BOREL SETS AND SEPARATION AXIOMS

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Abstract

The interactions of the Borel classes with separation axioms between T_0 and T_1 are considered. A topological space is said to be GT_D (resp. $GT_{\frac{1}{2}}$) if each point is an intersection of a closed set with a G_δ-set (resp. each point is closed or a G_δ-point). Minimal GT_D- and minimal $GT_{\frac{1}{2}}$-spaces are characterised. Relevant examples are presented.

1. Introduction

For each topological space, the collection of its Borel sets is defined to be the smallest σ-algebra containing the topology. They have been well studied in the area of metrisable spaces, and especially that of Polish spaces (separable completely metrisable spaces) [7]. The Borel sets on regular Hausdorff spaces were studied via universal sets in [4], and spaces for which every singleton is Borel were considered by Harley and McNulty in [5].

This paper is devoted to the study of interactions of Borel sets with separation axioms between T_0 and T_1. The axioms T_D and $T_{\frac{1}{2}}$ (or T_{ES}) are considered (see [1] and [13]). A space is T_D if every point is the intersection of a closed set with an open set. It is $T_{\frac{1}{2}}$ if every singleton is either open or closed. The axiom that every point is Borel is denoted here by GT_D, and a new axiom $GT_{\frac{1}{2}}$ is defined. A space is $GT_{\frac{1}{2}}$ if each point is closed or a G_δ-point.

1.1. *Topologies*

A survey article by Larson and Andima [10] contains many of the undefined lattice notions in this paper. We shall denote the lattice of all topologies on a given set X by $LT(X)$. The cofinite topology (all non-empty elements of the topology have finite complements in X) is denoted by $\mathscr{C}$, the x-inclusion topology (a non-empty subset is in the topology if and only if it contains x) by $\mathscr{I}(x)$, the x-exclusion topology (a proper subset is in the topology if and only if it excludes x) by $\mathscr{E}(x)$, and the discrete topology by $\mathbb{P}(X)$.

We shall also consider duplicates. Let X be a fixed set. Suppose that $\sigma \in LT(X)$ is a T_1-topology. We say that τ is a $T_{\frac{1}{2}}$-duplicate of σ if it is a topology on $X \times 2$, with basic open sets of the form $U \times 2$ (for some $U \in \sigma$) and $\{(x,0)\}$ (for some $x \in X$). These are $T_{\frac{1}{2}}$-spaces.

Suppose that σ is a topology such that all points in X are countable intersections

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of elements of σ. Then τ is said to be a $GT_{\frac{1}{2}}$-duplicate of σ if it is a topology on $X \times 2$, with basic open sets of the form $U \times 2$ (for some $U \in \sigma$) and $U \times \{0\}$ (for some $U \in \sigma$). These are $GT_{\frac{1}{2}}$-spaces. A $GT_{\frac{1}{2}}$-duplicate was used previously in [4, example 34].

Suppose that σ is any topology. We say that $\tau \in LT(X \times 2)$ is the Alexandroff duplicate of σ if the open sets are of the form $\mathrm{cl}_\sigma(x) \times \{0\}$ (for some x in X) and $(U \times 2) - (\mathrm{cl}_\sigma(x) \times \{0\})$ (for some x in X and $U \in \sigma$). Note that this is the usual Alexandroff duplicate if X is the closed unit interval with the Euclidean topology.

We use the notation $\mathrm{cl}_\upsilon(A)$ (resp. $\mathrm{cl}_\upsilon(x)$) to denote the υ-closure of the subset A (resp. the point x). The υ-point-derived sets are denoted by $\mathrm{pd}_\upsilon(x)$. Thus $\mathrm{pd}_\upsilon(x) = \mathrm{cl}_\upsilon(x) - \{x\}$.

1.2. Borel sets

Given a topology $\tau \in LT(X)$, the collection $\mathbb{B}(\tau)$ of Borel sets is defined to be the smallest σ-algebra of subsets of X containing τ. We now define the Borel classes: we denote τ by $\Sigma^0_1(\tau)$, the set of all complements of elements of $\Sigma^0_\alpha(\tau)$ by $\Pi^0_\alpha(\tau)$ (for $\alpha \in \omega_1 - \{0\}$), the set of all countable unions of elements of $\Pi^0_\alpha(\tau)$ by $\Sigma^0_{\alpha+1}(\tau)$ (for $\alpha \in \omega_1 - \{0\}$), and the set of all countable unions of elements of $\bigcup_{\alpha\in\lambda} \Pi^0_\alpha(\tau)$ by $\Sigma^0_\lambda(\upsilon)$ (for $\lambda \in \omega_1$ a limit ordinal). We know that $\mathbb{B}(\tau)$ is the union of all $\Sigma^0_\alpha(\tau)$ with α varying in $\omega_1 - \{0\}$. Following Kechris [7], we also use $\Delta^0_\alpha(\tau)$ to denote $\Sigma^0_\alpha(\tau) \bigcap \Pi^0_\alpha(\tau)$.

It is clear that the collection of all G_δ-sets is Π^0_2, and that of all F_σ-sets is Σ^0_2. For finite n, if n is odd, we let $G_n = \Sigma^0_n$ and $F_n = \Pi^0_n$. If n is even, we let $G_n = \Pi^0_n$ and $F_n = \Sigma^0_n$. Given a Borel class Γ, we say that a set (or a point) is a Γ-set (or a Γ-point) if that set (or point) is in Γ. Hence the G_δ-sets are the G_2-sets. Depending on the circumstances, for a topological space (X, τ), the notations $\Gamma(X)$ and $\Gamma(\tau)$ will mean the same set.

