \frac{\eta(\lambda)}{\lambda}\, d\lambda - te_1, \tag{4}$$

where $\Gamma$ is the boundary of any disc centred at 0 with radius $0 < r < 1$. Hence (4) depends on $t$ and $f[T]y$ is not defined.

We shall need the following results proved in [3].

**Proposition 2.1.** *Assume that $T \in L(X)$ has the SVEP and let $f$ and $g$ be holomorphic functions. Then the following assertions hold:*

- (i) *If $S \in L(X)$ commutes with $T$, then $S$ commutes with $f[T]$, i.e. $SD(f[T]) \subset D(f[T])$ and $Sf[T]x = f[T]Sx$ for all $x \in D(f[T])$.*
- (ii) *If $x \in D(f[T])$ and $y := f[T]x$, then $f[T]\hat{x}_T = \hat{y}_T$ in $\rho(x, T)$, and $\sigma(f[T]x, T) \subset \sigma(x, T)$.*
- (iii) *If $x \in D(f[T]) \bigcap D(g[T])$, then $(fg)[T]x = f[T]g[T]x = g[T]f[T]x$.*

## 3. Properties of the local functional calculus

Note that the operator provided by the local functional calculus is neither continuous nor closed in general, as the following easy example shows.

*Example 3.1.* Let $T$ be the operator on $\ell_2(\mathbb{N})$ defined by $Te_n = \frac{1}{n}e_n$. It is easy to show that if $x = (x_1, x_2, \ldots)$ then

$$\sigma(x, T) = \left\{ \frac{1}{n} \colon\ x_n \neq 0 \right\}.$$

If we take $f(\lambda) := 1/\lambda$, $D(f[T])$ is the subspace of all finitely non-zero sequences, and

$$f[T](x_1, x_2, \ldots, x_n, 0, \ldots) = (x_1, 2x_2, \ldots, nx_n, 0 \ldots).$$

Hence the operator $f[T]$ is neither continuous nor closed.

*Remark 3.1.* If $T \in L(X)$ has the SVEP and $f$ is a holomorphic function such that $D(f[T])$ is closed, then it is almost clear that $f[T]$ is continuous. In fact, $f[T] = f(T|D(f[T]))$ and $\sigma(T|D(f[T])) = \bigcup_{x \in D(f[T])} \sigma(x, T)$. The above equality of the operators follows from $\sigma(x, T) = \sigma(x, T|D(f[T]))$ and $\hat{x}_T = \hat{x}_{T|D(f[T])}$ for all $x \in D(f[T])$.

Next we show that in some cases $f[T]$ closed implies $f[T]$ continuous.

**Proposition 3.1.** *Assume that $T \in L(X)$ has property (C), and let $f$ be a holomorphic function on $\triangle(f)$. If $f[T]$ is a closed operator, then it is continuous.*

Proof. Suppose that the graph of $f[T]$ is closed. Let $(K_n)$ be a sequence of compact sets such that $K_n \subset int K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = \triangle(f)$. We have

$$D(f[T]) = \bigcup_{n=1}^{\infty} X(T, K_n),$$

hence the graph of $f[T]$ can be written as $G(f[T]) = \bigcup_{n=1}^{\infty} G(f[T]|X(T, K_n))$.

Since the graphs $G(f[T]|X(T, K_n))$ are closed, by Baire's category theorem there exists $m$ such that $G(f[T]) = G(f[T]|X(T, K_m))$. Hence, $D(f[T]) = K(T, K_m)$ is closed and $f[T] = f[T]|X(T, K_m)$, and the closed graph theorem allows us to conclude that $f[T]$ is continuous. ■

*Remark 3.2.* The converses of Remark 3.1 and Proposition 3.1 are not true. Taking $T$ as in Example 3.1, and $g(\lambda) = \sin \frac{1}{\lambda}$, we have again that $D(g[T])$ is the subspace of finitely non-zero sequences, and for $x = (x_n) \in D(f[T])$ we have $g[T]x = (x_n \sin n)$. Consequently $g[T]$ is continuous, but it is not closed.

By Proposition 2.1 we have that $\sigma(f[T]x, T) \subset \sigma(x, T)$. The next proposition gives the relation between $\sigma(x, T)$ and $\sigma(f[T]x, T)$.

