APPROXIMATE HOMOGENEITY OF C^*-ALGEBRAS WITH FINITELY MANY IDEALS

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Abstract

A number of questions regarding closure properties of the class of approximately homogeneous C^*-algebras were resolved in the negative by our examples with Dădărlat. Since these examples were C^*-algebras with infinitely many ideals, it remains to be determined whether good closure properties can be found for C^*-algebras with only finitely many ideals.

We offer positive results in this direction, based on an analysis of the interplay between the order on K-theory with coefficients and the splitting maps for Künneth sequences provided by Bödigheimer.

1. Introduction

Recent work by Dădărlat and the author has shown that the local criterion for AH algebras fails. To be more precise, let us denote by $\mathbf{H}$ the class of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} p_i\mathbf{M}_{n_i}(C(X_i))p_i,$$

where every X_i is a compact metrisable space and p_i is a projection in $\mathbf{M}_{n_i}(C(X_i))$.

A C^*-algebra A is then *approximately homogeneous* or *AH* if it can be written as a countable inductive limit of algebras from $\mathbf{H}$. A is *locally homogeneous* if for every $\varepsilon > 0$ and every finite set of elements $\mathscr{F} \subseteq A$ there exists a C^*-algebra $B \in \mathbf{H}$ and a $*$-homomorphism $\varphi : B \to A$ with $\mathrm{dist}(\mathscr{F}, \varphi(B)) < \varepsilon$.

It is clear that any approximately homogeneous C^*-algebra is also locally homogeneous. We proved in [6] what had long been expected, that there exists a C^*-algebra which is locally homogeneous but not approximately homogeneous.

In general, a locally homogeneous C^*-algebra must be nuclear. The C^*-algebra constructed in [6] had many other nice properties: it was separable, unital, of real

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rank zero, and of stable rank one. It could also be written as an inductive limit of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(D),$$

where D was a certain C^*-algebra having only irreducible representations of dimensions 1 and 2. However, it was *non-simple*; indeed, it had a rather complicated ideal structure.

There is limited evidence that the implication

$$A \text{ is locally homogeneous} \Longrightarrow A \text{ is } AH$$

could be true under the extra assumption of *simplicity* of A. The strongest result of this type can be found in [14], in which it is proved that every locally homogeneous C^*-algebra which is also simple and separable and which has real rank zero, stable rank one, weak unperforation and a unique trace is in fact AH. Other results, e.g. in [11] and [7], are of the form

$$A \text{ is } AX \Longrightarrow A \text{ is } AH,$$

where AX is defined as the inductive limits over $\mathbf{X}$, for some class of C^*-algebras containing $\mathbf{H}$. Again, such results are based on the extra assumption of simplicity.

The example mentioned above was a counterexample to general statements of both of these kinds. It is the purpose of the present paper to generalise the results of the second category above to the case of finitely many ideals. By doing so, we also demonstrate that the phenomenon exposed in [6] depends essentially on the existence of many ideals in the constructed C^*-algebras. More precisely, we shall prove that every ASH algebra of real rank zero and with slow dimension growth is in fact AH, provided that it has only finitely many ideals.

It seems to be a general principle of C^*-algebra theory that one cannot in any way directly pass from information about simple C^*-algebras to similar results about C^*-algebras with finitely many ideals. The reader who finds this hard to believe can consult [15] for a good example of how much extra effort and machinery goes into understanding the elements of a certain class of C^*-algebras with *one* non-trivial ideal, even though the simple elements of that same class are completely understood.

In particular, there is no obvious way to extend Lin's results even to C^*-algebras with just one non-trivial ideal. Our approach is based on an algebraic analysis of the K-theoretic invariants of the C^*-algebras in question. A direct approach, modelled on that used in [13], could probably also lead to these results, but basing the argument on the invariant gives the advantage that the results will automatically pass to larger classes, as soon as more general classification results become available.

To reach our goals, we must carry out two tasks, both of which are interesting in themselves:

(i) clarify the interrelations between the *ideal splitting* property from [9] and the existence of *IMI*-splittings considered in [6], and
(ii) prove the result stated in [9] that any *ASH* algebra of real rank zero and slow dimension growth has the ideal splitting property. This requires a close analysis of the original proof by Bödigheimer on the existence of coherent families of splitting maps for Künneth sequences.

