MAXIMAL CONNECTED PRINCIPAL TOPOLOGIES

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Abstract

The internal structure of maximal connected principal topologies is revealed within the lattice of all topologies for a given set, thereby providing an easily applied identification test. Some examples, including the Khalimsky line, are realised.

1. Introduction

The study of maximal connected topologies was initiated by Thomas [8] in 1968, and they were subsequently characterised by Clark and Schneider [1] in 1988 (as being those topologies which are submaximal and nearly maximal connected). The work of Neumann-Lara and Wilson [7] was also significant in this regard. Recently, the present authors have obtained a more general characterisation involving singular sets [4]. However, the full identification of such topologies remains a problem. Thomas contributed to a solution when, in the case of a principal topology, he established necessary and sufficient conditions for it to be maximal connected. Accordingly, he was able to determine the structure of all maximal connected principal topologies for small finite sets. Furthermore, Das [2] quantified the number of homeomorphic classes of maximal connected topologies definable for any finite set. The purpose of this paper is to identify all maximal connected principal topologies for a set of arbitrary cardinality.

Recall that, given a non-empty set X and a member $\mathscr{T}$ of $LT(X)$ (the lattice of all topologies definable for X), the topological space $(X, \mathscr{T})$ is *principal* when every point in X has a minimum $\mathscr{T}$-neighbourhood (equivalently, every intersection of $\mathscr{T}$-open subsets X is $\mathscr{T}$-open). Assuming that $(X, \mathscr{T})$ is principal, Thomas proved that $(X, \mathscr{T})$ is maximal connected if and only if (a) X is the union of all minimum $\mathscr{T}$-neighbourhoods of non-$\mathscr{T}$-isolated points in X, (b) the intersection of any two distinct such neighbourhoods is degenerate, and (c) every pair of points in X can be linked by means of a unique simple chain of such neighbourhoods.

We shall adapt and extend this result.

2. Degenerative covers

Given a non-empty proper subset A of X, $\mathscr{M}(\mathrm{A})$ denotes the member $\{U : either\, U \subseteq A\, or\, A \subseteq U\}$ of $LT(X)$. It is of some interest to observe that topologies of this nature are, of course, connected and indeed are precisely the principal minimal submaximal members of $LT(X)$ which are not T_1 (see [3]).

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Definition 1. A family $\mathscr{G}$ of subsets of X is called an *A-degenerate cover* of X when each of the following conditions is satisfied:

(i) $X = \cup\{G : G \in \mathscr{G}\}$;

(ii) $G_i \cap G_j$ is a degenerate (that is, it has at most one element) subset of A, for all $G_i \neq G_j$ in $\mathscr{G}$;

(iii) $\mathscr{G}$ is associated with A (that is, $G \cap A \neq \emptyset$ for all $G \in \mathscr{G}$);

(iv) $\mathscr{G}$ is uniquely associated with $X \backslash A$ (that is, $G \cap (X \backslash A)$ is a singleton for all $G \in \mathscr{G}$);

(v) $\mathscr{G}$ cannot be a non-trivial partition base for X (that is, there is no non-empty proper subset V of X such that both V and $X \backslash V$ are unions of members of $\mathscr{G}$).

Some observations regarding the above are pertinent at this stage. Because of (i), (ii) and (iv), for each $x \in X \backslash A$ there exists a unique $G_x \in \mathscr{G}$ with $x \in G_x$, and so $\mathscr{G} = \{G_x : x \in X \backslash A\}$; because of (iii), $\emptyset \subset G_x \backslash \{x\} \subseteq A$ for all $x \in X \backslash A$.

Also, if we write $\mathscr{B}(\mathscr{G})$ to denote the member of $LT(X)$ which has $\mathscr{G}$ as an open subbase, then condition (v) above essentially proclaims that $\mathscr{B}(\mathscr{G})$ is a connected topology. Because of (ii) and (v), there exists a subset I of A (the set of 'meeting points' of distinct pairs of elements from $\mathscr{G}$) which is either empty or $\mathscr{B}(\mathscr{G})$-dense. Consequently, with reference to the space $(X, \mathscr{B}(\mathscr{G}))$, for each $x \in X \backslash A$, G_x is the minimum neighbourhood of x, while if $x \in I$, $\{x\}$ is the minimum neighbourhood, and if $x \in A \backslash I$, there exists a unique $y \in X \backslash A$ such that $x \in G_y$, whence G_y is the minimum neighbourhood of x. That is, $\mathscr{B}(\mathscr{G})$ is principal.

