TWISTED CENTRAL PRODUCTS AND HAMILTONIAN GROUPS

By

Russell J. Higgs*
Department of Mathematics, University College Dublin

and

John F. Humphreys
Department of Mathematical Sciences, University of Liverpool

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Abstract

In this note the representation theory of the twisted central product of two finite groups is used to find the degrees of the irreducible projective representations of Hamiltonian 2-groups for all possible cohomology classes of the groups.

1. Introduction

For fuller details of the concepts mentioned in this section the reader is referred to the two books by Karpilovsky [6; 7], although all of the ideas mentioned here were originally due to Schur [8; 9]. All groups considered in this paper are finite.

Let G be a group. Then $M(G)$ is the Schur multiplier of G, and $\mathrm{Proj}(G, \alpha)$ denotes the set of all irreducible projective characters of G with 2-cocycle α.

Now suppose that G is a p-group. Then we say that G has *projective character degree pattern* $(x_0, x_1, x_2, \ldots)$ if there exists a 2-cocycle α such that $\mathrm{Proj}(G, \alpha)$ contains exactly x_i elements of degree p^i for all i. The degrees of the elements of $\mathrm{Proj}(G, \alpha)$ are unaffected by the choice of 2-cocycle from $[\alpha] \in M(G)$. Thus we say that the projective character degree pattern occurs with *multiplicity* n, if there are exactly n distinct cohomology classes for which G has the pattern.

A *Hamiltonian* group is a non-abelian group in which every subgroup is normal. Such groups were classified in [2, theorem 12.5.4]:

G is a Hamiltonian group if and only if G is isomorphic to $Q \times C_2^n \times A$, where Q is the quaternion group of order 8, *C_2^n denotes the elementary abelian group of order* 2^n, *and A is an abelian group of odd order.*

Let G_1 and G_2 be groups of coprime order. Then $M(G_1 \times G_2) \cong M(G_1) \times M(G_2)$, and it follows from [6, theorem 7.1.13] that $\{\xi \times \zeta : \xi \in \mathrm{Proj}(G_1, \alpha_1), \zeta \in \mathrm{Proj}(G_2, \alpha_2)\} = \mathrm{Proj}(G_1 \times G_2, \alpha_1\alpha_2)$. Thus we may regard $\mathrm{Proj}(G_1 \times G_2, \alpha)$ as having been constructed in this manner for all $[\alpha] \in M(G_1 \times G_2)$. Let $[\beta] \in M(B)$, where B is an abelian group. Then from [1, section 3] the elements of $\mathrm{Proj}(B, \beta)$ all have the same degree, which can be found for any given B and β. For these reasons we

* Corresponding author, e-mail: russel.higgs@ucd.ie

will find the projective character degree patterns of certain 2-groups in this paper, namely $Q \times C_2^n$ and $D_4 \times C_2^n$ for $n \geq 0$, where D_4 denotes the dihedral group of order 8. The groups Q and D_4 are related in the sense that their character tables are identical. On the other hand $M(Q)$ is trivial, whereas $M(D_4) \cong C_2$.

The motivation for this paper is that generally it is very difficult to describe the degrees of the irreducible projective representations of an infinite family of groups, the two exceptions to date being abelian and symmetric groups. In this paper we deal with some groups which are 'nearly' abelian, and using both the techniques of [1] and [5] are able to add two further families of groups, namely $D_4 \times C_2^n$ and $Q \times C_2^n$ for $n \geq 0$, to the list of those with known irreducible projective character degrees.

We will begin section 2 of this paper by reviewing the representation theory of the twisted central product of two groups. This is then used to explicitly describe the projective character tables of the groups $D_4 \times C_2$ and $Q \times C_2$. In section 3, using these examples as models, we will describe all of the projective character degree patterns of $D_4 \times C_2^n$ and $Q \times C_2^n$ for $n \geq 0$. The crucial difference between these two groups is that $U(Q \times C_2^n) = (Q \times C_2^n)'$, whereas $U(D_4 \times C_2^n)$ is trivial; here $U(G)$ denotes the central elements of G which are α-regular for all $[\alpha] \in M(G)$. This result firstly implies that $|\mathrm{Proj}(Q \times C_2^n, \alpha)| \geq 2$ for all $[\alpha] \in M(Q \times C_2^n)$, and secondly that if equality occurs the two elements of $\mathrm{Proj}(Q \times C_2^n, \alpha)$ are constructed from self-associate characters of a covering group.

2. Twisted central products

The main tool we will use in this section is the twisted central product $G \,\tilde{\curlyvee}\, H$ of two groups G and H, which is discussed in detail in [5].