2. Preliminary results

2.1. The T_0- and T_1-axioms on countable X

This section considers the condition when two topologies τ and υ on *countable* X are related to each other via $\upsilon \subseteq \tau \subseteq \Gamma(\upsilon)$, where Γ is a Borel class.

Proposition 2.1. *Let $n \geq 2$ be a positive integer and $\tau \in LT(X)$. Then the following are equivalent:*

(1) *τ is T_1;*

(2) *$\Sigma^0_n(\tau) = \mathbb{P}(X)$.*

Proof. Suppose that τ is T_1. Then each of the countably many singletons of X is closed. Therefore all subsets of X must be in $\Sigma^0_2(\tau)$ and also in $\Pi^0_2(\tau)$. It is easy to see that (2) is then true.

Conversely, suppose that (2) is true. This condition is the same as saying $\Delta^0_n(\tau) = \mathbb{P}(X)$. Let x and y be two distinct points in X. Then $\{x\}$ is in $\Delta^0_n(\tau)$. Suppose that it is the countable intersection of some $G_{n-1}(\tau)$-sets. There is at least one of these sets, A, which contains x and excludes y. The set A, being a $G_{n-1}(\tau)$-set, is the countable union of $G_{n-2}(\tau)$-sets. There is one of them which contains x but not y. Continuing this process, we will be able to obtain an open set containing x and not containing y. The case when x is the countable union of $G_{n-1}(\tau)$-sets is similar. ■

Proposition 2.2. *Let $\tau \in LT(X)$. Then the following are equivalent:*
(1) *τ is T_0;*
(2) *$\Sigma^0_{\omega+1}(\tau) = \mathbb{P}(X)$;*
(3) *$\mathbb{B}(\tau) = \mathbb{P}(X)$.*

Proof. Suppose that (1) is true, and let $A \in \mathbb{P}(X)$. There are only countably many elements of A, so it will be enough to show that all the points of X are $\Pi^0_\omega(\tau)$-points. Indeed, since X is countable, and since it is T_0, the point x must be the intersection of countably many sets that are either open or closed. Hence x is the intersection of a closed set with a G_δ-set.

If (2) is true, then certainly (3) will be true.

Suppose that (1) does not hold, and let x and y be distinct points of X. Suppose that they are both contained in every open or closed set. Then it is clear, since the Borel sets are generated by the open sets, that no Borel set will contain one and not the other. Therefore, $\mathbb{B}(\tau) \neq \mathbb{P}(X)$, which implies that (3) does not hold. ■

Clearly the above propositions hold only for countable X. It is known that the real line has $2^{2^{\aleph_0}}$ many subsets, while there are only $2^{\aleph_0}$ many Borel sets with respect to the Euclidean topology.

Together with Proposition 3.4 later, Propositions 2.1 and 2.2 might be interpreted as saying that the attempt to classify separation axioms on a countable set by the condition $\Gamma = \mathbb{P}$ (where Γ is a Borel class) collapses to just four cases: T_0, $GT_{\frac{1}{2}}$, T_1 and discrete.

Corollary 2.3. *Let $\upsilon, \tau \in LT(X)$ satisfy $\upsilon \subseteq \tau \subseteq \Sigma^0_n(\upsilon)$ for some finite n. If τ is T_1, then so is υ.*

Proof. By Proposition 2.1, we only have to show that $\mathbb{P}(X) = \Sigma^0_m(\upsilon)$ for some finite m. Let $A \in \mathbb{P}(X)$. As τ is T_1, $A \in \Sigma^0_2(\tau)$. Each τ-closed set is in $\Pi^0_n(\upsilon)$. Therefore $A \in \Sigma^0_{n+1}(\upsilon)$. ■

Note that the above corollary looks twice as powerful as it really is. The following lemma shows that we need to consider the G_n-classes and not the F_n-classes for finite n.

Lemma 2.4. *Let $\upsilon \in LT(X)$ be a T_0-topology such that $\upsilon \subseteq F_n$ for some $n \in \omega$. Then υ is T_1.*

Proof. The lemma follows from the proof of Proposition 2.1. ■

Note that any T_0-topology $\upsilon \in LT(X)$ has the property that for all finer T_1 $\tau \in LT(X)$, $\upsilon \subseteq \tau \subseteq \Sigma^0_{\omega+1}(\upsilon)$ (using Proposition 2.2). Together with Corollary 2.3, we only have the ω case to consider, where we have an example.

Example 2.5. Let the countable set X be $\mathbb{Q}\times 2$. Let σ be the usual Euclidean topology on $\mathbb{Q}$. Let υ be the $GT_{\frac{1}{2}}$-duplicate of σ, and τ be the Alexandroff duplicate of σ. Then $\upsilon \subseteq \tau \subseteq \Delta^0_\omega(\upsilon)$, υ is T_0 but not T_1, and τ is T_1.

Proof. Note that the subspace $\mathbb{Q} \times \{1\}$ with the υ-topology has the Euclidean topology. Let $A \in \tau$, and set $A^1 = A \cap (\mathbb{Q} \times \{1\})$, $A^0 = A \cap (\mathbb{Q} \times \{0\})$. Then A^i is a Euclidean-open set of $\mathbb{Q} \times \{i\}$ for $i = 1, 2$.

