SPATIAL BEHAVIOUR OF SOLUTIONS FOR A CLASS OF NON-LINEAR FOURTH-ORDER PARABOLIC EQUATIONS

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Abstract

In this paper we establish a spatial growth estimate for a class of non-linear fourth-order parabolic equations. Alternatively, the results may be viewed as theorems of Phragmen–Lindelöf type. We conclude the paper by extending results for a type of non-linear diffusivity term.

1. Introduction

In this paper we are concerned with the spatial behaviour of solutions of initial-boundary value problems for a class of non-linear parabolic equations. More precisely we will prove a Phragmen–Lindelöf-type theorem for a class of parabolic equations of the form

$$u_t + \Delta^2 u - \Delta f(u) = 0.$$

Under a growth condition on non-linearity f, the growth rate of the non-trivial solutions of an initial-boundary value problem for the above equation in unbounded cylindrical domains with homogeneous boundary conditions is established.

Phragmen–Lindelöf-type theorems for some classes of non-linear elliptic and parabolic equations have been obtained previously (for relevant references one may consult [1–8 and 10–14]).

Our results are established mainly following the ideas of [3; 5; 7], in which Phragmen–Lindelöf-type theorems for some semi-linear fourth-order elliptic, second-order parabolic and Navier–Stokes equations have been derived.

Let us point out that our results obtained here are different from the Phragmen–Lindelöf-type theorems obtained in [2; 14] under the condition that the relevent solutions tend to zero.

We finish the paper by describing some possible extensions of the methods and results for non-linear diffusive parabolic equations.

2. Preliminaries

Let

$$\Omega = \{x \in \mathbf{R}^n : \ x_1 \in \mathbf{R}^+, \ x' = (x_2, x_3, \ldots, x_n) \in \tilde{\sigma}_{x_1} \subset \mathbf{R}^{n-1}\}$$

be the interior of a semi-infinite cylindrical domain where

$$\sigma_\tau = \{(x_1, x') \in \Omega : \ x_1 = \tau\},$$

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and let $\tau \to \tilde{\sigma}_\tau$ be a mapping from $\mathbf{R}^+$ into the family of bounded domain subsets of $\mathbf{R}^{n-1}$; suppose that

$$0 < \alpha \leq \inf_\tau mes\, \tilde{\sigma}_\tau \leq \sup_\tau mes\, \tilde{\sigma}_\tau \leq \beta,$$

and $\partial\sigma_\tau$ is smooth. Let

$$\Omega_\tau = \{(x_1, x') \in \Omega : \quad 0 < x_1 < \tau\}.$$

We will consider the following initial-boundary value problem:

$$u_t + \Delta^2 u = \Delta f(u), \quad (x_1, x', t) \in Q_T := \Omega \times [0, T), \tag{2.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \quad (x_1, x', t) \in \partial\Omega \times [0, T), \tag{2.2}$$

$$u(x_1, x', 0) = 0, \quad (x_1, x') \in \Omega. \tag{2.3}$$

We suppose that $f \in C^2(\mathbf{R})$, $f(0) = 0$,

$$f'(u) \geq 0 \quad \forall\, u \geq 0, \tag{2.4}$$

$$|f'(u)| \leq A_0 |u|^{\frac{m}{2}-1} \quad \forall\, u \in \mathbf{R}, \tag{2.5}$$

where $2 < m \leq \frac{2n}{n-2}$ for $n > 2$, $2 < m < \infty$ for $n = 2$, and A_0 is a positive constant.

Throughout the article we will employ the following notations:

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_i^k = \frac{\partial^k}{\partial {x_i}^k}, \quad |\nabla u|^2 = \sum_{i=1}^n (\partial_i u)^2$$

$$|\nabla^2 u|^2 = \sum_{i,j=1}^n (\partial_i \partial_j u)^2, \quad \Delta = \sum_{i=1}^n \partial_i^2$$

$$\Delta^2 = \sum_{i,j=1}^n \partial_i^2 \partial_j^2, \quad ||u||_\Omega^2 = \int_\Omega u^2 dx$$

and $\frac{\partial u}{\partial \nu}$ will represent the exterior normal derivative of u.