Let $\mathscr{G}$ denote the category of triples (G, s, w), where G is a finite group, s is a group homomorphism from G to $\mathbf{Z}_2$, and $w \in Z(G)$ of order 2 with $s(w) = 0$. We call $\langle w \rangle$ the *distinguished central subgroup* of G. A morphism in $\mathscr{G}$ from (G, s, w) to (H, t, z) is a group homomorphism f from G to H with $f(w) = z$ and $s(g) = t(f(g))$ for all $g \in G$. We define an irreducible representation R of G to be *w-positive* (or just *positive* if no ambiguity exists) if the restriction of R to $\langle w \rangle$ is a multiple of the trivial representation, and to be *w-negative* otherwise.

Let (G, s, w_1) and (H, t, w_2) be objects in $\mathscr{G}$, and $\langle z \rangle$ be a group of order 2. We consider the subgroups $G_0 = \langle (g, 1, z) : g \in G \rangle$ and $H_0 = \langle (1, h, z) : h \in H \rangle$ of the direct product $G \times H \times \langle z \rangle$. We make G_0 into an object in $\mathscr{G}$ by taking the distinguished central subgroup to be $\langle (1, 1, z) \rangle$ and defining the homomorphism $s_0 : G_0 \to \mathbf{Z}_2$ by $s_0(g, 1, z^j) = s(g)$. Similarly H_0 may be made into an object in $\mathscr{G}$. We now define K to be the group which is the Cartesian product $G \times H \times \langle z \rangle$, together with the binary operation given by

$$(g_1, h_1, z^i)(g_2, h_2, z^j) = (g_1 g_2, h_1 h_2, z^{s_0(g_2) t_0(h_1)} z^{i+j}).$$

It is easy to check that K is $\mathscr{G}$-isomorphic to $(G \times \langle z \rangle) \,\tilde{\curlyvee}\, (H \times \langle z \rangle)$ where we have identified G and H with the obvious subgroups of K. Finally we note that if $R = \langle (w_1, 1, 1), (1, w_2, 1), (1, 1, z) \rangle$, then $K/R \cong G/\langle w_1 \rangle \times H/\langle w_2 \rangle$.

Suppose that we are considering a group which is a twisted central product

TABLE 1—Character table of D_8.

1	x^4	x	x^2	x^3	y	yx
1	1	1	1	1	1	1
1	1	1	1	1	-1	-1
1	1	-1	1	-1	1	-1
1	1	-1	1	-1	-1	1
2	2	0	-2	0	0	0
2	-2	$\sqrt{2}$	0	$-\sqrt{2}$	0	0
2	-2	$-\sqrt{2}$	0	$\sqrt{2}$	0	0

TABLE 2—Projective character table of D_4.

1	x
2	$\sqrt{2}$
2	$-\sqrt{2}$

$G \tilde{\curlyvee} H$ of two objects in $\mathscr{G}$. Then there is a 'twisted tensor product' construction which yields negative irreducible representations of our twisted central product from the negative irreducible representations of G and H. If P and Q are negative irreducible representations of G and H respectively, then this twisted tensor product $P\tilde{\otimes}Q$ has dimension equal to the product of the dimensions of P and Q except when they are both non-self-associate. In the exceptional case P^a and Q^a are not equivalent to P and Q respectively, and $P\tilde{\otimes}Q$ has dimension equal to twice the product of the dimensions of P and Q. It is also possible to determine in all cases the character values of the twisted tensor product construction; the details are given in [5].

We now apply these ideas to construct the projective character tables of the group $D_4 \times C_2$ and $Q \times C_2$.

Example 2.1. Consider the construction of a representation group for $D_4 \times C_2$, where

$$D_n = \langle x, y : x^n = y^2 = 1, yxy = x^{-1} \rangle \text{ and } C_2 = \langle z : z^2 = 1 \rangle.$$

We note that $M(D_4) \cong C_2$ and $M(C_2) = \langle 1 \rangle$. Also a representation group for D_4 is D_8, which has Table 1 as its character table. Thus the projective character tables of D_8 are the character table of D_4 and Table 2. Now since $D_8' = \langle x^2 \rangle$, the cosets of the quotient group D_8/D_8' have representatives 1, x, y and xy. Thus J in the theory of [3, section 3] is determined by two elements $j_1 = j_{x,z}$ and $j_2 = j_{y,z}$, with $j_{xy,z} = j_1 j_2$. Also $j_1^2 = j_{x,z}^2 = j_{x^2,z} = 1$ and similarly $j_2^2 = 1$, so that j_1 and j_2 are central elements of order 2. Thus a presentation for a representation group K for the direct product

TABLE 3—A projective character table of $D_4 \times C_2$.

1	x	xz	z
2	$\sqrt{2}$	$\sqrt{2}$	2
2	$-\sqrt{2}$	$-\sqrt{2}$	2
2	$-\sqrt{2}$	$-\sqrt{2}$	-2
2	$\sqrt{2}$	$\sqrt{2}$	-2

TABLE 4—A projective character table of $D_4 \times C_2$.