By perfectness of the Euclidean topology, every open subset of $\mathbb{Q} \times \{1\}$ is a Euclidean-F_σ. Since the set $\mathbb{Q} \times \{1\}$ is closed in (X, υ) and the set $\mathbb{Q} \times \{0\}$ is open in (X, υ), we deduce that $A^1 \in \Sigma^0_2(\upsilon)$ and $A^0 \in \Sigma^0_1(\upsilon)$. Then A is in $\Sigma^0_\omega(\upsilon)$.

We may run a similar argument for the complement of A and obtain that $X - A \in \Sigma^0_\omega(\upsilon)$, and that $A \in \Delta^0_\omega(\upsilon)$. ■

2.2. Weak separation axioms

From now on X may have any cardinality, unless otherwise stated.

Definition 2.6. With regard to any member of $LT(X)$, a subset U of X is *regular open if it is open and it is the interior of its closure.*

We now define three axioms between T_0 and T_1 in order of strength. (See [1], [13] and [2].)

Definition 2.7. A topology $\tau \in LT(X)$ is T_D if each singleton is an intersection of an open set and a closed set. It is $T_{\frac{1}{2}}$ if each singleton is either open or closed. It is $T_{\frac{3}{4}}$ if each singleton is either regular open or closed.

It is clear that both the T_D- and $T_{\frac{1}{2}}$-axioms are expansive, i.e. for all $\upsilon, \tau \in LT(X)$, if $\upsilon \subseteq \tau$ and υ is T_D (resp. $T_{\frac{1}{2}}$), then τ is T_D (resp. $T_{\frac{1}{2}}$). This property is also shared by the T_0- and T_1-axioms. The $T_{\frac{3}{4}}$-axiom is not expansive.

Continuing in the same way as in Section 2.1, we look at how the condition $\upsilon \subseteq \tau \subseteq \Gamma(\upsilon)$ (for some Borel class Γ) affects υ for different τ's.

Example 2.8. Let X be the countable set $\mathbb{Q}$. There is a topology υ on X that is T_0 and not T_D, and a topology τ which is T_D. These topologies satisfy $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Proof. Let $\upsilon = \{\varnothing, X\} \cup \{(-\infty, x) : x \in \mathbb{Q}\}$ and $\tau = \{\varnothing, X\} \cup \{(-\infty, x) : x \in \mathbb{Q}\} \cup \{(-\infty, x) : x \in \mathbb{Q}\}$. ■

Example 2.9. Let X be the countable set $\mathbb{Q} \times 2$. There are, on X, a T_D- but not $T_{\frac{1}{2}}$-topology υ, and a $T_{\frac{1}{2}}$-topology τ such that $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Proof. Let υ be the $GT_{\frac{1}{2}}$-duplicate of the Euclidean topology on $\mathbb{Q}$, and τ be the $T_{\frac{1}{2}}$-duplicate of the same topology. ■

Proposition 2.10. *Let X be any set, υ and τ be elements of $LT(X)$ such that $\upsilon \subseteq \tau \subseteq G_n(\upsilon)$, for some $n \in \omega$. For any $x \in X$, $\mathrm{cl}_\tau(x) = \mathrm{cl}_\upsilon(x)$.*

Proof. Suppose $\mathrm{cl}_\tau(x) \neq \mathrm{cl}_\upsilon(x)$ (it is true that $\mathrm{cl}_\tau(x) \subseteq \mathrm{cl}_\upsilon(x)$). Then the set $X - \mathrm{cl}_\tau(x)$ is open in τ and is therefore in $G_n(\upsilon)$. Note that there is a point $\xi \in \mathrm{cl}_\upsilon(x) - \mathrm{cl}_\tau(x)$. By a similar technique as in the proof of Lemma 2.4, there is an υ-open set containing ξ and not containing x. This is not possible, and we conclude that $\mathrm{cl}_\tau(x) = \mathrm{cl}_\upsilon(x)$. ■

Proposition 2.11. *Suppose that υ and τ are in $LT(X)$ such that υ is $T_{\frac{1}{2}}$ and τ is $T_{\frac{3}{4}}$. If, for some $n \in \omega$, $\upsilon \subseteq \tau \subseteq G_n(\upsilon)$, then υ is $T_{\frac{3}{4}}$.*

Proof. Let $x \in X$ be υ-open and not υ-closed. Since each τ-closed set is in $F_\sigma(\upsilon)$, it is not possible for the singleton $\{x\}$ to be τ-closed. Therefore it must be τ-regular open.

Note that the υ-interior of any set A is contained in the τ-interior of A. Therefore if the singleton $\{x\}$ is not the υ-interior of $\mathrm{cl}(x)$ (the closures are the same, by Proposition 2.10), then it cannot be the τ-interior of $\mathrm{cl}(x)$. Hence, x must be υ-regular open. ■

By Lemma 2.4, we have the following corollary.