Combining this with established classification results, we get the result outlined above. Along the way we can also give a new proof of a result by Gong, explaining in terms of the invariants why K-theory with coefficients is redundant in the classification of *AH* algebras without infinitesimals.

2. Notation

We follow the notational conventions of [5] and [6]. In particular, the *ASH* algebras are those given by inductive limits over

$$\left[\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i\right] \oplus \left[\bigoplus_{j=1}^{l} \mathbf{M}_{m_j}(\mathbb{I}^{\sim}_{d_j})\right].$$

When G is an abelian group, $\mathrm{tor}(G)$ is the subgroup of torsion elements, and $G[n]$ the subgroup of elements annihilated by n.

3. Definitions

In this section we shall review notation from [6] and [9]. To show most clearly the similarities between the properties considered in these two papers, we first single out two important properties which are essential in both.

3.1. Coherence and ideal preservation

For every m and n, the group homomorphism

$$\kappa_{n,m} : x + m\mathbb{Z} \mapsto \frac{n}{(n,m)} x + n\mathbb{Z}$$

is a generator of $\mathrm{Hom}(\mathbb{Z}/m, \mathbb{Z}/n)$. One has

$$\kappa_{n,m}\kappa_{m,k} = \frac{m(n,k)}{(n,m)(m,k)} \kappa_{n,k}. \tag{3.1}$$

It is clear that for any abelian group G these homomorphisms induce homomorphisms

$$\underline{\kappa} : G \otimes \mathsf{Z}/m \to G \otimes \mathsf{Z}/n \qquad \overline{\kappa} : \mathrm{Tor}(G, \mathsf{Z}/m) \to \mathrm{Tor}(G, \mathsf{Z}/n),$$

where, if one identifies $\mathrm{Tor}(G, \mathsf{Z}/m)$ with $G[m]$ in the canonical way, $\overline{\kappa}_{n,m}x = \frac{m}{(n,m)}x$. Furthermore, as described in [16], the maps $\kappa_{n,m}$ also induce natural transformations

$$\kappa^i_{n,m} : K_i(A; \mathsf{Z}/m) \to K_i(A; \mathsf{Z}/n).$$

We now say that a family of homomorphisms indexed by N is *coherent* if it intertwines the homomorphisms induced by $\kappa_{n,m}$. For instance, a family $\sigma_n : K_1(A)[n] \to K_0(A; \mathsf{Z}/n)$ is coherent if

$$\kappa^0_{n,m}\sigma_m = \sigma_n\overline{\kappa}_{n,m}.$$

We shall leave it to the reader to extract from the context the exact meanings of the concrete coherence properties we need in the following. Note that the remaining natural transformations of K-theory with coefficients, β^i_n and ρ^i_n, form coherent families by [16] or [3].

As another general property of group homomorphisms between K-groups, we consider *ideal preservation.* Recall from [6] that $F(A\|I)$ denotes the image of

$$F(I) \xrightarrow{F(\iota)} F(A)$$

whenever $F(-)$ is a functor defined on C^*-algebras and $*$-homomorphisms, and I is a closed two-sided ideal of A, embedded by ι. When two such functors $F(-)$ and $G(-)$ are given, and α maps from $F(A)$ to $G(A)$, we say that α is *ideal-preserving* if

$$\alpha(F(A\|I)) \subseteq G(A\|I)$$

for every ideal of A *generated by projections*. The reason for this slightly abusive choice of terminology is explained in [6, 3.1]; when F and G are of a K-theoretic nature, it makes sense to consider only those ideals which are clearly reflected in the K_0-groups. Note that any natural transformation will be ideal-preserving.

3.2. *IMI-splitting maps*

As in [6], we denote by $K_0^{\mathrm{Inf}}(A)$ the quotient of $K_0(A)$ by the subgroup of infinitesimals therein. We shall work with the group homomorphisms in the following diagrams.

Diagram 3.1.