1	x^2	x	xz
2	-2	0	$-2i$
2	-2	0	$2i$
2	2	2	0
2	2	-2	0

$D_4 \times C_2$ is

$$\langle x, y, z, j_1, j_2 : x^8 = y^2 = z^2 = 1, yxy = x^{-1}, [x, z] = j_1, [y, z] = j_2 \rangle$$

where j_1, j_2 and x^4 are central elements of order 2. We now use this group to construct the projective characters of $D_4 \times C_2$ corresponding to distinct elements of $M(G)$ by factoring out K in turn by the eight subgroups of order 4 or 8 of $\langle x^4, j_1, j_2 \rangle$.

(1) K/N, where $N = \langle x^4, j_1, j_2 \rangle$. This quotient group is obviously isomorphic to $D_4 \times C_2$.

(2) K/N, where $N = \langle j_1, j_2 \rangle$. This quotient group is clearly isomorphic to $D_8 \times C_2$, in which x^4 is a central element of order 2. The projective character table of $D_4 \times C_2$ corresponding to the x^4-negative character table of $D_8 \times C_2$ is given in Table 3.

(3) K/N, where $N = \langle x^4, j_1 \rangle$. It is clear that in this case the quotient group has presentation

$$\langle x, y, z, j_2 : x^4 = y^2 = z^2 = [x, z] = 1, yxy = x^{-1}, [y, z] = j_2 \rangle,$$

where j_2 is a central element of order 2. This group can be expressed as a twisted central product of $D_4 \times C_2 = \langle x, y, j_2 \rangle$ with $C_2 \times C_2 = \langle z, j_2 \rangle$. The first of these has the structure of a group in $\mathcal{G}$ by taking $\ker s = \langle x, j_2 \rangle$; the corresponding homomorphism in the second group has $\ker t = \langle j_2 \rangle$. It is now clear that K/N is isomorphic to a twisted central product in this case. The projective character table of $D_4 \times C_2$ corresponding to the negative character table of K/N is given in Table 4.

(4) K/N, where $N = \langle x^4, j_2 \rangle$. This case is essentially the same as (2), but with the roles of j_1 and j_2 interchanged, so again the quotient group K/N

TABLE 5—A projective character table of $D_4 \times C_2$.

1	x^2	y	yz
2	-2	0	2
2	-2	0	-2
2	2	2	0
2	2	-2	0

TABLE 6—A projective character table of $D_4 \times C_2$.

1	x^2z	x	xz
2	$2i$	$\sqrt{2}$	$i\sqrt{2}$
2	$2i$	$-\sqrt{2}$	$-i\sqrt{2}$
2	$-2i$	$\sqrt{2}$	$-i\sqrt{2}$
2	$-2i$	$-\sqrt{2}$	$i\sqrt{2}$

is isomorphic to a twisted central product of $D_4 \times C_2 = \langle x, y, j_1\rangle$ with $C_2 \times C_2 = \langle z, j_1\rangle$. This time we regard the groups $D_4 \times C_2$ and $C_2 \times C_2$ as objects in $\mathscr{G}$ by taking $\ker s = \langle x^2, y, j_1\rangle$ and $\ker t = \langle j_1\rangle$ respectively. The projective character table of $D_4 \times C_2$ in this case is given in Table 5.

(5) K/N, where $N = \langle x^4, j_1 j_2\rangle$. In this case the quotient group has presentation

$$\langle x, y, z, w : x^4 = y^2 = z^2 = 1, yxy = x^{-1}, [x, z] = [y, z] = w\rangle,$$

where w is a central element of order 2. This group can also be expressed as a twisted central product of $D_4 \times C_2 = \langle x, y, w\rangle$ with $C_2 \times C_2 = \langle z, w\rangle$. This time in $D_4 \times C_2$ we take $\ker s = \langle x^2, yx, w\rangle$, and in $C_2 \times C_2$ we take $\ker t = \langle w\rangle$. We see that the multiplication on the twisted central product gives us $[x, z] = [y, z] = w$, so that this group is again isomorphic to K/N. In this case the projective character table of $D_4 \times C_2$ is the same as Table 5 but with y and yz replaced by xy and xyz respectively.

(6) K/N, where $N = \langle j_1, j_2 x^4\rangle$. Here let $D_8 = \langle x, y\rangle$ regarded as an object in $\mathscr{G}$ by taking $\ker s = \langle x\rangle$; also give $C_2 \times C_2 = \langle z, j_2\rangle$ the structure of an object in $\mathscr{G}$ by taking $\ker t = \langle j_2\rangle$. It is clear that K/N is then isomorphic to a twisted central product of D_8 with $C_2 \times C_2$. The two irreducible negative representations of D_8 are then each self-associate, since x is in $\ker s$, so the negative representations of K/N are all of degree 2, and the corresponding projective character table of $D_4 \times C_2$ is given in Table 6.

(7) Next take $N = \langle j_2, j_1 x^4\rangle$ and repeat the discussion in (6) with j_1 and j_2 interchanged to see that K/N is again isomorphic to a twisted central product of D_8 with $C_2 \times C_2$. However, in this case the kernel of s on D_8 is the subgroup generated by y and x^2. This means that the two irre-

TABLE 7—Character table of Q.