Corollary 2.12. *Suppose that υ and τ are in $LT(X)$ such that υ is $T_{\frac{1}{2}}$ and τ is $T_{\frac{3}{4}}$. If, for some $n \in \omega$, $\upsilon \subseteq \tau \subseteq \Sigma^0_n(\upsilon) \cup \Pi^0_n(\upsilon)$, then υ is $T_{\frac{3}{4}}$.*

As in Section 2.1, the remaining interesting case is the ω case. This is solved by the following observation. It implies that any $T_{\frac{1}{2}}$-topology that is not $T_{\frac{3}{4}}$ will be an example. Note that the two-point Sierpinski space is such a space. This is the two-point space with underlying set $\{0,1\}$ and topology $\{\varnothing, \{0\}, X\}$.

Lemma 2.13. *If X is countable and $\upsilon \in LT(X)$ is $T_{\frac{1}{2}}$, then $\mathbb{P}(X) = \Sigma^0_\omega(\upsilon)$.*

It is necessary to consider the following example in the investigation as the $T_{\frac{3}{4}}$-axiom is not expansive.

Example 2.14. Let the countable set X be $\mathbb{Q} \times 2$. Then there are, on X, a topology υ which is $T_{\frac{1}{2}}$ not $T_{\frac{3}{4}}$, and a topology τ which is $T_{\frac{3}{4}}$, satisfying $\tau \subseteq \upsilon \subseteq \Pi^0_2(\tau)$.

Proof. Let x be a point fixed in $\mathbb{Q}$. Define υ as the join of the $T_{\frac{1}{2}}$-duplicate of the Euclidean topology on $\mathbb{Q}$ with $\{\varnothing, \{(x,0),(x,1)\}, X\}$. Define τ to be the $T_{\frac{1}{2}}$-duplicate. ■

Example 2.15. Let X be the countable set $\mathbb{Q} \times 2$. There are, on X, a topology υ which is $T_{\frac{3}{4}}$ not T_1, and a topology τ which is T_1, satisfying $\upsilon \subseteq \tau \subseteq \Delta^0_\omega(\upsilon)$.

PROOF. Let υ be the $T_{\frac{1}{2}}$-duplicate of the Euclidean topology on $\mathbb{Q}$. Let τ be the Alexandroff duplicate of the Euclidean topology on $\mathbb{Q}$. See the proof of Example 2.5 for the proof that $\upsilon \subseteq \tau \subseteq \Delta^0_\omega(\upsilon)$. ■

3. The GT_D- and $GT_{\frac{1}{2}}$-axioms

Definition 3.1. Let $\tau \in LT(X)$. It is GT_D if every singleton is a Borel set. It is $GT_{\frac{1}{2}}$ if, for all $x \in X$, the singleton $\{x\}$ is either a closed set or a G_δ-set.

From the definition of these axioms, it can be deduced straightaway that they are hereditary and expansive properties. There are many examples of $GT_{\frac{1}{2}}$-spaces. Indeed, all $GT_{\frac{1}{2}}$-duplicates are $GT_{\frac{1}{2}}$.

Proposition 3.2. *Let (X, τ) be a topological space in which all the singletons are either closed or in $G_n(\tau)$ for some $n \in \omega$. Then (X, τ) is a $GT_{\frac{1}{2}}$-space.*

PROOF. Let the singleton $\{x\}$ be a G_{2n}-set (the G_{2n-1} case is similar). Then it is the countable intersection of G_{2n-1}-sets, U_m, with x being an element of each of them. For any one U_m of these countably many sets, being a G_{2n-1}-set, it is the countable union of G_{2n-2}-sets, with x being in at least one of them. Choose one $V(U_m)$ of them which contains x. Note that $\{x\} = \bigcap\{V(U_m) : m \in \omega\}$. Continuing in this way, we will be able to choose countably many open sets which intersect down to $\{x\}$. ■

In a similar spirit, since any F_n-points are just countable intersections of closed sets, which are closed, the $GT_{\frac{1}{2}}$-axiom includes the property of points being F_n-sets or G_n-sets for some finite n. Note also that there is no incentive to generalise, and consider the case when all singletons are of higher Borel classes, as the following result tells us (taken from [5, theorem 0.1]).

Theorem 3.3 ([5]). *The following are equivalent for a topological space (X, τ):*
(1) *τ is GT_D;*
(2) *each point is the intersection of a closed set with a G_δ-set;*
(3) $\mathrm{pd}(x) \in \mathbb{B}(\tau)$ *for all x;*
(4) $\mathrm{pd}(x) \in F_\sigma(\tau)$ *for all x.*

In Theorems 2.1 and 2.2 the meaning of $\Sigma^0_\alpha(\tau) = \mathbb{P}(X)$ was considered for countable X, $\tau \in LT(X)$ and $\alpha \in \omega_1 - \{\omega\}$. We now see that the condition $\Sigma^0_\omega(\tau) = \mathbb{P}(X)$ characterises an axiom strictly between T_0 and T_1.

Proposition 3.4. *Let X be countable and $\tau \in LT(X)$. Then $\Sigma^0_\omega(\tau) = \mathbb{P}(X)$ if and only if τ is $GT_{\frac{1}{2}}$.*

Proof. If τ is $GT_{\frac{1}{2}}$, then each subset of the countable set X is a countable union of closed or G_δ-points. If each singleton is a Σ^0_ω-set, then it must be the case that each of them is an F_n- or G_n-set for some finite n. By the same method as in the proof of Proposition 3.2, the singletons are either closed or G_δ-sets. ■

3.1. Relationship with other axioms

It is clear that both GT_D and $GT_{\frac{1}{2}}$ lie between T_0 and T_1. They are generalisations of the T_D- and $T_{\frac{1}{2}}$-axioms respectively. A T_D-space is GT_D and a $T_{\frac{1}{2}}$-space is $GT_{\frac{1}{2}}$. In this section we consider examples which distinguish the T_0-, GT_D-, T_D-, $GT_{\frac{1}{2}}$- and $T_{\frac{1}{2}}$-axioms from each other.