For further considerations, the following technical lemmas will be important to us.

Lemma 1 ([9]). *Let $D \subset \mathbf{R}^n$ be a bounded region. Then $W_0^{2,2}(D) \subset L^p(D)$, for $2 < p \leq \frac{2n}{n-4}$ if $n > 4$ and $1 \leq p < \infty$ if $n \leq 4$. That is to say, a constant C depends upon D, and n and p exist such that*

$$\int_D |v|^p dx \leq C \left\{ \int_D |\nabla^2 v|^2 dx \right\}^{\frac{p}{2}}$$

for every $u \in W_0^{2,2}(D)$.

Lemma 2 ([8]). *Let* Ψ *be a monotone increasing function with* $\Psi(0) = 0$, $\lim_{\tau\to\infty}\Psi(\tau) = +\infty$. *Then* $\mathbf{z}(\tau) > 0$ *satisfying* $\mathbf{z}(\tau) \leq \Psi(\mathbf{z}'(\tau))$, $\tau \geq 0$, *tends to* $+\infty$ *when* $\tau \to \infty$. *If* $\Psi(\tau) \leq c_0\tau^p$ *for some* c_0 *and* $p > 1$ *for* $\tau \geq \tau_1$, *then*

$$\underline{lim}_{\tau\to\infty}\, \tau^{\frac{-p}{p-1}}\mathbf{z}(\tau) > 0.$$

3. A Phragmen–Lindelöf-type alternative

Let u be any non-trivial solution of the initial-boundary value problem (2.1)–(2.3). Multiplying (2.1) by u and integrating with respect to x over Ω_τ yields

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_\tau} u^2\, dx + \int_{\Omega_\tau} u\Delta^2 u\, dx = \int_{\Omega_\tau} u\Delta f(u)dx. \tag{3.1}$$

It is not difficult to see that

$$\begin{aligned} u\Delta^2 u &= \sum_{i=1}^{n} u\partial_i^4 u + 2\sum_{\substack{i,j=1\\ j>i}}^{n} u\partial_i^2\partial_j^2 u \\ &= |\nabla^2 u|^2 + \sum_{i=1}^{n}\partial_i\left[u\partial_i^3 u - \partial_i u\partial_i^2 u - 2\sum_{j=i+1}^{n}\partial_j u\partial_i\partial_j u\right] \\ &\quad + 2\sum_{\substack{i,j=1\\ j>i}}^{n}\partial_j(u\partial_i^2\partial_j u). \end{aligned}$$

By the Stokes formula and boundary conditions (2.2), we have

$$\begin{aligned} I_1 &:= \int_{\Omega_\tau} u\Delta^2 u\, dx \\ &= \int_{\Omega_\tau} |\nabla^2 u|^2 dx + \int_{\sigma_\tau}\left[u\partial_1^3 u - \partial_1 u\partial_1^2 u - 2\sum_{j=2}^{n}\partial_j u\partial_1\partial_j u\right] dx' \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} I_2 &:= \int_{\Omega_\tau} u\Delta f(u)dx \\ &= -\int_{\Omega_\tau} f'(u)\sum_{i=1}^{n}(\partial_i u)^2 dx + \int_{\sigma_\tau} uf'(u)\partial_1 u dx'. \end{aligned} \tag{3.3}$$

Taking into account conditions (2.4) and (2.5), from (3.3) we obtain the inequality

$$I_2 \leq A_0\int_{\sigma_\tau} |u|^{\frac{m}{2}}|\partial_1 u|dx'. \tag{3.4}$$

On the other hand,

$$\begin{aligned} u\partial_1^3 u - \partial_1 u \partial_1^2 u - 2\sum_{j=2}^{n} \partial_j u \partial_1 \partial_j u &= \partial_1(u\partial_1^2 u) - \sum_{j=1}^{n} \partial_1(\partial_j u)^2 \\ &= \partial_1 \left[u\partial_1^2 u - \sum_{j=1}^{n} (\partial_j u)^2 \right]. \end{aligned}$$