**Proposition 3.2.** *Assume that $T \in L(X)$ has the SVEP, let $f$ be a holomorphic function on $\triangle(f)$ and let $x \in D(f[T])$. If $f$ is not identically zero on every component of $\triangle(f)$ that intersects $\sigma(x,T)$, then*

$$\sigma(x,T) = \sigma(f[T]x,T) \cup \{z \in \sigma(x,T) \cap \sigma_p(T) : f(z) = 0\}.$$

PROOF. We write $f(z) = g(z)p(z)$, where $g(z) \neq 0$ on $\sigma(x,T)$ and $p$ is a polynomial with all zeros of $f$ on $\sigma(x,T)$. Then $f[T]x = g[T]p(T)x = p(T)g[T]x$, and by [3, proposition 5] we have

$$\sigma(f[T]x,T) = \sigma(g[T]p(T)x,T) = \sigma(p(T)x,T).$$

Moreover, using [4, proposition 3.1 and corollary 3.2], we obtain

$$\sigma(x,T) = \sigma(p(T)x,T) \cup \{z \in \sigma(x,T) \cap \sigma_p(T) : p(z) = 0\}.$$

Therefore,

$$\sigma(x,T)\sigma(f[T]x,T) \cup \{z \in \sigma(x,T) \cap \sigma_p(T) \ : \ f(z) = 0\}.$$

Notice that if $f$ is analytic in a neighbourhood of $\sigma(T)$ then $f(\sigma(x,T)) \subseteq \sigma(x,f(T))$ for every $x \in X$, and equality obtains if $T$ has the SVEP or if $T \in L(X)$ is arbitrary and $f$ is non-constant on each component of $\triangle(f)$; see [12, theorem 3.3.8].

The following result is a local spectral mapping theorem for a certain class of operators.

**Theorem 3.1.** *Assume that $T \in L(X)$ has property (C), and let $f$ be a holomorphic function on $\triangle(f)$ and injective on $\sigma_p(T) \cap \triangle(f)$. Then $f(\sigma(x,T)) = \sigma(x,f[T])$ for every $x \in D(f[T])$.*

PROOF. We fix $x \in D(f[T])$ and assume in the first case that $f$ is not identically zero on any component of $\triangle(f)$ that intersects $\sigma(x,T)$. In order to show that $f(\sigma(x,T)) \subset \sigma(x,f[T])$, we take $\lambda_0 \in \triangle(f)$ such that $f(\lambda_0) \in \rho(x,f[T])$, and we have to show that $\lambda_0 \in \rho(x,T)$. Since $f(\lambda_0) \in \rho(x,f[T])$, there are a neighbourhood $V$ of $f(\lambda_0)$ and an analytic function $u : V \longrightarrow D(f[T])$ such that $(\mu - f[T])u(\mu) = x$ for every $\mu \in V$. Let $U$ be an open neighbourhood of $\lambda_0$ such that $f(U) \subseteq V$. Therefore, we have

$$(f(\lambda) - f[T])u(f(\lambda)) = x, \text{ for every } \lambda \in U.$$

The function $g : \triangle(f) \times \triangle(f) \longrightarrow \mathbb{C}$ defined by

$$g_\lambda(\mu) = \begin{cases} \dfrac{f(\lambda) - f(\mu)}{\lambda - \mu} & \text{if } \mu \neq \lambda \\ f'(\lambda) & \text{if } \mu = \lambda \end{cases}$$

is (separately) holomorphic in both $\lambda, \mu \in \triangle(f)$. Using part (iii) of Proposition 2.1, we get

$$(f(\lambda) - f[T])u(f(\lambda)) = (\lambda - T)g_\lambda[T]u(f(\lambda)) = x.$$

Hence, by part (ii) of Proposition 2.1 applied to the function $f(\lambda) - f(.)$, we have

$$\sigma(x, T) = \sigma((f(\lambda) - f[T])u(f(\lambda)), T) \subset \sigma(u(f(\lambda)), T)$$

for all $\lambda \in U$. Now, since $f$ is injective on $\sigma_p(T) \bigcap \triangle(f)$, Proposition 3.2 gives

$$\sigma(u(f(\lambda)), T) \subset \sigma(x, T) \bigcup \{\lambda\},$$

for each $\lambda \in U$. Therefore, there exist an open neighbourhood $U(\lambda_0)$ of $\lambda_0$ contained in $U$ and a closed set $F \subset \triangle(f)$ such that $\sigma(u(f(\lambda)), T) \subset F$, for all $\lambda \in U(\lambda_0)$.