Lemma 3.5. *If X is countable, then a topology $\tau \in LT(X)$ is GT_D if and only if it is T_0.*

Proof. Suppose that it is T_0. Fix $x \in X$. For each of the countably many points y different from x, there is a set A_y, which is either open or closed, containing x and not containing y. The intersection of all these A_y's is the singleton $\{x\}$ and is the intersection of a closed set with a G_2-set. ■

Example 3.6. Let X be an uncountable set, and $x \in X$ be a fixed point of X. Then the topology $\mathscr{C} \cap \mathscr{I}(x)$ is T_0 and not GT_D.

The space $(\mathbb{Q}, \upsilon)$ of Example 2.8 is GT_D and not T_D and not $GT_{\frac{1}{2}}$. The space $(\mathbb{Q}, \tau)$ of Example 2.8 is T_D and not $GT_{\frac{1}{2}}$.

Example 3.7. Let X be a countable set, and $x \in X$ be a fixed point of X. Then the topology $\mathscr{C} \cap \mathscr{I}(x)$ is $GT_{\frac{1}{2}}$ and not T_D.

The space (X, υ) of Example 2.9 is $GT_{\frac{1}{2}}$ and T_D but not $T_{\frac{1}{2}}$.

3.2. The condition $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$

The GT_D- and $GT_{\frac{1}{2}}$-axioms defined above are related to the condition $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$ introduced in Section 2.

Proposition 3.8. *Let X be a set and $\upsilon \in LT(X)$. Suppose that there is a topology $\tau \in LT(X)$ such that $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$. If τ is T_D (resp. $T_{\frac{1}{2}}$), then υ is GT_D (resp. $GT_{\frac{1}{2}}$).*

Proof. Let $x \in X$. If τ is T_D, then $\mathrm{cl}_\tau(x) - \{x\}$ is τ-closed, and therefore is in $F_\sigma(\upsilon)$. From Theorem 3.3 it can be deduced that υ is GT_D.

If τ is $T_{\frac{1}{2}}$, then any τ-open singleton would be in $G_\delta(\upsilon)$. If the singleton $\{x\}$ is τ-closed, then it must be in $F_\sigma(\upsilon)$, and therefore is a countable union of υ-closed sets. It is then immediate that $\{x\}$ is υ-closed. ■

When only the topologies on a countable set are considered, the converse of the above proposition is true.

Theorem 3.9. *Let* $\upsilon \in LT(X)$*, where* X *is a countable set. If* υ *is* GT_D *(resp.* $GT_{\frac{1}{2}}$*), then there is a* T_D*- (resp.* $T_{\frac{1}{2}}$*-) topology* $\tau \in LT(X)$ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Proof. *Ad* (1): Let τ be the topology on X generated by υ and the complements of point-derived sets. This topology is certainly finer than υ. As τ is finer than υ, the τ-closures are contained in υ-closures.

We now prove that any τ-open set is a $G_\delta(\upsilon)$-set. Let $V \in \tau$. Then for some υ-open sets U_α ($\alpha \in A$) and some points $\xi_{\alpha i} \in X$ (where i ranges from 1 to $n_\alpha \in \omega$),

$$V = \bigcup_{\alpha \in A} \bigcap_{i=1}^{n_\alpha} U_\alpha \cap (X - \mathrm{pd}_\upsilon(\xi_{\alpha i})).$$

Suppose $x \notin V$. Then, for all $\alpha \in A$, $x \notin \bigcap_{i=1}^{n_\alpha} U_\alpha \cap (X - \mathrm{pd}_\upsilon(\xi_{\alpha i}))$. Hence, for each α, either $x \notin U_\alpha$ or, for some i, $x \in \mathrm{pd}_\upsilon(\xi_{\alpha i})$. Then, by the definition of closures, it is clear that for all α either $\mathrm{cl}_\upsilon(x) \subseteq X - U_\alpha$ or $\mathrm{cl}_\upsilon(x) \subseteq \mathrm{pd}_\upsilon(\xi)$. Immediately, this gives $\mathrm{cl}_\upsilon(x) \subseteq X - V$. As the space is countable, $X - V \in F_\sigma(\upsilon)$, as required.

From Proposition 2.10 it is clear that τ is T_D.

Ad (2): Suppose that υ is $GT_{\frac{1}{2}}$. Let Y be the set of all those $x \in X$ such that x is not υ-closed. Define a finer topology τ to be that generated by υ and all sets of the form $\{y\}$ (where $y \in Y$). This is clearly a $T_{\frac{1}{2}}$-topology. We now only need to show that all τ-open sets are in $G_\delta(\upsilon)$.

Let $V \in \tau - \upsilon$, and note that there necessarily exists an υ-open set W and a subset A of Y such that $V = W \cup A$. Now each point in $X - Y$ is υ-closed, and, for each $x \in Y - A$, $\mathrm{cl}_\upsilon(x) \cap A = \varnothing$ (since each point of A is a $G_\delta(\upsilon)$-point). Consequently, $X - A$ is in $F_\sigma(\upsilon)$, so that $X - V = (X - W) \cap (X - A)$ is also in $F_\sigma(\upsilon)$, as required. ■

Theorem 3.10. *Let* $\upsilon \in LT(X)$ *have a finer* $T_{\frac{1}{2}}$*- (resp.* T_D*-) topology* τ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$*. Then a least such* τ *exists in* $LT(X)$.