Given an A-degenerate cover $\mathscr{G}$ of X, another A-degenerate cover $\mathscr{H}$ of X is said to be *finer* than $\mathscr{G}$ if and only if $H_x \subseteq G_x$ for all $x \in X \backslash A$. Then $\mathscr{G}$ is said to be *final* if and only if $\mathscr{H}$ is finer than $\mathscr{G}$ implies $\mathscr{H} = \mathscr{G}$.

Example 2. Let R denote the set of real numbers, let Q be the set of rational numbers, and let $Q_1(Q_2)$ be the set of non-negative (non-positive) rationals respectively, so that $Q_1 \cap Q_2 = \{0\}$. Then

$$\mathscr{K} = \{Q_1 \cup \{\sqrt{2}\}, Q_2 \cup \{\sqrt{5}\}, \{1, -1, \sqrt{3}\}, \{1, x\} : x \in R \backslash (Q \cup \{\sqrt{2}, \sqrt{3}, \sqrt{5}\})\}$$

is a Q-degenerate cover of R which is not final since

$$\mathscr{G} = \{Q_1 \cup \{\sqrt{2}\}, Q_2 \cup \{\sqrt{5}\}, \{1, y\} : y \in R \backslash (Q \cup \{\sqrt{2}, \sqrt{5}\})\}$$

is a strictly finer Q-degenerate cover of R. Incidentally, $\mathscr{G}$ is final.

Lemma 3. *Given a non-empty subset A of X and an A-degenerate cover $\mathscr{G}$ of X, the topology $\mathscr{M}(A) \vee \mathscr{B}(\mathscr{G})$ (the supremum of $\mathscr{M}(A)$ and $\mathscr{B}(\mathscr{G})$ in $LT(X)$) is connected and principal.*

Proof. That $\mathscr{T} = \mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$ is principal is fairly immediate, since, for each $a \in A$, $\{a\}$ is the minimum $\mathscr{T}$-neighbourhood of a, while, for each $x \in X \backslash A$, the unique $G_x \in \mathscr{G}$ is the minimum $\mathscr{T}$-neighbourhood of x (note that $G_x \subseteq A \cup \{x\}$).

To show that $\mathscr{T}$ is connected, suppose otherwise; then there exists a non-empty proper subset V of X such that $V \in \mathscr{T}$ and $X \backslash V \in \mathscr{T}$. Let $U_1 = \cup\{G_x : x \in V \cap (X \backslash A)\}$ and $U_2 = \cup\{G_x : x \in (X \backslash V) \cap (X \backslash A)\}$. Since $U_1 \subseteq V$, $U_2 \subseteq X \backslash V$ and $U_1 \cup U_2 = X$, it follows that $U_1 = V$ and $U_2 = X \backslash V$. This contradicts the connectedness of $\mathscr{B}(\mathscr{G})$. ■

3. Maximal connectedness

Theorem 4. *Given a non-empty subset A of X and an A-degenerate cover $\mathscr{G}$ of X, the topology $\mathscr{M}(A) \vee \mathscr{B}(\mathscr{G})$ is maximal connected if and only if $\mathscr{G}$ is final.*

PROOF. Suppose that $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$ is maximal connected and $\mathscr{G}$ is *not* final; then there exists an A-degenerate cover $\mathscr{H}$ of X which is strictly finer than $\mathscr{G}$. In particular, there exists $x \in X \backslash A$ with $x \in H_x \subset G_x \subseteq A \cup \{x\}$. Thus $H_x \notin \mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$ and so $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G}) \subset \mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{H})$ in $LT(X)$ (observe that $G_y = H_y \cup (G_y \backslash H_y)$ and $G_y \backslash H_y \subseteq A$ for all $y \in X \backslash A$). But $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{H})$ is connected by Lemma 3, thereby contradicting the maximality of $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$.