To prove that $g_\lambda[T]u(f(\lambda))$ is holomorphic at $\lambda_0$, note that since $\sigma(T|X(T,F)) \subseteq F \subset \triangle(f) = \triangle(g_\lambda)$, for every $\lambda \in \triangle(f)$, the operator $g_\lambda(T|X(T,F))$ is defined by the Riesz–Dunford functional calculus,

$$g_\lambda(T|X(T,F)) = \frac{1}{2\pi i}\int_\Gamma g_\lambda(z)(z - T|X(T,F))^{-1}dz,$$

for some contour $\Gamma$ surrounding $F$ in $\triangle(f)$. A standard argument shows that $\lambda \longmapsto g_\lambda(T|X(T,F))$ is analytic on $\triangle(f)$. For any analytic function $\varphi : U \longrightarrow X(T,F)$, it follows that $g_\lambda[T]\varphi(\lambda) = g_\lambda(T|X(T,F))\varphi(\lambda)$ is analytic on $U$.

To prove the opposite inclusion let us denote $G := \sigma(x,T)$. Then $X(T,G)$ is closed and $\sigma(T|X(T,G)) = G \subset \triangle(f)$; hence $f(T|X(T,G)) = f[T]|X(T,G)$. Using the local spectral mapping theorem for the holomorphic functional calculus [2], we conclude that

$$\sigma(x, f[T]) \subset \sigma(x, f(T|X(T,G))) = f(\sigma(x, T|X(T,G)) = f(\sigma(x,T)).$$

In the remaining case, when $f$ is identically zero on some components of $\triangle(f)$ that meet $\sigma(x,T)$, we can denote by $g$ the function $f$ restricted to the components of $\triangle(f)$ in which it is not identically zero. By [4, theorem 2.1] we can write $x = x_1 + x_2$, with $\sigma(x_1, T) \subset \Delta(g)$ and $\sigma(x_2, T) \subset \triangle(f) \setminus \Delta(g)$. Then we can apply the previous arguments to $g$ and get

$$\sigma(x, f[T]) = \sigma(x_1, g[T]) \bigcup \sigma(x_2, 0) = g(\sigma(x_1, T)) \bigcup \{0\} = f(\sigma(x,T)). \quad \blacksquare$$

In the next theorem, we prove that the SVEP is stable under the local functional calculus (with certain conditions). Our result is an improvement of one implication of [6, theorem 1.5], using a different argument. Indeed, we prove it as a consequence of the local spectral mapping theorem.

**Theorem 3.2** (stability of the SVEP). *Assume that $T \in L(X)$ has the SVEP and let $f$ be a holomorphic function such that $f[T]$ is closable and $\sigma(x, f[T]) = f(\sigma(x,T))$ for all $x \in D(f[T])$. Then $f[T]$ has the SVEP.*

PROOF. Let us suppose that $f[T]$ fails the SVEP. Then there exist an open set $U$ and a holomorphic function $u : U \subset \mathbb{C} \to D(f[T])$, which is not identically 0 and satisfies

$$(\lambda - f[T])u(\lambda) = 0.$$

Let $\lambda_0 \in U$ such that $u(\lambda_0) \neq 0$; hence

$$u(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n x_n,$$

on a neighbourhood $V \subset U$ of $\lambda_0$. Arguing as in [8, p. 298], we will prove that $\sigma(x_0, f[T]) = \varnothing$. In fact, observe that $(\lambda_0 - f[T])x_0 = 0$. Therefore, $\sigma(x_0, f[T]) \subset \{\lambda_0\}$ since

$$(\mu - f[T])\frac{x_0}{\mu - \lambda_0} = x_0$$

for all $\mu \in \mathbb{C}\backslash\{\lambda_0\}$. On the other hand, using that $f[T]$ is closable, that is, there exists a minimal closed extension of $f[T]$, $\overline{f[T]}$, we obtain that

$$\begin{aligned}(\lambda_0 - f[T])x_{n+1} = (\lambda_0 - \overline{f[T]})x_{n+1} &= \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda_0 - \overline{f[T]})u(\lambda)}{(\lambda - \lambda_0)^{n+2}}\, d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda_0 - \lambda)u(\lambda)}{(\lambda - \lambda_0)^{n+2}}\, d\lambda = -x_n.\end{aligned}$$