Coupling these identities with (3.2), we obtain

$$I_1 = \int_{\Omega_\tau} |\nabla^2 u|^2 dx + \int_{\sigma_\tau} \partial_1 \left[u\partial_1^2 u - \sum_{j=1}^{n} (\partial_j u)^2 \right] dx'. \tag{3.5}$$

Equation (3.1), with (3.4) and (3.5), yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}||u(.,t)||^2_{\Omega_\tau} + ||\ |\nabla^2 u(.,t)|\ ||^2_{\Omega_\tau} \\ &\qquad \leq -\int_{\sigma_\tau} \partial_1 \left[u\partial_1^2 u - \sum_{j=1}^{n} (\partial_j u)^2 \right] dx' + A_0 \int_{\sigma_\tau} |u|^{\frac{m}{2}} |\partial_1 u| dx'. \end{aligned}$$

After an integration with respect to τ, we obtain

$$\begin{aligned} &\int_0^\tau \left\{ \frac{1}{2}\frac{d}{dt}||u(.,t)||^2_{\Omega_s} + ||\ |\nabla^2 u(.,t)|\ ||^2_{\Omega_s} \right\} ds \\ &\qquad \leq \int_{\sigma_\tau} [|\nabla u|^2 - u\partial_1^2 u] dx' + A_0 \int_{\Omega_\tau} |u|^{\frac{m}{2}} |\partial_1 u| dx. \end{aligned} \tag{3.6}$$

Integrating (3.6) with respect to t in the interval $[0, T)$ and utilising initial-condition (2.3) give

$$\begin{aligned} &\int_0^\tau \left\{ \frac{1}{2}||u(.,T)||^2_{\Omega_s} + \int_0^T ||\ |\nabla^2 u(.,t)|\ ||^2_{\Omega_s} dt \right\} ds \\ &\qquad \leq \int_0^T \left(\int_{\sigma_\tau} [|\nabla u|^2 - u\partial_1^2 u] dx' \right) dt \\ &\qquad +A_0 \int_0^T \left(\int_{\Omega_\tau} |u|^{\frac{m}{2}} |\partial_1 u| dx \right) dt. \end{aligned} \tag{3.7}$$

Now we will consider each term on the right-hand side of (3.7) separately. We use the Schwarz inequality to get

$$\begin{aligned} J_1 &:= \int_0^T \left\{ \int_{\sigma_\tau} [|\nabla u|^2 - u\partial_1^2 u] dx' \right\} dt \\ &\leq \int_0^T \left\{ \int_{\sigma_\tau} |\nabla u|^2 dx' + \left(\int_{\sigma_\tau} u^2 dx' \right)^{\frac{1}{2}} \left(\int_{\sigma_\tau} |\partial_1^2 u|^2 dx' \right)^{\frac{1}{2}} \right\} dt \end{aligned} \tag{3.8}$$

and

$$J_2 := A_0 \int_0^T \left\{ \int_{\Omega_\tau} |u|^{\frac{m}{2}} |\partial_1 u| dx \right\} dt$$
$$\leq A_0 \int_0^T \left\{ \left(\int_{\Omega_\tau} |u|^m dx \right)^{\frac{1}{2}} \left(\int_{\Omega_\tau} |\partial_1 u|^2 dx \right)^{\frac{1}{2}} \right\} dt. \tag{3.9}$$

Recall the inequality $2|A|\,|B| \leq \varepsilon(A)^2 + \frac{1}{\varepsilon}(B)^2$, which holds for positive A, B and ε; thus we have

$$J_1 \leq \int_0^T \left\{ \int_{\sigma_\tau} |\nabla u|^2 dx' + \frac{\varepsilon}{2} \int_{\sigma_\tau} u^2 dx' + \frac{1}{2\varepsilon} \int_{\sigma_\tau} |\partial_1^2 u|^2 dx' \right\} dt \tag{3.10}$$

and

$$J_2 \leq A_0 \int_0^T \left\{ \frac{\varepsilon}{2} \int_{\Omega_\tau} |u|^m dx + \frac{1}{2\varepsilon} \int_{\Omega_\tau} |\partial_1 u|^2 dx \right\} dt. \tag{3.11}$$