Proof. Let $Y = \{x \in X : X - \{x\} \notin \upsilon\}$. Let τ be a finer topology such that $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$. Consider $y \in Y$. If $\{y\} \notin \tau$, then it must be closed in τ, and therefore an $F_\sigma(\upsilon)$-point. This is a contradiction, and we must conclude that $\{y\}$ cannot be closed in τ and must be open in τ.

Now suppose that ρ is the topology generated by υ and all $\{y\}$ (for $y \in Y$). This topology is $T_{\frac{1}{2}}$. Moreover, as we have seen above, any other $T_{\frac{1}{2}}$-topologies τ which satisfy $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$ must be finer than ρ. It now remains to show that ρ also satisfies $\upsilon \subseteq \rho \subseteq G_\delta(\upsilon)$.

Let $A \in \rho$. Then it is the union of an υ-open set with a subset of Y. Suppose $A \subseteq Y$. We know that $A \in \tau \subseteq G_\delta(\upsilon)$, where τ is as mentioned in the statement of the theorem. Therefore $\rho \subseteq G_\delta(\upsilon)$.

We now consider the case when τ is T_D. Consider ρ, the member of $LT(X)$ generated by υ and the complements of the υ-point-derived sets. Now, suppose that τ is any finer T_D-topology such that $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$. Then Proposition 2.10 tells us that the υ-point-derived sets coincide with the τ-point-derived sets. Therefore, all υ-point-derived sets are τ-closed. Hence $\rho \subseteq \tau \subseteq G_\delta(\upsilon)$. ■

Corollary 3.11. *Let* (X, υ) *be a* $GT_{\frac{1}{2}}$*-space, and* $Y = \{x \in X : X - \{x\} \notin \upsilon\}$. *Then there is a* $T_{\frac{1}{2}}$*-topology* $\tau \in LT(X)$ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$ *if and only if* $\mathbb{P}(Y) \subseteq G_\delta(\upsilon)$.

When we consider uncountable X, Theorem 3.9 (2) does not hold. We note that in the proof τ was the coarsest possible topology to satisfy the required properties. Note also that the real line with its usual Euclidean topology is first countable, but there is a non-G_δ-subset (e.g. $\mathbb{Q}$).

Proposition 3.12. *Let* (X, υ) *be an uncountable space, in which all points are* G_δ*-points. If there is a set* A *in* X *which is not a* G_δ*, then the* $GT_{\frac{1}{2}}$*-duplicate of this space is* $GT_{\frac{1}{2}}$*, but does not have a finer* $T_{\frac{1}{2}}$*-topology* τ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Proof. No point on $X \times \{0\}$ is closed in the $GT_{\frac{1}{2}}$-duplicate. So if there is a finer $T_{\frac{1}{2}}$-topology τ satisfying the condition, it is easy to see that all these points must be τ-open. If $A \subseteq X$ is not in $G_\delta(\upsilon)$, then $A \times \{0\}$ cannot be a G_δ-set of the $GT_{\frac{1}{2}}$-duplicate. We have arrived at a contradiction when we note that A must be a τ-open set. ■

There are (in **ZFC**) Q-spaces, spaces in which *all* subsets are G_δ. Clearly, any discrete space is a Q-space. See [17, example 4.1.16] for an example of a Q-space which is not σ-discrete. It is well known that under **MA** every subset of the real line of cardinality less than continuum has this property [16, theorem 12]. The following lemma can be deduced from the above proof.

Lemma 3.13. *Let* X *be a* Q*-space. Then its* $GT_{\frac{1}{2}}$*-duplicate has a finer* $T_{\frac{1}{2}}$*-topology* τ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Lemma 3.14. *For every* T_0*-topology* υ *on a countable set, there is a* T_D*-topology* τ *such that* $\upsilon \subseteq \tau \subseteq G_\delta(\upsilon)$.

Proof. All T_0-topologies on a countable set are GT_D, by Lemma 3.5. ■

Problem 3.15. Does Theorem 3.9 (1) still hold for uncountable X?

4. Minimal GT_D- and $GT_{\frac{1}{2}}$-topologies

Characterisations of minimal GT_D- and $GT_{\frac{1}{2}}$-topologies are recorded in this section. For comparisons, the characterisations of minimal T_D- and $T_{\frac{1}{2}}$-axioms are included. Recall that a topology σ is nested if, for any A and B in σ, either $A \subseteq B$ or $B \subseteq A$. The following is taken from [9] (see also [14] and [12]).

Theorem 4.1 ([9]; [14]). *Let* $\tau \in LT(X)$.

(1) *It is minimal* T_D *if and only if it is* T_D *and nested.*

(2) *It is minimal* T_0 *if and only if it is* T_0*, nested, and the family* $\{X - \mathrm{cl}(x) : x \in X\}$ *is a base for* τ*.*

The following theorem is due to Dunham [3, section 6] and McCartan [11].