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[dr, "q"'] & K_0(A) \otimes \mathsf{Z}/n \arrow[r, "\tilde{\rho}_n^0"] \arrow[d, "\tilde{q}"'] & K_0(A;\mathsf{Z}/n) \arrow[dl, dashed, "\alpha_n"] \\
 & K_0^{\mathrm{Inf}}(A) \otimes \mathsf{Z}/n &
\end{tikzcd}$$

Diagram 3.2.

$$\begin{tikzcd}
K_0(A) \otimes \mathsf{Q} \arrow[r] \arrow[dr, "q"'] & K_0(A) \otimes \mathsf{Q}/\mathsf{Z} \arrow[r, "\tilde{\rho}_\infty^0"] \arrow[d, "\tilde{q}"'] & K_0(A;\mathsf{Q}/\mathsf{Z}) \arrow[dl, dashed, "\alpha_\infty"] \\
 & K_0^{\mathrm{Inf}}(A) \otimes \mathsf{Q}/\mathsf{Z} &
\end{tikzcd}$$

With the diagrams and the ideal-preserving property laid out we can now, in just a few lines, recall the (rather lengthy) definitions from [6].

Definition 3.3. Let us say that a C^*-algebra A possesses an *IMI_n-splitting* when there exists an ideal-preserving group homomorphism

$$\alpha_n : K_0(A;\mathsf{Z}/n) \to K_0^{\mathrm{Inf}}(A) \otimes \mathsf{Z}/n$$

fitting into Diagram 3.1, making it commutative.

We are particularly interested in the case where there is an IMI_n-splitting for every $n \in \mathsf{N}$, and this family is coherent in the sense described in Section 3.1. We shall refer to such a family as a *coherent family of IMI-splittings.*

Definition 3.4. Let us say that a C^*-algebra A possesses an *IMI_∞-splitting* when there exists an ideal-preserving group homomorphism

$$\alpha_\infty : K_0(A;\mathsf{Q}/\mathsf{Z}) \to K_0^{\mathrm{Inf}}(A) \otimes \mathsf{Q}/\mathsf{Z}$$

fitting into Diagram 3.2, making it commutative.

3.3. IP-splitting maps

In [9] the related property of *ideally split* algebras was studied. To correlate this with the terminology introduced above we need to revise the definitions somewhat. We do this using the following diagrams.

Diagram 3.5.

$$\begin{tikzcd}
K_i(A) \arrow[r, "\rho^i_n"] & K_i(A;\mathsf{Z}/n) \arrow[r, "\beta^i_n"] & K_{i+1}(A) \\
 & & K_{i+1}(A)[n] \arrow[u, hook] \arrow[ul, dashed, "\sigma^i_n"]
\end{tikzcd}$$

Diagram 3.6.

$$\begin{tikzcd}
K_0(A) \otimes \mathsf{Q} \arrow[r, "\rho^0_\infty"] & K_0(A;\mathsf{Q}/\mathsf{Z}) \arrow[r, "\beta^0_\infty"] & K_1(A) \\
 & & \mathrm{tor}K_1(A) \arrow[u, hook] \arrow[ul, dashed, "\sigma_\infty"]
\end{tikzcd}$$

Definition 3.7. Let us say that a C^*-algebra A possesses an *$IP_{i,n}$-splitting* when there exists an ideal-preserving group homomorphism

$$\sigma^i_n : K_{i+1}(A)[n] \to K_i(A;\mathsf{Z}/n)$$

fitting into Diagram 3.5, making it commutative.

We are particularly interested in the case where there is an $IP_{i,n}$-splitting for every i and every $n \in \mathsf{N}$, and this family is coherent in the sense described in Section 3.1. We shall refer to such a family as a *coherent family of IP-splittings.*

Definition 3.8. Let us say that a C^*-algebra A possesses an *IP_∞-splitting* when there exists an ideal-preserving group homomorphism

$$\sigma_\infty : \mathrm{tor}(K_1(A)) \to K_0(A;\mathsf{Q}/\mathsf{Z})$$

fitting into Diagram 3.6, making it commutative.