Recall the Poincaré–Friedrichs inequality

$$\lambda \int_D u^2 dx \leq \int_D |\nabla u|^2 dx, \tag{3.12}$$

where λ is the first eigenvalue of the Laplacian operator in D with homogeneous Dirichlet boundary conditions. Because of (3.12),

$$J_1 \leq \left(\lambda_1^{-1}(\tau) + \frac{\varepsilon}{2} \lambda_1^{-2}(\tau) + \frac{1}{2\varepsilon} \right) \int_0^T \left(\int_{\sigma_\tau} |\nabla^2 u|^2 dx' \right) dt \tag{3.13}$$

and

$$J_2 \leq A_0 \int_0^T \left\{ \frac{\varepsilon}{2} \int_{\Omega_\tau} |u|^m dx + \frac{1}{2\varepsilon} \lambda_2^{-1}(\tau) \int_{\Omega_\tau} |\nabla^2 u|^2 dx \right\} dt, \tag{3.14}$$

where $\lambda_2(\tau)$ depends on Ω_τ.

Lemma 1 and inequality (3.14) yield

$$J_2 \leq A_0 \left\{ \frac{\varepsilon}{2} C(\tau) \left(\int_0^T \|\, |\nabla^2 u(.,t)|\, \|_{\Omega_\tau}^2 dt \right)^{\frac{m}{2}} + \frac{\lambda_2^{-1}(\tau)}{2\varepsilon} \int_0^T \|\, |\nabla^2 u(.,t)|\, \|_{\Omega_\tau}^2 dt \right\}. \tag{3.15}$$

Neglecting the first term in the left-hand side of (3.7) and using the inequalities (3.13) and (3.15) we deduce that

$$\int_0^\tau \left(\int_0^T \|\, |\nabla^2 u(.,t)|\, \|_{\Omega_s}^2 dt \right) ds$$
$$\leq d_1(\tau) \int_0^T \left(\int_{\sigma_\tau} |\nabla^2 u|^2 dx' \right) dt + d_2(\tau) \int_0^T \|\, |\nabla^2 u(.,t)|\, \|_{\Omega_\tau}^2 dt$$
$$+ \frac{\varepsilon A_0 C(\tau)}{2} \left(\int_0^T \|\, |\nabla^2 u(.,t)|\, \|_{\Omega_\tau}^2 dt \right)^{\frac{m}{2}}, \tag{3.16}$$

where

$$d_1(\tau) := \lambda_1^{-1}(\tau) + \frac{\varepsilon}{2}\lambda_1^{-2}(\tau) + \frac{1}{2\varepsilon}, \tag{3.17}$$

$$d_2(\tau) := \frac{A_0\lambda_2^{-1}(\tau)}{2\varepsilon}. \tag{3.18}$$

Plugging (3.17) and (3.18) into (3.16) gives an inequality involving

$$E(\tau, T) := \int_0^T \| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} dt, \tag{3.19}$$

the strain energy contained in Ω_τ. Precisely, we have the following non-linear integro-differential inequality,

$$\int_0^\tau E(s,T)ds \le d_1(\tau)E'(\tau,T) + d_2(\tau)E(\tau,T) + \frac{\varepsilon A_0 C(\tau)}{2}[E(\tau,T)]^{\frac{m}{2}}, \tag{3.20}$$

where

$$E'(\tau,T) = \frac{d}{d\tau}\int_0^T \| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} dt \int_0^T \left(\int_{\sigma_\tau} |\nabla^2 u|^2 dx' \right) dt. \tag{3.21}$$

Our next objective is to solve the non-linear integro-differential inequality (3.20). Introducing

$$F(\tau,T) = E(\tau,T) + \int_0^\tau E(s,T)ds$$

in (3.20), we get

$$F(\tau,T) \le (1 + d_1(\tau) + d_2(\tau))F'(\tau,T) + \frac{\varepsilon A_0 C(\tau)}{2}[F'(\tau,T)]^{\frac{m}{2}}. \tag{3.22}$$

Hence Lemma 2 gives

$$\underline{lim}_{\tau\to\infty}\, \tau^{-\frac{m}{m-2}} F(\tau,T) > 0$$

and

$$\underline{lim}_{\tau\to\infty}\, \tau^{-\frac{2}{m-2}} E(\tau,T) > 0. \tag{3.23}$$

We have thus established the following theorem.