1	x^2	x	y	xy
1	1	1	1	1
1	1	−1	1	−1
1	1	−1	−1	1
1	1	1	−1	−1
2	−2	0	0	0

ducible negative representations of D_8 are associates of one another, and so the twisted central product has one irreducible negative representation of degree 4.

(8) $K/\langle j_1x^4, j_2x^4\rangle$. In this case, the quotient group has j_1 and j_2 each identified with x^4, so is again a twisted central product of D_8 with $C_2 \times C_2$, with xy centralising z. Here D_8 is given its $\mathcal{G}$ structure by taking $\ker s$ to be $\langle x^2, xy\rangle$. This means that the two irreducible negative representations of D_8 are associates of one another, and so the twisted central product has one irreducible negative representation of degree 4.

Thus we have shown in particular that $D_4 \times C_2$ has Schur multiplier elementary abelian of order 8, and its projective character degree patterns are $(8, 2)$ with multiplicity one, $(0, 2)$ with multiplicity five, and $(0, 0, 1)$ with multiplicity two.

Example 2.2. Consider the construction of a representation group for $Q \times C_2$, where

$$Q = \langle x, y : x^4 = 1, y^2 = x^2, y^{-1}xy = x^{-1}\rangle \text{ and } C_2 = \langle z : z^2 = 1\rangle.$$

We note that $M(Q) = \langle 1\rangle$. So a representation group for Q is Q, which has Table 7 as its character table. We now leave the reader, using Example 2.1 as a model, to check that a presentation for a representation group for the direct product $Q \times C_2$ is

$$\langle x, y, z, j_1, j_2 : x^4 = z^2 = 1, y^2 = x^2, y^{-1}xy = x^{-1}, [x, z] = j_1, [y, z] = j_2\rangle,$$

where j_1 and j_2 are central elements of order 2, and also that the projective character tables of $Q \times C_2$ are identical to those in Example 2.1 (1), (3), (4) and (5). We note for future reference that these are precisely the tables of $D_4 \times C_2$ for which x^2 is α-regular.

Thus we have shown in particular that $Q \times C_2$ has Schur multiplier elementary abelian of order 4, and its projective character degree patterns are $(8, 2)$ with multiplicity one, and $(0, 4)$ with multiplicity three.

Using the above examples as guides, we will ultimately be able to describe the degrees of all the irreducible projective characters of $D_4 \times C_2^n$ and $Q \times C_2^n$ for $n \geq 0$.

Lemma 2.3. *Suppose that* $K = (G, s, z)\tilde{\curlyvee}(H, t, z)$. *Let* χ *and* χ' *be irreducible negative characters of* G *such that* $\chi' = \lambda\chi$ *for some linear representation* λ *of* G. *Suppose also* *that ψ and ψ' are irreducible negative characters of H such that $\psi' = \mu\psi$ for some linear representation μ of H. Then $(\widetilde{\chi'\psi'})(gh) = \lambda(g)\mu(h)(\widetilde{\chi\psi})(gh)$ for all $g \in G$ and all $h \in H$, where $\widetilde{\chi\psi}$ denotes the irreducible negative character of K constructed from χ and ψ.*

PROOF. We first note that χ and χ' are both self-associate (SA) or both non-self-associate (NSA), since $\chi' = \lambda\chi$, and similarly for ψ and ψ'. We consider the four possibilities which can arise, with reference to [5, table 5.7].

(1) χ and ψ are SA.
In this case

$$(\widetilde{\chi'\psi'})(gh) = \chi'(g)\psi'(h) = \lambda(g)\chi(g)\mu(h)\psi(h) = \lambda(g)\mu(h)(\widetilde{\chi\psi})(gh)$$

if $\sigma(g) = 0 = \sigma(h)$, and $(\widetilde{\chi'\psi'})(gh) = 0 = (\widetilde{\chi\psi})(gh)$ otherwise.

(2) χ and ψ are NSA.
In this case

$$(\widetilde{\chi'\psi'})(gh) = 2\chi'(g)\psi'(h) = 2\lambda(g)\chi(g)\mu(h)\psi(h) = \lambda(g)\mu(h)(\widetilde{\chi\psi})(gh)$$

if $\sigma(g) = 0 = \sigma(h)$, and $(\widetilde{\chi'\psi'})(gh) = 0 = (\widetilde{\chi\psi})(gh)$ otherwise.