By [4] a C^*-algebra A of real rank zero is ideally split in the sense of [9] precisely when it possesses an IP_∞-splitting as defined above.

4. Results

4.1. Correlations of splitting properties

Lemma 4.1. *If A possesses a coherent family of IP-splittings, then A possesses an IP_∞-splitting. If* $\mathrm{tor}K_0(A) = 0$*, then the converse implication also holds.*

PROOF. The first claim is proved by taking inductive limits; cf. [5, 2.3]. For the second claim, first note that since $K_0(A)[n] = 0$ there is no need for splitting maps σ_n^1. This also accounts for the fact that the sequence

$$\begin{tikzcd} 0 \arrow[r] & K_0(A;\mathsf{Z}/n) \arrow[r, "\mu_n^0"] & K_0(A;\mathsf{Q}/\mathsf{Z}) \arrow[r, "\times n"] & K_0(A;\mathsf{Q}/\mathsf{Z}) \end{tikzcd}$$

is exact; cf. [5, 2.5 (ii)]. Hence we may identify $K_0(A;\mathsf{Z}/n)$ with $K_0(A;\mathsf{Q}/\mathsf{Z})[n]$ (as noted with a less elegant proof in [8, 2.4.8]). Hence maps σ_n^0 are uniquely induced by σ_∞, and they must automatically preserve every ideal preserved by σ_∞. ■

Lemma 4.2. *If A possesses a coherent family of IMI-splittings, then it possesses an IMI_∞-splitting.*

PROOF. This is clear on passing to limits; cf. [5, 2.3]. ■

Proposition 4.3. *If A possesses a coherent family of IP-splittings, then A possesses a coherent family of IMI-splittings. If A possesses an IMI_∞-splitting, then A possesses an IP_∞-splitting. If $\mathrm{Inf} K_0(A) = 0$, then the converse implications also hold.*

PROOF. All of these claims are proved by similar reasoning. As an example, assume that $\sigma_n^0 : K_1(A)[n] \to K_0(A;\mathsf{Z}/n)$ is a splitting map for $\tilde{\beta}_n$ and that it sends $K_1(A\|I)[n]$ to $K_0(A\|I;\mathsf{Z}/n)$. Then

$$\alpha_n(x) = \tilde{q}((\tilde{\rho}_n^0)^{-1}(x - \sigma_n^0(\tilde{\beta}_n^0(x))))$$

fits into Diagram 3.1 and sends $K_0(A\|I;\mathsf{Z}/n)$ to $K_0(A\|I) \otimes \mathsf{Z}/n$. ■

Corollary 4.4. *If $K_0(A)$ has no infinitesimals, then the following four statements are equivalent:*

(i) *A possesses a coherent family of IP-splittings;*
(ii) *A possesses an IP_∞-splitting;*
(iii) *A possesses a coherent family of IMI-splittings;*
(iv) *A possesses an IMI_∞-splitting.*

PROOF. As $\mathrm{Inf} K_0(A) = 0$ implies $\mathrm{tor} K_0(A) = 0$, all the results above are applicable. ■

4.2. Coherent Künneth splittings

The main result of this section, Theorem 4.6 below, was claimed for algebras with finitely many ideals with a sketch of a proof in [9]. With the concrete application in this paper in mind, we believe that it is pertinent to give a complete proof of this fact, but we choose to give a slightly different proof, which can also be used to say something about certain infinite ideal lattices.

Lemma 4.5. *Let I be an ideal generated by projections in a C^*-algebra A. When a coherent family of IP-splittings (τ^i_n) for I is given, it extends to a coherent family of IP-splittings (σ^i_n) for A.*

Proof. When we are given a coherent Künneth splitting $(\sigma^i_{p_n})$ at the prime powers, it is easy to define a coherent family of Künneth splittings (σ^i_n) defined for every $n \in \mathsf{N}$ proceeding as in [5, 3.2]. Hence we only need to define the splitting maps on prime powers and to check that the splitting maps corresponding to powers of the same prime are coherent.