Theorem 1. *If* (2.4) *and* (2.5) *hold, then for every positive t each non-trivial solution u of initial-boundary value problem* (2.1)–(2.3) *satisfies*

$$\underline{lim}_{\tau\to\infty} \tau^{-\frac{2}{m-2}} \int_0^T \| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} dt > 0. \tag{3.24}$$

4. An extension for non-linear diffusivity term

In this section we discuss some possible extensions of the previous methods and results to equation (1.1) with non-linear diffusivity term. Let us consider the non-linear parabolic equation of the form

$$\delta \Delta g(u_t) + \Delta^2 u = \Delta f(u), \tag{4.1}$$

where $\delta < 0$ and g is a non-linear function. Now we suppose that $g \in C(\mathbf{R}), g(0) = 0$,

$$g'(u) \geq c_0 > 0, \tag{4.2}$$

$$|g'(u)| \leq B_0 |\nabla u|^{\frac{m}{2}-1}, \tag{4.3}$$

where $2 < m < \frac{2n}{n-2}$ for $n > 2$, $2 < m < \infty$ for $n = 2$, and B_0 is a positive constant. We consider the problem determined by equation (4.1), with initial and boundary conditions (2.2), (2.3). Similar calculations to those used in the previous section lead to the equality

$$\begin{aligned}\int_{\Omega_\tau} |\nabla^2 u|^2 dx = &- \int_{\sigma_\tau} \partial_1 \left[u \partial_1^2 u - \sum_{j=1}^{n} (\partial_j u)^2 \right] dx' \\ &- \int_{\Omega_\tau} f'(u) \sum_{i=1}^{n} (\partial_i u)^2 dx + \int_{\sigma_\tau} u f'(u) \partial_1 u dx' \\ &+ \frac{\delta}{2} \int_{\Omega_\tau} g'(u_t) \frac{d}{dt} \left(\sum_{i=1}^{n} (\partial_i u)^2 \right) dx - \delta \int_{\sigma_\tau} u g'(u_t) \partial_1 u_t dx'.\end{aligned} \tag{4.4}$$

By conditions (4.2) and (4.3), we obtain the inequalities

$$\begin{aligned}\tilde{I}_1 &:= \frac{\delta}{2} \int_{\Omega_\tau} g'(u_t) \frac{d}{dt} \sum_{i=1}^{n} (\partial_i u)^2 dx \\ &\leq \frac{\delta}{2} c_0 \int_{\Omega_\tau} \frac{d}{dt} |\nabla u|^2 dx\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\tilde{I}_2 &:= -\delta \int_{\sigma_\tau} u g'(u_t) \partial_1 u_t dx' \\ &\leq |\delta| B_0 \int_{\sigma_\tau} u |\nabla u_t|^{\frac{m}{2}} dx'.\end{aligned} \tag{4.6}$$

From (4.4)–(4.6), and after integration from 0 to T and 0 to τ, we obtain

$$\begin{aligned}\int_0^\tau\int_0^T \|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau}dtds \le{}& \int_0^T\int_{\sigma_\tau}(|\nabla u|^2 - u\partial_1^2 u)dx'dt\\ &+\int_0^\tau\int_0^T\int_{\Omega_\tau} f'(u)|\nabla u|^2 dxdtds\\ &+\int_0^T\int_{\Omega_\tau} uf'(u)\partial_1 u\, dxdt\\ &+\frac{\delta}{2}c_0\int_0^\tau\int_{\Omega_\tau}|\nabla u(.,T)|^2 dxds\\ &+|\delta|B_0\int_0^T\int_{\Omega_\tau} u|\nabla u_t|^{\frac{m}{2}}dxdt. \qquad (4.7)\end{aligned}$$