(3) χ is SA and ψ is NSA.
This is the same as (1) unless $\sigma(g) = 0 \neq \sigma(h)$. In the case $\sigma(g) = 0 \neq \sigma(h)$, $\chi_K = \zeta + \zeta^c$, where K is the kernel of s and ζ^c is the distinct G-conjugate of the irreducible character ζ. Now $\chi'_K = \lambda_K\chi_K = \lambda_K\zeta + \lambda_K\zeta^c = \lambda_K\zeta + (\lambda_K\zeta)^c$, since λ_K is G-invariant. Thus

$$\begin{aligned}(\widetilde{\chi'\psi'})(gh) &= (\lambda_K(g)\zeta(g) - \lambda_K(g)\zeta^c(g))\psi'(h)\\ &= \lambda(g)\mu(h)(\zeta(g) - \zeta^c(g))\psi(h) = \lambda(g)\mu(h)(\widetilde{\chi\psi})(gh).\end{aligned}$$

(4) χ is NSA and ψ is SA.
This is the same as case 3 but with the roles of χ and ψ interchanged.

Finally we note that the case when $st = 0$ is covered by the above using the remarks following [5, table 5.7]. ■

The application of this result to projective characters is as follows. Two projective characters of a group G are said to be *projectively equivalent* if they are in the same orbit under the action (by multiplication) of the linear representations of G on the projective characters of G.

Using the notation of Lemma 2.3, let ξ and ξ' be the two projective characters of $G/\langle z\rangle$ linearised by χ and χ'; then ξ and ξ' are projectively equivalent. Similarly, if ζ and ζ' are the two projective characters of $H/\langle z\rangle$ linearised by ψ and ψ', then ζ and ζ' are also projectively equivalent. Lemma 2.3 then states that the projective characters of $G/\langle z\rangle \times H/\langle z\rangle$ linearised by $\widetilde{\chi'\psi'}$ and $\widetilde{\chi\psi}$ are also projectively equivalent. The remaining results in this paper all have similar interpretations about the projective characters of $G/\langle z\rangle \times H/\langle z\rangle$.

Corollary 2.4. *Suppose that $K = (G, s, z)\tilde{\curlyvee}(H, t, z)$. Let A be a subgroup of the linear* *representations of $G/\langle z\rangle$ regarded as linear representations of G. Let B be a subgroup of the linear representations of $H/\langle z\rangle$ regarded as linear representations of H. Suppose that A has m orbits on the irreducible negative characters of G and B has n orbits on the irreducible negative characters of H. Then the group AB of linear representations of $G/\langle z\rangle \times H/\langle z\rangle$ of the form $\lambda\mu$ for $\lambda \in A$ and $\mu \in B$, regarded as linear representations of K, have $\leq mn$ orbits on the irreducible negative characters of K.*

Proof. By [5, theorem 5.9] the irreducible negative characters of K are all of the form $\widetilde{\chi\psi}$, where χ is an irreducible negative character of G and ψ is an irreducible negative character of H. However, by Lemma 2.3 if χ and χ' are in the same A-orbit, and ψ and ψ' are in the same B-orbit, then $\widetilde{\chi\psi}$ and $\widetilde{\chi'\psi'}$ are in the same AB-orbit. Thus AB has at most mn orbits on the irreducible negative characters of K. ■

3. Projective character degree patterns

We begin this section by considering the irreducible characters of $T \times C_2^n$, where T is one of the two groups D_4 or Q. These are obtained by multiplying the irreducible characters of T by the linear representations of C_2^n. It follows that the linear representations of $T \times C_2^n$ act transitively on the linear representations of $T \times C_2^n$, and on the remaining set of irreducible characters of degree 2. This result essentially extends to the situation when we are dealing with projective characters of $T \times C_2^n$, as shown below.

Proposition 3.1. *Let α be a 2-cocycle of $T \times C_2^n$, where T is one of the two groups D_4 or Q. Then the linear representations of $T \times C_2^n$ have at most two orbits on* $\mathrm{Proj}(T \times C_2^n, \alpha)$.

Proof. We proceed by induction on n. The result is true for $n = 0$ or $n = 1$ from Example 2.1 for $T = D_4$ or Example 2.2 for $T = Q$.

Let $R(n)$ denote the set of objects (H, t, z) in $\mathscr{G}$ with $H/\langle z\rangle \cong T \times C_2^n$.

Let $(G, s, w) \in R(n-1)$ and $\langle z\rangle$ be a group of order 2. Then the set of irreducible characters of $(G \times \langle z\rangle)\,\tilde{\curlyvee}\,(C_2 \times \langle z\rangle)$ which are w-positive but z-negative may be regarded as the z-negative characters of $\big((G \times \langle z\rangle)\,\tilde{\curlyvee}\,(C_2 \times \langle z\rangle)\big)/\langle w\rangle$. If σ denotes the homomorphism of $(G \times \langle z\rangle)\,\tilde{\curlyvee}\,(C_2 \times \langle z\rangle)$ constructed from the component homomorphisms in the twisted central product, then $\big((G \times \langle z\rangle)\tilde{\curlyvee}(C_2 \times \langle z\rangle)\big)/\langle w\rangle$ can be made naturally into an object in $\mathscr{G}$, and hence into an element of $R(n)$, by defining the kernel of the homomorphism to be the image of $\ker\sigma$ in the quotient group. Also the set of irreducible characters of $(G \times \langle z\rangle)\tilde{\curlyvee}(C_2 \times \langle z\rangle)$ which are w-negative and z-negative may be regarded as the z'-negative characters of $\big((G \times \langle z\rangle)\tilde{\curlyvee}(C_2 \times \langle z\rangle)\big)/\langle wz\rangle$, where $\langle z'\rangle = \langle w, z\rangle/\langle wz\rangle$. Again $\big((G \times \langle z\rangle)\,\tilde{\curlyvee}\,(C_2 \times \langle z\rangle)\big)/\langle wz\rangle$ may be made into an object in $\mathscr{G}$, and hence into an element of $R(n)$, by defining the kernel of the homomorphism to be the image of $\ker\sigma$ in the quotient group and noting that $\langle w, z\rangle \leq \ker\sigma$.