We fix a prime p and $i \in \{0, 1\}$. We define $\sigma^i_{p^k}$ inductively, satisfying

(i) $\beta^i_{p^k}\sigma^i_{p^k} = \mathrm{id}$, $\quad k \geq 1$;
(ii) $\sigma^i_{p^{k-1}}\overline{\kappa}_{p^{k-1},p^k} = \kappa^i_{p^{k-1},p^k}\sigma^i_{p^k}$, $\quad k > 1$;
(iii) $\sigma^i_{p^k}\overline{\kappa}_{p^k,p^{k-1}} = \kappa^i_{p^k,p^{k-1}}\sigma^i_{p^{k-1}}$, $\quad k > 1$;
(iv) $\sigma^i_{p^k}|_{K_{i+1}(A\|I)[p^k]} = \tau^i_{p^k}$, $\quad k \geq 1$.

For $k = 1$, all the groups in the Künneth sequence are vector spaces, and so $K_{i+1}(A\|I)[p]$ must be a subspace. We choose a basis for $K_{i+1}(A\|I)[p]$ and augment it to a basis (y_j) for $K_{i+1}(A)[p]$. We may then define σ_p by specifying it on y_j. If $y_j \in K_{i+1}(A\|I)[p]$, we define σ_p using τ_p; if not, we choose any pre-image of y_j. This map will then satisfy (i) and (iv).

Let maps $\sigma^i_{p^k}$ be given, satisfying (i)–(iv) above for all $k \leq n$. Applying [5, 4.4 (ii)], $K_{i+1}(A\|I)$ is a pure subgroup of $K_{i+1}(A)$, so by [12, 29.1, remark 1] we can write

$$K_{i+1}(A)[p^{n+1}] = K_{i+1}(A\|I)[p^{n+1}] \oplus X.$$

Employing [12, 17.2], we can find a *basis* for $K_{i+1}(A\|I)[p^{n+1}]$—i.e. a set such that $K_{i+1}(A\|I)[p^{n+1}]$ is a direct sum of the cyclic groups they generate—and then augment it to a basis (y_j) for $K_{i+1}(A)[p^{n+1}]$. We may then define a map

$$\hat{\sigma}^i_{p^{n+1}} : K_{i+1}(A)[p^{n+1}] \to K_i(A; \mathsf{Z}/p^{n+1})$$

by

$$y_j \mapsto \begin{cases} \kappa^i_{p^{n+1},p^n}\sigma^i_{p^n}y_j & \text{when order}(y_j) \leq p^n, \\ \tau^i_{p^{n+1}}y_j & \text{when order}(y_j) = p^{n+1} \text{ and } y_j \in K_{i+1}(A\|I), \\ \hat{z}_j & \text{when order}(y_j) = p^{n+1} \text{ and } y_j \in X, \end{cases} \tag{4.1}$$

where $\hat{z}_j$ is an arbitrary lift of y_j in the last case. The map will satisfy (i) and (iv) by construction, but not necessarily (ii) and (iii).

We apply the two-step procedure described in [2] and [3] to replace $\hat{\sigma}^i_{p^{n+1}}$ by a map which also satisfies (ii) and (iii). We leave the actual computations to the reader, but we will record along the way whenever there is more than coherence involved in them. We will use that for *any* group homomorphism $\psi : K_{i+1}(A)[p^r] \longrightarrow K_i(A;\mathsf{Z}/p^r)$ we have

$$\psi\overline{\kappa}_{p^r,p^s}\overline{\kappa}_{p^s,p^r} = \kappa^i_{p^r,p^s}\kappa^i_{p^s,p^r}\psi, \tag{4.2}$$

simply because both of the endomorphisms are multiplications by the same power of p.

Defining $\delta_j \in K_i(A;\mathsf{Z}/n)$ by

$$\delta_j = \sigma^i_{p^n}\overline{\kappa}_{p^n,p^{n+1}}y_j - \kappa^i_{p^n,p^{n+1}}\hat{\sigma}^i_{p^{n+1}}y_j,$$

we get by the fact that $\sigma^i_{p^n}$ satisfies (iii) and by coherence of $\tau^i_{p^{n+1}}$ that $\delta_j = 0$ for y_j in the two first cases of (4.1). As in [2, 2.8], we may replace $\hat{z}_j$ by

$$\dot{z}_j = \hat{z}_j + \rho^i_{p^{n+1}}x_j,$$

where x_j is chosen so that $\rho^i_{p^n}x_j = \delta_j$. The map $\dot{\sigma}^i_{p^{n+1}}$ thus defined will then also satisfy (i), and we can check that we have obtained a map which also satisfies (ii). We postpone checking (iv) until after step two below.