Conversely, suppose that $\mathscr{G}$ is final yet $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G}) \subset \mathscr{T}$ in $LT(X)$, where $\mathscr{T}$ is connected, so that there exists a non-empty proper subset V of X with $V \in \mathscr{T}$ and $V \notin \mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$. Evidently, $V \nsubseteq A$ and $V = X \cap V = \cup\{G_x \cap V : x \in X \backslash A\}$, so there exists some $x^* \in (X \backslash A) \cap V$ with $x^* \in G_{x^*} \cap V \subset G_{x^*}$; write $G = G_{x^*} \cap V$ and consider the family of subsets $\mathscr{H} = \{G, \{G_y : y \in X \backslash A, y \neq x^*\}\}$. Since $\mathscr{T}$ is connected, $\mathscr{H}$ satisfies conditions (ii)–(v) of Definition 1 above. Consequently $\mathscr{H}$ fails to satisfy condition (i) (otherwise $\mathscr{H}$ is a strictly finer A-degenerate cover of X than $\mathscr{G}$). That is, $U = G \cup \{\cup\{G_y : y \in X \backslash A, y \neq x^*\}\}$ is a proper $\mathscr{T}$-open subset of X; but then $X \backslash U \subseteq A$ is also non-empty $\mathscr{T}$-open, contradicting the connectedness of $\mathscr{T}$. ■

Lemma 5. *Let $(X, \mathscr{T})$ be a maximal connected and principal space. If A denotes the subset of X consisting of all $\mathscr{T}$-isolated points and $\mathscr{G}$ denotes the family of minimum $\mathscr{T}$-neighbourhoods of points in $X \backslash A$, then $\mathscr{G}$ is a final A-degenerate cover of X.*

PROOF. That $\mathscr{G}$ is an A-degenerate cover of X is an immediate consequence of Thomas's result. Also, by hypothesis, $\mathscr{T} \subseteq \mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$ in $LT(X)$, so the maximality of $\mathscr{T}$ and Lemma 3 ensures equality here, while the finality of $\mathscr{G}$ follows from Theorem 4. ■

Theorem 6. *A topological space $(X, \mathscr{T})$ is maximal connected and principal if and only if there exists a non-empty proper subset A of X and a final A-degenerate cover $\mathscr{G}$ of X such that $\mathscr{T} = \mathscr{M}(A) \vee \mathscr{B}(\mathscr{G})$. (Moreover, the construction is canonical: A is the subset of $\mathscr{T}$-isolated points in X and $\mathscr{G}$ is the family of minimum $\mathscr{T}$-neighbourhoods of points in $X \backslash A$).*

PROOF. Immediate by Theorem 4 and Lemma 5. ■

Example 7. (i) When $A = \{a\}$ and $\mathscr{G} = \{\{a, x\} : x \in X \backslash A\}$, then we have

$$\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G}) = \mathscr{I}(\mathrm{a}) = \{G \subseteq X : a \in G\} \cup \{\emptyset\}$$

(an 'included point' topology).

(ii) When $A = X \backslash \{b\}$ and $\mathscr{G} = \{X\}$, then we have

$$\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G}) = \mathscr{E}(\mathrm{b}) = \{G \subseteq X : b \notin G\} \cup \{X\}$$

(an 'excluded point' topology).

Of course, the included point and excluded point members of $LT(X)$ are precisely those which are principal, maximal connected, and *door*.

Example 8. When $a \in A,\ b \in X \backslash A$ and $\mathscr{G} = \{A \cup \{b\}, \{a, x\} : x \in X \backslash (A \cup \{b\})\}$, then we have $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G}) = \mathscr{M}(\mathrm{A}) \cup (\mathscr{I}(\mathrm{a}) \cap \mathscr{E}(\mathrm{b}))$.

These are the principal maximal connected topologies first mentioned by Larson [6]. Indeed, they are submaximal and nearly maximal connected principal members of $LT(X)$, whose predominant feature is that there exists a single unique boundary point (namely b) for all non-empty proper *regular* open sets (see [1] and [4]).

Example 9. When X is the set of (real) integers, A is the set of odd integers, and $\mathscr{G} = \{\{2n-1, 2n, 2n+1\} : n \in X\}$, then $\mathscr{M}(\mathrm{A}) \vee \mathscr{B}(\mathscr{G})$ is the topology of the so-called 'Khalimsky line', much in vogue recently with digital topologists (see, for example, [5]).

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