Conversely, by [3, corollary 3.4] the negative characters of any element of $R(n)$ may be constructed as above, starting with some element of $R(n-1)$.

Continuing with the above notation, we have already said that the group of linear representations of G lifted from the linear representations of $G/\langle w\rangle$ have exactly two orbits on the w-positive irreducible characters of G, so that the group of linear representations of $G \times \langle z \rangle$ lifted from the linear representations of $(G \times \langle z \rangle)/\langle w, z \rangle$ have two orbits on the w-positive, z-negative irreducible characters of $G \times \langle z \rangle$. Clearly the linear representations of $C_2 \times \langle z \rangle$ have just one orbit on the z-negative representations of $C_2 \times \langle z \rangle$. We conclude by Corollary 2.4 and the above remarks that the group of linear representations of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle w, z \rangle$ regarded as linear representations of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle w \rangle$ have at most two orbits on the z-negative irreducible characters of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle w \rangle$.

In the other case the inductive hypothesis gives that the group of linear representations of G lifted from the linear representations of $G/\langle w \rangle$ have at most two orbits on the w-negative irreducible characters of G, so that the group of linear representations of $G \times \langle z \rangle$ lifted from the linear representations of $(G \times \langle z \rangle)/\langle w, z \rangle$ have at most two orbits on the w-negative, z-negative irreducible characters of $G \times \langle z \rangle$. We conclude by Corollary 2.4 and the above remarks that the group of linear representations of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle w, z \rangle$ regarded as linear representations of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle wz \rangle$ have at most two orbits on the z'-negative irreducible characters of $\big((G \times \langle z \rangle) \tilde{\curlyvee} (C_2 \times \langle z \rangle)\big)/\langle wz \rangle$. We thus have the desired conclusion by translating these results about covering groups and characters into those about $T \times C_2^n$ and projective characters. ∎

Corollary 3.2. *Let T be one of the two groups D_4 or Q. Then the projective character degree patterns of $G = T \times C_2^n$ are all of the form:*

(1) $x_i = 2^{n+2-2i}$, $x_{i+1} = 2^{n-2i}$, *and* $x_j = 0$ *for* $j \neq i, i+1$, *for some* $i = 0, \dots, [n/2]$;

(2) $x_i = 2^{n+3-2i}$, *and* $x_j = 0$ *for* $j \neq i$, *for some* $i = 1, \dots, [(n+3)/2]$.

If case 1 *occurs for some* 2*-cocycle* α, *then the linear representations of* G *act transitively on each of the two subsets of* $\mathrm{Proj}(G, \alpha)$ *containing projective characters of the same degree.*

If case 2 *occurs then either the linear representations of* G *act transitively on* $\mathrm{Proj}(G, \alpha)$ *or there are two orbits of the same length.*

PROOF. By Proposition 3.1 the linear representations of G have at most two orbits on $\mathrm{Proj}(G, \alpha)$ and each orbit has 2-power length. Next we note that G has an abelian subgroup of index 2, which implies by [4, lemma 2.4] that the degree of the elements of $\mathrm{Proj}(G, \alpha)$ are chosen from $\{2^i, 2^{i+1}\}$, for some integer i with $0 \leq i \leq [n/2]$ if two degrees occur, or $1 \leq i \leq [(n+3)/2]$ if just one degree occurs. In the first case let $x_i = 2^a$ and $x_{i+1} = 2^b$; then we require that $2^a 2^{2i} + 2^b 2^{2i+2} = 2^{n+3}$, from which we conclude that $a = b + 2$ and $b = n - 2i$. In the second case suppose that the linear representations of G have two orbits on $\mathrm{Proj}(G, \alpha)$, which all have the same degree; then a similar argument to that just given shows that the orbit lengths must be equal. ∎

In fact, as we will show below, we have exactly described the projective character degree patterns of $D_4 \times C_2^n$ in Corollary 3.2, but for $Q \times C_2^n$ there does not exist a 2-cocycle α for which $|\mathrm{Proj}(Q \times C_2^n, \alpha)| = 1$. This is a consequence of the following result, which represents a fundamental difference between the two groups. Let G be a group; then $U(G)$ denotes the central elements of G which are α-regular for all $[\alpha] \in M(G)$.