As in [3, 2], we can use that $\beta^i_{p^{n+1}}\kappa^i_{p^{n+1},p^n}\sigma^i_{p^n} = \overline{\kappa}_{p^{n+1},p^n}$ to write

$$\dot{\sigma}^i_{p^{n+1}}\overline{\kappa}_{p^{n+1},p^n} - \kappa^i_{p^{n+1},p^n}\sigma^i_{p^n} = \tilde{\rho}^i_{p^{n+1}}\Delta$$

for some group homomorphism $\Delta : K_{i+1}(A)[p^n] \to K_i(A)\otimes\mathsf{Z}/p^{n+1}$. Using (4.2) twice, we get

$$\tilde{\rho}^i_{p^n}\underline{\kappa}_{p^n,p^{n+1}}\Delta = 0, \tag{4.3}$$

$$\tilde{\rho}^i_{p^{n+1}}\Delta\overline{\kappa}_{p^n,p^{n+1}} = 0. \tag{4.4}$$

Defining, as we may, $\pi : K_{i+1}(A)[p^{n+1}] \to K_{i+1}(A)[p^n]$ by

$$y_j \mapsto \begin{cases} y_j & \text{when order}(y_j) \le p^n, \\ \overline{\kappa}_{p^n,p^{n+1}}y_j & \text{when order}(y_j) = p^{n+1}, \end{cases} \tag{4.5}$$

we will let

$$\sigma^i_{p^{n+1}} = \dot{\sigma}^i_{p^{n+1}} - \tilde{\rho}^i_{p^{n+1}}\Delta\pi$$

and claim that this map will have all the desired properties. Clearly, (i) remains unaltered, and (ii) follows if we use (4.3). To check that (iii) is also satisfied, we note that it suffices to check it on all elements of the form $\pi(y_j)$ in $K_{i+1}(A)[p^n]$. In the first case of (4.5), (iii) holds by construction, and in the second we use (4.4) and (4.2) again. Finally, using (4.4) once more, we note that this two-step procedure has not altered our initial choice of pre-images of basis elements y_j of $K_{i+1}(A\|I)[p^{n+1}]$ of maximal order, and hence those are mapped into $\tau^i_{p^{n+1}}y_j$ by (4.1). Other basis elements for $K_{i+1}(A\|I)[p^{n+1}]$ are of the form $\overline{\kappa}_{p^{n+1},p^n}y'$, and we get

$$\sigma^i_{p^{n+1}}\overline{\kappa}_{p^{n+1},p^n}y' = \tau^i_{p^{n+1}}\overline{\kappa}_{p^{n+1},p^n}y'$$

by (iii) combined with coherence of the family (τ^i_n). ■

At the suggestion of the referee, we formulate the following result in the setting of C^*-algebras with ideal lattices which have the decreasing chain condition [1, III.4]—every non-empty set of ideals has a minimal element. This condition is obviously true in the case of a finite ideal lattice.

Theorem 4.6. *Consider an ASH algebra A with real rank zero and slow dimension growth. If the ideal lattice of A has the decreasing chain condition, then A possesses a coherent family of IP-splittings.*

Proof. By [1, 38] the generalised induction principle is available, beginning from the zero ideal (grey on Fig. 1). Let us thus fix a non-zero ideal I, set $\mathscr{J} = \{J \subsetneq I\}$, and assume that IP-splittings

$$\tau^i_{n,J} : K_{i+1}(A\|J)[n] \longrightarrow K_i(A\|J;\mathsf{Z}/n)$$

exist for all ideals in $\mathscr{J}$, in such a way that $\tau^i_{n,J_1}|_{K_{i+1}(A\|J_2)[n]} = \tau^i_{n,J_2}$ whenever $J_2 \subsetneq J_1 \subsetneq I$. We need to prove that IP-splittings

$$\sigma^i_n : K_{i+1}(A\|I)[n] \longrightarrow K_i(A\|I;\mathsf{Z}/n)$$

exist with the same compatibility condition. There are three cases to consider:

1° $\overline{\sum_{J\in\mathscr{J}} J} \in \mathscr{J}$ [I white on Fig. 1];
2° $\overline{\sum_{J\in\mathscr{J}} J} = I$ and $J_1 + J_2 = I$ for some $J_1, J_2 \in \mathscr{J}$ [I black on Fig. 1];
3° $\overline{\sum_{J\in\mathscr{J}} J} = I$ and $J_1 + J_2 \in \mathscr{J}$ for any $J_1, J_2 \in \mathscr{J}$ [not occurring in the finite case].

In case 1° the ideal $J_0 = \overline{\sum_{J\in\mathscr{J}} J}$ is maximal in I, and we can apply Lemma 4.5 to extend the IP-splittings from J_0 to I in a coherent way. In case 2° we define σ^i_n

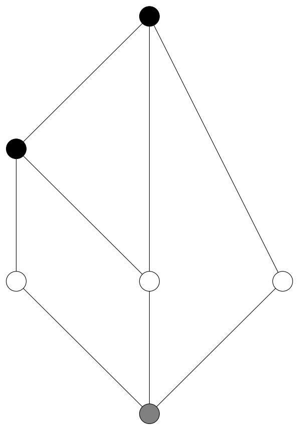

FIG. 1—A finite ideal lattice.

using the diagram

$$\begin{tikzcd}
0 & & 0 \\
K_i(A\|I;\mathsf{Z}/n) \arrow[u] & & K_{i+1}(A\|I)[n] \arrow[u] \arrow[ll, dashed] \\
K_i(A\|J_1;\mathsf{Z}/n) \oplus K_i(A\|J_2;\mathsf{Z}/n) \arrow[u] & & K_{i+1}(A\|J_1)[n] \oplus K_{i+1}(A\|J_2)[n] \arrow[u] \arrow[ll, "\tau^i_{n,J_1} \oplus \tau^i_{n,J_2}"'] \\
K_i(A\|J_1 \cap J_2;\mathsf{Z}/n) \arrow[u] & & K_{i+1}(A\|J_1 \cap J_2)[n] \arrow[u] \arrow[ll, "\tau^i_{n,J_1 \cap J_2}"']
\end{tikzcd}$$

as in [9, 4.5], using this time the purity assured by [5, 4.4] to prove exactness of the rightmost column. We need to check that σ^i_n restricts to τ^i_{n,J_3} for any $J_3 \in \mathscr{J}$. Since J_3 is the sum of $J_3 \cap J_1$ and $J_3 \cap J_2$ by distributivity, we can use purity again to write $x_3 \in K_{i+1}(A\|J_3)[n]$ as a sum $x_1 + x_2$ with $x_j \in K_{i+1}(A\|J_3 \cap J_j)[n]$. And then

$$\sigma^i_n(x_3) = \tau^i_{n,J_3 \cap J_1}(x_1) + \tau^i_{n,J_3 \cap J_2}(x_2) = \tau^i_{n,J_3}(x_1 + x_2) = \tau^i_{n,J_3}(x_3),$$

as desired.

In case 3° we have

$$K_{i+1}(A\|I)[n] = \bigcup_{J \in \mathscr{J}} K_{i+1}(A\|J)[n] \qquad K_i(A\|I;\mathsf{Z}/n) = \bigcup_{J \in \mathscr{J}} K_i(A\|J;\mathsf{Z}/n)$$

by continuity of the functors, so we can attempt to define σ_n^i by

$$\sigma_n^i(x) = \tau_{n,J}^i(x) \text{ if } x \in K_{i+1}(A\|J)[n].$$

This will be well defined since, if $x \in K_{i+1}(A\|J_1)[n] \cap K_{i+1}(A\|J_2)[n]$, both possible defining maps are restrictions of $\tau_{n,J_1+J_2}^i$. ■

5. Consequences

5.1. Classification results

The practical importance of the existence of *IP*-splittings is that it allows us, under favourable circumstances, to bypass the large invariant $\underline{\mathbf{K}}(-)$ and work instead with the much more tractable classical invariant $K_\bullet(-)$. This is done as follows.