Theorem 4.2 ([3]; [11]). *Let* $\tau \in LT(X)$*.*

(1) *If* X *is infinite, then* τ *is minimal* $T_{\frac{1}{2}}$ *if and only if there is a proper subset* A *of* X *such that* τ *is the set of all* U *with the property that either* $U \subseteq A$ *or* $A \subseteq U$*, in which case* U *is cofinite.*

(2) *Suppose that* X *is finite. Then* τ *is minimal* $T_{\frac{1}{2}}$ *if and only if there is a proper non-empty subset* A *of* X *such that* τ *is the set of all* U *with the property that either* $U \subseteq A$ *or* $A \subseteq U$*.*

By Lemma 3.5 we see that every minimal GT_D-topology on a countable set must be minimal T_0. Larson's theorem tells us that such topologies must be nested. In fact, this latter fact is true whether X is countable or not. The following is a modification of Larson's in [9].

Proposition 4.3. *If* $\tau \in LT(X)$ *is minimal* GT_D*, then it is nested.*

Proof. Let $U \in \tau$, and let $\upsilon_U = \{V \in \tau : V \subseteq U$ or $U \subseteq V\}$. Let $x \in X$ such that $\{x\} = C \cap \bigcap_{n\in\omega} V_n$, where C is τ-closed and each V_n is τ-open. If $x \in U$, then let $V_n' = V_n \cap U$ and $C' = C \cup (X - U)$; if $x \notin U$, then let $V_n' = V_n \cup U$ and $C' = C \cap (X - U)$. In either case, $\{x\} = C' \cap \bigcap_{n\in\omega} V_n'$, with C' being υ_U-closed and V_n' being υ_U-open.

The topology υ_U is then a coarser GT_D-topology, and, by minimality of τ, $\tau = \upsilon_U$. Since this is true for any $U \in \tau$, we conclude that τ is nested. ■

Theorem 4.4. *For a space* (X, τ)*, let* Z_τ *be the set of all* $x \in X$ *such that* $\mathrm{pd}_\tau(x)$ *is not the union of countably infinitely many closed sets. A space* (X, τ) *is minimal* GT_D *if and only if* τ *is nested* GT_D *and generated by*

$$\mathscr{B} = \{X - \mathrm{cl}_\tau(x) : x \in X\} \cup \{X - \mathrm{pd}_\tau(z) : z \in Z_\tau\}.$$

Proof. Suppose that (X, τ) is minimal GT_D. Consider υ to be the topology generated by $\mathscr{B}$. Let $x \in X - Z_\tau$ (if $x \in Z_\tau$, then $\mathrm{pd}_\tau(x)$ is τ-closed and $\mathrm{pd}_\upsilon(x)$ is υ-closed), and let $\mathrm{pd}_\tau(x)$ be the strictly increasing union of the τ-closed sets C_n's, with $y_n \in C_n - C_{n+1}$ for each n. Since all the closed sets are nested by Proposition 4.3, for all n, $\mathrm{cl}_\tau(y_n) \subseteq C_n$, and, for all $m > n$, $\mathrm{cl}_\tau(y_n) \supseteq C_m$. It is clear that $\bigcup_{n\in\omega} C_n = \bigcup_{n\in\omega} \mathrm{cl}_\tau(y_n)$ and hence $\mathrm{pd}_\upsilon(x) \in F_\sigma(\upsilon)$. Then υ is GT_D, and, by minimality of τ, $\tau = \upsilon$.

Conversely, suppose that τ is nested GT_D and generated by $\mathscr{B}$, and let υ be a coarser GT_D-member of $LT(X)$. Let $x \in X$ and $y \in \mathrm{cl}_\upsilon(x) - \mathrm{cl}_\tau(x)$. There is a $U \in \upsilon$ such that $y \notin U$ and $x \in U$ since υ is GT_D. Now since τ is nested and $y \in (X - \mathrm{cl}_\tau(x)) - U$, $U \subseteq X - \mathrm{cl}_\tau(x)$. This contradicts the fact that $x \in U$ and

$x \notin X - \mathrm{cl}_\tau(x)$. Therefore, for all $x \in X$, $\mathrm{cl}_\upsilon(x) = \mathrm{cl}_\tau(x)$. Clearly, $Z_\upsilon \supseteq Z_\tau$, and, since υ is GT_D, $\mathrm{pd}_\upsilon(z) = \mathrm{pd}_\tau(z)$ is closed for each $z \in Z_\tau$. Consequently $\upsilon = \tau$. ■

The above characterisation does indeed reduce to the characterisation for minimal T_0-topologies when one considers countable X's.

An investigation of minimal $GT_{\frac{1}{2}}$-topologies is recorded below firstly for uncountable X and then for countable X. We shall make use of the following result of J.R. Porter [15, theorem 3.5]. Recall that an E_0-space is a topological space with the property that each of its points is a G_δ-point.

Theorem 4.5 ([15]). *A topology $\sigma \in LT(X)$ is minimal E_0 if and only if X is countable and σ is the cofinite topology, $\mathscr{C}$.*

Fix an uncountable set X. For each $\tau \in LT(X)$, let Y_τ be the set of all $x \in X$ which are not τ-closed.

Proposition 4.6. *If $\tau \in LT(X)$ is minimal $GT_{\frac{1}{2}}$, then $Y = Y_\tau$ is τ-open.*

Proof. Suppose for a contradiction that there is an $\eta \in Y$ (necessarily a non-open G_δ-point) such that $\eta \in \mathrm{cl}_\tau(X - Y)$. The singleton $\{\eta\}$ is the intersection of a strictly decreasing sequence, $\{U_n\}_{n\in\omega}$, of τ-open sets. We may assume that the difference between each pair, U_n and U_{n+1}, is witnessed by a (τ–closed) $\xi_n \in X - Y$.