Lemma 3.3. $U(T \times C_2^n) = \begin{cases} \langle x^2 \rangle, & \textit{if } T = Q; \\ \langle 1 \rangle, & \textit{if } T = D_4. \end{cases}$

Proof. This is immediate from [3, proposition 3.6] and the fact that $U(Q) = Z(Q)$, since $M(Q)$ is trivial, whereas $U(D_4)$ is trivial from Example 2.1. ■

This result firstly implies that $|\text{Proj}(Q \times C_2^n, \alpha)| \geq 2$ for all $[\alpha] \in M(Q \times C_2^n)$, and secondly that if equality occurs the two elements of $\text{Proj}(Q \times C_2^n, \alpha)$ are constructed from self-associate characters of a covering group.

Theorem 3.4. *The projective character degree patterns of* $D_4 \times C_2^n$ *are:*

(1) $x_i = 2^{n+2-2i}$, $x_{i+1} = 2^{n-2i}$, *and* $x_j = 0$ *for* $j \neq i, i+1$, *for* $i = 0, \ldots, [n/2]$;

(2) $x_i = 2^{n+3-2i}$, *and* $x_j = 0$ *for* $j \neq i$, *for* $i = 1, \ldots, [(n+3)/2]$.

Proof. Let $R(n)$ once again denote the set of objects (G, s, w) in $\mathscr{G}$ with $G/\langle w \rangle \cong D_4 \times C_2^n$ for $n \geq 0$. Then we will prove the following statements by induction on n, from which the theorem directly follows. If $(G, s, w) \in R(n)$ then the irreducible negative characters of G are as follows, with all possible degree and associativity structures occurring for at least one $(G, s, w) \in R(n)$:

(1) 2^{n-2i} of degree 2^{i+1} and 2^{n+2-2i} of degree 2^i, the possible values of i being $0, \ldots, [n/2]$. The possible associativity structures are that the characters are either (a) all self-associate (SA); (b) all non-self-associate (NSA) if $n-2i \geq 1$; or (c) just those of degree 2^{i+1} are SA for $n - 2i \geq 0$.

(2) 2^{n+3-2i} of degree 2^i, the possible values of i being $1, \ldots, [(n+3)/2]$. The possible associativity structures are that the characters are either (a) all SA; (b) all NSA if $n + 3 - 2i \geq 1$; or (c) half are SA and half are NSA if $n + 3 - 2i \geq 2$.

The result is true for $n = 0$ and also for $n = 1$ from Example 2.1. Assume that the result is true for $n-1$. Then it follows as in the proof of Proposition 3.1 that all the irreducible negative characters of elements of $R(n)$ can be found by considering for all $(G, s, w) \in R(n-1)$ the cohort of irreducible characters of $(G \times \langle z \rangle) \tilde{\Upsilon} (C_2 \times \langle z \rangle)$ which are positive with respect to w and negative with respect to z, and the cohort which are negative with respect to both w and z.

The inductive hypothesis and the twisted tensor product construction yield that the cohorts of irreducible characters of $(G \times \langle z \rangle) \tilde{\Upsilon} (C_2 \times \langle z \rangle)$ under consideration are as follows, each possibility of degree structure and associativity occurring for some choice of $(G, s, w) \in R(n-1)$. The numbering and lettering below refer to the z-negative irreducible characters constructed in the twisted central product using those irreducible z-negative characters in the first factor with the same case number and letter in the inductive hypothesis above.

Case 1: The two negative linear representations of $C_2 \times \langle z \rangle$ are SA.

(1) 2^{n+2-2i} of degree 2^i and 2^{n-2i} of degree 2^{i+1} for $i = 0, \ldots, [(n-1)/2]$. These irreducible characters are either all SA, or all NSA if $n-2i \geq 2$, or just those of degree 2^{i+1} are SA for $n - 2i \geq 1$.

(2) 2^{n+3-2i} of degree 2^i for $i = 1, \ldots, [(n+2)/2]$. These irreducible characters are either all SA, or all NSA if $n + 3 - 2i \geq 2$, or half are SA and half are NSA if $n + 3 - 2i \geq 3$.

Case 2: The two negative linear representations of $C_2 \times \langle z \rangle$ are NSA.

(1a) 2^{n+2-2i} NSA irreducible characters of degree 2^i and 2^{n-2i} NSA of degree 2^{i+1} for $i = 0, \ldots, [(n-1)/2]$.

(1b) 2^{n-2i} SA irreducible characters of degree 2^{i+1} and 2^{n-2-2i} SA of degree 2^{i+2} for $i = 0, \ldots, [(n-1)/2]$, provided that $n - 2i \geq 2$. That is, there are 2^{n+2-2j} SA irreducible characters of degree 2^j and 2^{n-2j} SA irreducible characters of degree 2^{j+1} for $j = 1, \ldots, [n/2]$.