Proposition 5.1. *Let A and B be ASH algebras of real rank zero and with slow dimension growth which both possess coherent families of IP-splittings. If*

$$[K_\bullet(A), K_\bullet(A)_+, \Sigma(A)] \simeq [K_\bullet(B), K_\bullet(B)_+, \Sigma(B)],$$

then $A \simeq B$.

Proof. Let the isomorphism be given by (φ^0, φ^1). Using the splitting maps, one may induce isomorphisms

$$\psi_n^i : K_i(A; \mathsf{Z}/n) \to K_i(B; \mathsf{Z}/n)$$

from the isomorphisms $\varphi^i : K_i(A) \to K_i(B)$. Since, according to the positivity, the φ^i must be ideal-preserving in the sense

$$\varphi^0(K_0(A\|I)) = K_0(B\|J) \Longrightarrow \varphi^1(K_1(A\|I)) = K_1(B\|J),$$

and since the splitting maps are ideal-preserving by definition, the ψ_n^i maps are also ideal-preserving in the sense

$$\varphi^0(K_0(A\|I)) = K_0(B\|J) \Longrightarrow \psi_n^i(K_i(A\|I; \mathsf{Z}/n)) = K_i(B\|J; \mathsf{Z}/n).$$

Apply [5, 6.5] to get that this gives an order-preserving isomorphism $\underline{\mathbf{K}}(A) \simeq \underline{\mathbf{K}}(B)$, and then [7, 6.5] to conclude that $A \simeq B$. ■

It was proved by Gong that ordered K-theory was sufficient to classify all *AH* algebras with real rank zero and slow dimension growth, provided that their K_0-groups had no infinitesimals. In view of the subsequent classification results which invoked—and had to invoke—the groups $K_0(-; \mathsf{Z}/n)$, Gong's result is quite surprising.

We note here that Proposition 5.1 combines with the main theorem of [6] to yield an alternative proof of this result, which can be viewed as an explanation of why, in this case, there is no extra information in the augmented ordered K-groups. Compare to the note added in proof of [13].

Corollary 5.2 [13, 4.17]. *Let A and B be AH algebras of real rank zero and with slow dimension growth, and assume that* $\mathrm{Inf} K_0(A) = \mathrm{Inf} K_0(B) = 0$. *If*

$$[K_\bullet(A), K_\bullet(A)_+, \Sigma(A)] \simeq [K_\bullet(B), K_\bullet(B)_+, \Sigma(B)],$$

then $A \simeq B$.

PROOF. By [6, 3.2], A and B both possess coherent families of *IMI*-splittings. Applying Corollary 4.4, we may transform these into *IP*-splittings, and Proposition 5.1 proves the isomorphism. ■

5.2. *C^*-algebras with finitely many ideals*

We are now ready to state and prove the main theorem of the paper.

Theorem 5.3. *Let A be an ASH algebra of real rank zero and with slow dimension growth. If the ideal lattice of A has the decreasing chain condition, in particular if A has only finitely many ideals, then A is an AH algebra.*

PROOF. By [10, 4.11, 4.12], $[K_\bullet(A), K_\bullet(A)_+, \Sigma(A)]$ is a graded ordered group satisfying the Riesz interpolation property as well as the weak unperforation property, and by [10, 4.18] there is then an *AH* algebra of real rank zero with $K_\bullet(A) \simeq K_\bullet(B)$. Since the ideal lattices of A and B are isomorphic, both algebras possess a coherent family of *IP*-splittings by Theorem 4.6. Consequently, $A \simeq B$ by Proposition 5.1. ■

Remark 5.4. The C^*-algebras considered in [6] have ideal lattices isomorphic to the topology of the one-point compactification of the integers. It is easy to see directly that such lattices do not have the decreasing chain condition—for instance, the set of neighbourhoods of the point at infinity has no minimal elements.

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