For each $n \geq 1$, define

$$\tau_n = (\mathscr{E}(\xi_n) \cap \tau_{n-1}) \vee \bigvee \{\{X - \{x\}, \varnothing, X\} : x \in X - Y\}$$

and $\tau_0 = \tau$. Clearly, for all $n \in \omega$, $\tau_{n+1} \subseteq \tau_n$. If $x \in X - Y$, then it is closed in τ_n for all n. Suppose $y \in Y$. Then it cannot be τ_n-closed for any n since this would imply that it is τ-closed. If it is a $G_\delta(\tau_n)$-point, then it is easily seen to be a $G_\delta(\tau_{n+1})$-point also. Therefore, Y consists of all the points of X which are not closed in all τ_n. At the same time, it is also the set of all points of X which are not closed in some τ_n. Moreover, each τ_n is $GT_{\frac{1}{2}}$.

We now note that, for all n, $U_n \in \tau_n$ and $U_n \notin \mathscr{E}(\xi_n) \cap \tau_n$. Since the topology τ_0 is minimal $GT_{\frac{1}{2}}$, it must be the case that $U_n \in \tau_{n+1}$ and is cofinite (the only τ_{n+1}-open sets containing ξ_n are cofinite). Clearly, this cannot happen for all n in this uncountable space X. ■

Proposition 4.7. *Let $\tau \in LT(X)$ be minimal $GT_{\frac{1}{2}}$, and $U \in \tau$. Then either*

(1) *U contains Y and is cofinite; or*
(2) *U is contained in Y.*

Proof. Define the following weaker topology $\upsilon \in LT(X)$. It is generated by sets of the form $X - \{x\}$ (where $x \in X - Y$) and $G \cap Y$ (where G is a τ-open set). Note that Y is an open set. Then clearly this topology is $GT_{\frac{1}{2}}$, and is coarser than τ. Therefore it is τ, by minimality. This topology does indeed satisfy (1) and (2). ■

Theorem 4.8. *Let* X *be an uncountable set,* $\tau \in LT(X)$*. Then* τ *is minimal* $GT_{\frac{1}{2}}$ *if and only if*

(1) τ *is* $GT_{\frac{1}{2}}$*;*
(2) *the set* $Y = \{x \in X : X - \{x\} \notin \tau\}$ *is open;*
(3) *for all* $U \in \tau$*, either* U *contains* Y *and is cofinite, or* U *is contained in* Y*; and*
(4) *the subspace* Y *has the weakest topology such that each point is a* G_δ*.*

Proof. Suppose that τ is minimal $GT_{\frac{1}{2}}$. Then (1), (2) and (3) follow from the above propositions. Clause (4) is immediate by minimality of τ.

Conversely, suppose that (1)–(4) hold. If υ is any strictly coarser topology, then by (3) this is witnessed by: either an open set contained in Y, in which case a point of Y would not be υ-closed nor can it be a $G_\delta(\upsilon)$-point by (4); or an open set containing Y and cofinite in X, in which case a point outside Y ceases to be closed or G_δ. Therefore υ cannot be $GT_{\frac{1}{2}}$. ■

From Porter's theorem (Theorem 4.5) the following is true.

Corollary 4.9. *Let* X *be an uncountable set,* $\tau \in LT(X)$*. Then* τ *is minimal* $GT_{\frac{1}{2}}$ *if and only if*

(1) τ *is* $GT_{\frac{1}{2}}$*;*
(2) *the set* $Y = \{x \in X : X - \{x\} \notin \tau\}$ *is open;*
(3) *for all* $U \in \tau$*, either* U *contains* Y *and is cofinite, or* U *is contained in* Y*; and*
(4) *the subset* Y *is countable and* $\tau_Y = \mathscr{C}$*.*

Recall that a space (X, τ) is T_F if, for all $x \in X$, either $\{x\}$ is an intersection of open sets or $\{x\}$ is closed (see [1] and [6]). By the above result, if X is uncountable, then X is minimal $GT_{\frac{1}{2}}$ if and only if X is $GT_{\frac{1}{2}}$ and minimal with respect to the property of being T_D and T_F [8, section 3]. We now consider the case when X is countable.

Theorem 4.10. *Let* (X, τ) *be a countable space. Let* Y *be the set of all those* $x \in X$ *that are not* τ*-closed. Then the following are equivalent:*

(1) (X, τ) *is minimal* $GT_{\frac{1}{2}}$*;*
(2) (X, τ) *is minimal* T_F*; and*
(3) *if* Y *is a singleton then* $\tau = \mathscr{C} \cap \mathscr{I}(Y)$*; and if* $|Y| > 1$*, then, for all* $U \in \tau$*, either* $U \subseteq Y$ *and* $|Y - U| < \aleph_0$*, or* $U \supseteq Y$ *and* $|X - U| < \aleph_0$*.*

Proof. Statements (1) and (2) are equivalent since (when X is countable) (X, τ) is $GT_{\frac{1}{2}}$ if and only if it is T_F. Statement (3) is the characterisation of minimal T_F-spaces as described in [6, theorem 2]. ■

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