(1c) 2^{n+1-2i} irreducible characters of degree 2^{i+1}, of which 2^{n-2i} are SA and 2^{n-2i} are NSA, for $i = 0, \ldots, [(n-1)/2]$.

(2a) 2^{n+3-2i} NSA irreducible characters of degree 2^i for $i = 1, \ldots, [(n+2)/2]$.

(2b) 2^{n+1-2i} SA irreducible characters of degree 2^{i+1} for $i = 1, \ldots, [(n+2)/2]$, provided that $n + 3 - 2i \geq 2$. That is, there are 2^{n+3-2j} SA irreducible characters of degree 2^j for $j = 2, \ldots, [(n+3)/2]$.

(2c) 2^{n+2-2i} NSA irreducible characters of degree 2^i and 2^{n+1-2i} SA irreducible characters of degree 2^{i+1} for $i = 1, \ldots, [(n+2)/2]$, provided that $n+3-2i \geq 3$. So in fact this occurs for $i = 1, \ldots, [n/2]$.

Now in both cases all the degrees and structures are of the required form, and it can easily be checked that the possible required degree and associativity structures of $(G, s, w) \in R(n)$ which do not occur in case 1 do occur in case 2, and this observation means that all the described degree and associativity structures occur for at least one element of $R(n)$. It remains from Corollary 3.2 to show that no other associativity structures can occur. So let M be a subgroup of index 2 in an object $(G, s, w) \in R(n)$. By Corollary 3.2 the linear representations of $G/\langle w \rangle$ have at most two orbits on the irreducible w-negative characters of G. All the irreducible characters in such an orbit are either all SA or all NSA. Thus when only one degree occurs the only possible associativity structures which can occur are that all the characters are SA, or they are all NSA, or using Corollary 3.2 the characters in one orbit are all SA and the characters in the other orbit are all NSA. When two degrees 2^i and 2^{i+1} occur, we have from Corollary 3.2 that either all the characters are SA, or they are all NSA, or only those of degree 2^{i+1} are SA, or only those of degree 2^i are SA. Suppose that the last of these four cases occurs; then repeating the construction given above (with the z-negative representations of $C_2 \times \langle z \rangle$ NSA) will produce a 2-cocycle α of $D_4 \times C_2^{n+1}$ for which $\mathrm{Proj}(D_4 \times C_2^{n+1}, \alpha)$ contains both elements of degree 2^i and of degree 2^{i+2}, contrary to Corollary 3.2.

Thus we have shown as required that each of the possible degree and associativity structures in the inductive statement occur for some $(G, s, w) \in R(n)$, and that only one of these structures occurs for each $(G, s, w) \in R(n)$. ■

Theorem 3.5. *The projective character degree patterns of* $Q \times C_2^n$ *are:*

(1) $x_i = 2^{n+2-2i}$, $x_{i+1} = 2^{n-2i}$, *and* $x_j = 0$ *for* $j \neq i, i+1$, *for* $i = 0, \ldots, [n/2]$, *and* $n \geq 0$;

(2) $x_i = 2^{n+3-2i}$, *and* $x_j = 0$ *for* $j \neq i$, *for* $i = 1, \ldots, [(n+2)/2]$, *and* $n \geq 1$.

PROOF. The proof is essentially the same as that of Theorem 3.4 with two minor differences in the inductive hypothesis. These differences are that the largest possible value of i in (2) will be $[(n+2)/2]$ for $Q \times C_2^n$, whereas it was $[(n+3)/2]$ for $D_4 \times C_2^n$. The only difference in associativity structures occurs in (2b), where all the characters can be NSA for $n+3-2i \geq 2$ for $Q \times C_2^n$, whereas all the characters were NSA for $n+3-2i \geq 1$ for $D_4 \times C_2^n$. ■

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References

[1] R. Frucht, Über die Darstellung endlicher Abelscher Gruppen durch Kollineationen, *Journal für die reine und angewandte Mathematik* **166** (1931), 16–29.
[2] M. Hall, *The theory of groups*, Macmillan, New York, 1959.
[3] R.J. Higgs and J.F. Humphreys, Projective characters of direct products of certain *p*-groups, *Communications in Algebra* **27** (1999), 4493–514.
[4] R.J. Higgs and J.F. Humphreys, Projective character degree patterns of 2-groups, *Communications in Algebra* **28** (2000), 1189–210.
[5] P.N. Hoffman and J.F. Humphreys, *Projective representations of the symmetric groups*, Oxford University Press, Oxford, 1992.
[6] G. Karpilovsky, *Projective representations of finite groups*, Marcel Dekker, New York, 1985.
[7] G. Karpilovsky, *The Schur Multiplier*, Oxford University Press, Oxford, 1987.
[8] I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, *Journal für die reine und angewandte Mathematik* **127** (1904), 20–50.
[9] I. Schur, Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen, *Journal für die reine und angewandte Mathematik* **132** (1907), 85–137.