Hence $v(\lambda) := -\sum_{n=1}^{\infty}(\lambda - \lambda_0)^{n-1} x_n$ satisfies $(\lambda - f[T])v(\lambda) = x_0$ in a neighbourhood of $\lambda_0$,

$$\begin{aligned}(\lambda - f[T])v(\lambda) &= (\lambda - \lambda_0)v(\lambda) + (\lambda_0 - f[T])v(\lambda) \\ &= -\sum_{n=1}^{\infty}(\lambda - \lambda_0)^n x_n - \sum_{n=1}^{\infty}(\lambda - \lambda_0)^{n-1}(\lambda_0 - f[T])x_n = x_0;\end{aligned}$$

hence $\lambda_0 \notin \sigma(x_0, f[T])$. We have proved that $\sigma(x_0, f[T]) = f(\sigma(x_0, T)) = \varnothing$ and $x_0 = u(\lambda_0) = 0$. Hence $T$ fails the SVEP. ■

The definition of the holomorphic functional calculus was extended to meromorphic functions by Gindler [9]. Let $f$ be a meromorphic function on an open set $\triangle(f)$ containing $\sigma(T)$, and let $\alpha_1, \ldots, \alpha_k$ be the poles of $f$ in $\sigma(T)$, with multiplicities $n_1, \ldots, n_k$, respectively. We assume that the poles of $f$ are not eigenvalues of $T$, and consider the polynomial $p(\lambda) = \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}$. The function $g(\lambda) := f(\lambda)p(\lambda)$ is holomorphic on a neighbourhood of $\sigma(T)$. So we can define the operator $f\{T\}$ of the *meromorphic functional calculus* by

$$f\{T\} := g(T)p(T)^{-1},$$

obtaining a closed operator $f\{T\}$. Obviously, the meromorphic calculus is an extension of the holomorphic calculus.

In order to show the relations between the meromorphic calculus and the local functional calculus, the following result will be useful.

**Proposition 3.3.** *Assume that* $T \in L(X)$ *has the SVEP and let* $f$ *be a function of the meromorphic functional calculus. If* $p$ *is the polynomial of poles of* $f$*, then* $p(T)D(f[T]) = D(f[T])$.

Proof. If $f = g/p$ where $g$ is analytic in a neighbourhood of $\sigma(T)$ and $p(z) = \prod_{j=1}^{m}(z - a_j)^{n_j}$ with $\{a_j\}_{j=1}^{m} \cap \sigma_p(T) = \varnothing$, then $p(T)$ is injective $x \in D(f[T])$ if and only if $\{a_j\}_{j=1}^{m} \cap \sigma(x, T) = \varnothing$, equivalently, $x \in D(\frac{1}{p}[T])$. Proposition 2.1(iii) implies that $\frac{1}{p}[T]x = p(T)^{-1}x$, and 2.1(ii) implies that $\sigma(p(T)^{-1}x, T) \subset \sigma(x, T)$. ■

**Proposition 3.4.** *Assume that* $T \in L(X)$ *has the SVEP and let* $f$ *be a function of the meromorphic functional calculus. Then* $f\{T\}$ *is a closed extension of* $f[T]$*. Consequently* $f[T]$ *is closable.*

Proof. We consider the polynomial $p(\lambda) := \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}$ such that $g(\lambda) = p(\lambda)f(\lambda)$ is holomorphic on $\sigma(T)$. If $x \in D(f[T])$, then $\alpha_i \in \rho(x, T)$; hence $x \in R(\alpha_i - T)^{n_i}$ for all $i \in \{1, \ldots, k\}$. By [10, theorem 1.1], we have $D(f\{T\}) = R(p(T)) = \bigcap_{i=1}^{k} R(\alpha_i - T)^{n_i}$. Consequently $D(f[T]) \subset D(f\{T\})$.

By Proposition **??**, given $x \in D(f[T])$, there exists $y \in D(f[T])$ such that $x = p(T)y$; hence by part (iii) of Proposition 2.1 we obtain

$$f[T]x = f[T]p(T)y = (fp)[T]y = g(T)y = g(T)p(T)^{-1}x = f\{T\}x. \quad ■$$

The following example shows that the operator provided by the local functional calculus is not always closable.

*Example 3.2.* Let $T$ be the compact operator on the Hilbert space $\ell_2(\mathbb{N})$ determined by $Te_1 := 0$ and $T(e_n) := n^{-2}(ne_1 + e_n)$ for $n > 1$, and let $f(\lambda) := \lambda^{-1}$, for $\lambda \neq 0$.

Taking $x_n := n^{-3}(ne_1 + e_n)$, we have $Tx_n = n^{-2}x_n$. Thus $\sigma(x_n, T) = \{n^{-2}\}$, and $x_n \in D(f[T])$. Moreover, $x_n$ converges to 0 and $f[T]x_n = e_1 + n^{-1}e_n$ converges to $e_1$ as $n$ tends to $\infty$. Therefore $f[T]$ is not closable.

The closure of the operator $f[T]$ is known in some cases.

**Proposition 3.5.** *Let* $T$ *be a normal operator on a Hilbert space* $\mathscr{H}$ *and let* $f$ *be a function of the meromorphic functional calculus. Then* $\overline{f[T]} = f\{T\}$.

Proof. For simplicity, we give a proof for the case of $f$ having only one pole $\alpha_0$. In the general case, the proof is similar. By Proposition 3.4 we have that $\overline{G(f[T])} \subset G(f\{T\})$. We denote by $E$ the resolution of the identity of $T$. For every positive integer $n$ we consider the sets $F_n := \sigma(T)\backslash\{\lambda\colon |\lambda - \alpha_0| < 1/n\}$, and we define $G_1 := F_1$ and $G_n := F_n \backslash F_{n-1}$ for $n > 1$.

Let $x \in D(f\{T\})$ such that $\alpha_0 \in \sigma(x, T)$. From [5, theorem 3.3] we obtain $E(F_n)\mathscr{H} = \{x \in \mathscr{H}\colon \sigma(x, T) \subset F_n\}$. Hence $x_n := \sum_{k=1}^{n} E(G_k)x \in D(f[T])$, and $\sum_{k=1}^{n} E(G_k)x$ converges to $(I - E(\{\alpha_0\}))x$ and $\sigma(E(\{\alpha_0\})x, T) \subset \{\alpha_0\}$. If we suppose that $\sigma(E(\{\alpha_0\})x, T) = \{\alpha_0\}$, then by the comment in the Preliminaries we have that $(\alpha_0 - T)E(\{\alpha_0\})x = 0$; hence, $\alpha_0 \in \sigma_p(T)$. But this is a contradiction, because the pole $\alpha_0$ cannot be in $\sigma_p(T)$ by definition of the meromorphic functional calculus. Hence $E(\{\alpha_0\})x = 0$, and consequently $\sum_{k=1}^{n} E(G_k)x$ converges to $x$. Moreover,

$$f[T]x_n = \sum_{k=1}^{n} f[T]E(G_k)x = \sum_{k=1}^{n} f\{T\}E(G_k)x = \sum_{k=1}^{n} E(G_k)f\{T\}x.$$

Then $f[T]x_n = f\{T\}x_n \to (I - E(\{\alpha_0\})f\{T\}x = f\{T\}x$; hence $G(f\{T\}) \subset \overline{G(f[T])}$. ■

Finally we apply the local functional calculus to obtain local resolvent equations. Given operators $S, T \in L(X)$, for $\lambda, \mu \in \rho(T)$, the equality

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T) \tag{5}$$

is called the *first resolvent equation*, and for $\lambda \in \rho(S) \bigcap \rho(T)$ the equality

$$R(\lambda, S) - R(\lambda, T) = R(\lambda, S)(S - T)R(\lambda, T) \tag{6}$$

is called the *second resolvent equation.*

If we try to establish equalities (5) and (6) for the local resolvent function, we find that the 'products of vectors' $\hat{x}_T(\lambda)\hat{x}_T(\mu)$ and $\hat{x}_S(\lambda)(S - T)\hat{x}_T(\lambda)$ are not defined. This drawback may be solved using the local functional calculus.

For every $\lambda \in \mathbb{C}$, we denote by $f_\lambda$ the function given by $f_\lambda(\mu) := (\lambda - \mu)^{-1}$.

**Lemma 3.1.** *Assume that $T \in L(X)$ has the SVEP and let $x \in X$. If $\lambda \in \rho(x, T)$, then $\hat{x}_T(\lambda) = f_\lambda[T]x$.*

PROOF. Let $\lambda \in \rho(x, T)$. Applying Cauchy's theorem, we obtain

$$f_\lambda[T]x = \frac{1}{2\pi i}\int_\Gamma \frac{\hat{x}_T(\mu)}{\lambda - \mu}\, d\mu = \hat{x}_T(\lambda). \quad ■$$

This lemma will be the key to establishing the local resolvent equations.

**Theorem 3.3.** *Let $x \in X$ and let $S, T \in L(X)$ satisfying the SVEP.*

(i) (*the first local resolvent equation*). *If $\lambda, \mu \in \rho(x, T)$, then*

$$f_\lambda[T]x - f_\mu[T]x = (\mu - \lambda)f_\lambda[T]f_\mu[T]x. \tag{7}$$

(ii) (*the second local resolvent equation*). *If $S$ and $T$ commute and $\lambda \in \rho(x, T) \bigcap \rho(f_\lambda[T]x, S)$, then*

$$f_\lambda[S]x - f_\lambda[T]x = f_\lambda[S](S - T)f_\lambda[T]x. \tag{8}$$

Proof. (i) The functions $f_\lambda$ and $f_\mu$ are holomorphic on $\sigma(x, T)$. Moreover, $f_\mu[T]x = \hat{x}_T(\mu) \in D(f_\lambda[T])$, for all $\mu \in \rho(x, T)$ [6, proposition 1.2]. Analogously, $\hat{x}_T(\lambda) \in D(f_\mu[T])$. By part (iii) of Proposition 2.1 we obtain

$$(\mu - \lambda) f_\lambda[T] f_\mu[T]x = (\mu - T + T - \lambda) f_\lambda[T] f_\mu[T]x = f_\lambda[T]x - f_\mu[T]x.$$

(ii) By [6, proposition I.2.1] we have that

$$\sigma((S - T) f_\lambda[T]x, S) \subset \sigma(f_\lambda[T]x, S). \tag{9}$$

Hence $(S - T) f_\lambda[T]x \in D(f_\lambda[S])$. By (9), $\lambda \in \rho(f_\lambda[T]x, S)$ and, if we use that $\lambda \in \rho(x, S)$,

$$f_\lambda[S](S - T) f_\lambda[T]x = f_\lambda[S](S - \lambda + \lambda - T) f_\lambda[T]x = f_\lambda[S]x - f_\lambda[T]x.$$

Hence the result is proved. ∎

*Remark 3.3.* (1) Equations (7) and (8) may be written in the following way

$$\hat{x}_T(\lambda) - \hat{x}_T(\mu) = (\mu - \lambda)\widehat{\hat{x}_T(\lambda)}_T(\mu),$$

$$\hat{x}_S(\lambda) - \hat{x}_T(\lambda) = (S - T)\widehat{\hat{x}_T(\lambda)}_S(\lambda).$$

Observe that the local resolvent equations admit a more natural expression using the local functional calculus. Moreover, these equations are suggested by the local functional calculus.

(2) Once the expression of the first local resolvent equation is known, we can prove it just by means of algebraic computations. Further, we obtain the following equality (similar to the second local resolvent equation)

$$(\lambda - S)(\hat{x}_S(\lambda) - \hat{x}_T(\lambda)) = (S - T)\hat{x}_T(\lambda),$$

without the hypothesis that $T$ and $S$ commute and $\lambda \in \rho(\hat{x}_T(\lambda), S)$.

(3) Notice that in the hypotheses of part (ii) of Theorem 3.3 we obtain that

$$(S - T)\widehat{\hat{x}_T(\lambda)}_S(\mu) = \hat{x}_S(\mu) - \hat{x}_T(\lambda) + (\mu - \lambda)\widehat{\hat{x}_T(\lambda)}_S(\mu)$$

for $\mu$ in a neighbourhood of $\lambda$, since

$$(\mu - S)(\hat{x}_S(\mu) - \hat{x}_T(\lambda) + (\mu - \lambda)\widehat{\hat{x}_T(\lambda)}_S(\mu)) = (S - T)\hat{x}_T(\lambda).$$

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