On account of Lemma 1, the Poincaré–Friedrichs inequality (3.12) and conditions (2.4) and (2.5), (4.7) is estimated as

$$\begin{aligned}\int_0^\tau\int_0^T \|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau}dtds \le{}& \tilde d_1(\tau)\int_0^T(\|\,|\nabla^2 u(.,t)|\,\|^2_{\sigma_\tau} + \|\,|\nabla^2 u_t(.,t)|\,\|^2_{\sigma_\tau})dt\\ &+\tilde d_2(\tau)\int_0^T(\|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau} + \|\,|\nabla^2 u_t(.,t)|\,\|^2_{\Omega_\tau})dt\\ &+\tilde d_3(\tau)\left\{\int_0^T(\|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau} + \|\,|\nabla^2 u_t(.,t)|\,\|^2_{\Omega_\tau})dt\right\}^{\frac{m}{2}}, \qquad (4.8)\end{aligned}$$

where

$$\tilde d_1(\tau) := \max\left\{\lambda_1^{-1}(\tau) + \frac{\varepsilon}{2}\lambda_1^{-2}(\tau) + \frac{1}{2\varepsilon}\,,\ 1\right\}, \qquad (4.9)$$

$$\tilde d_2(\tau) := \max\left\{\frac{1}{2\varepsilon}\lambda_2^{-2}(\tau)\,,\ \frac{1}{2\varepsilon}|\delta|B_0\lambda_2^{-1}(\tau)\,,\ 1\right\}, \qquad (4.10)$$

$$\tilde d_3(\tau) := \max\left\{\frac{\varepsilon}{2}A_0C(\tau)\,,\ \frac{\varepsilon}{2}|\delta|B_0C(\tau)\lambda_2^{-1}(\tau)\,,\ 1\right\}. \qquad (4.11)$$

We define

$$\tilde E(\tau,T) = \int_0^T(\|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau} + \|\,|\nabla^2 u_t(.,t)|\,\|^2_{\Omega_\tau})dt \qquad (4.12)$$

as the strain energy contained in Ω_τ. Inserting (4.12) in (4.8) we get

$$\int_0^\tau\int_0^T \|\,|\nabla^2 u(.,t)|\,\|^2_{\Omega_\tau}dxds \le \tilde d_1(\tau)\tilde E'(\tau,t) + \tilde d_2(\tau)\tilde E(\tau,t) + \tilde d_3(\tau)\{\tilde E(\tau,t)\}^{\frac{m}{2}}, \qquad (4.13)$$

where

$$\tilde{E}'(\tau, T) = \int_0^T (\| \, |\nabla^2 u(.,t)| \, \|^2_{\sigma_\tau} + \| \, |\nabla^2 u_t(.,t)| \, \|^2_{\sigma_\tau}) dt.$$

In order to put the left-hand side of (4.13) into the energy term (4.12), we add some appropriate terms into the right-hand side. Thus,

$$\int_0^\tau \tilde{E}(s, T) ds \leq \gamma^{-1} \tilde{d}_1(\tau) \tilde{E}'(\tau, t) + \gamma^{-1} \tilde{d}_2(\tau) \tilde{E}(\tau, t) + \gamma^{-1} \tilde{d}_3(\tau) \{\tilde{E}(\tau, t)\}^{\frac{m}{2}}, \qquad (4.14)$$

where $\gamma := \beta - \nu > 0$ and $\beta = \max\{1 \, , \, \beta_0\}$, $\nu = \min\{\alpha_0, \beta_0\}$.

Now we may proceed in a way similar to section 3 to deduce spatial growth for the solutions to the problem determined by equation (4.1), boundary conditions (2.2) and initial-conditions (2.3). Therefore we have proved the following theorem.

Theorem 2. *If* (4.2) *and* (4.3) *hold, then for every positive t each non-trivial solution u of initial-boundary value problem* (4.1), (2.2) *and* (2.3) *satisfies*

$$\underline{lim}_{\tau \to \infty} \tau^{-\frac{2}{m-2}} \int_o^T (\| \, |\nabla^2 u(.,t)| \, \|^2_{\Omega_\tau} + \| \, |\nabla^2 u_t(.,t)| \, \|^2_{\Omega_\tau}) dt > 0. \qquad (4.15)$$

Remark. The estimates (3.24) and (4.15) will be valid also if we take Ω as a real cylinder of the form

$$\Omega = \{(x_1, x') \in \mathbf{R}^n : \; -\infty < x_1 < \infty, \;\; x' \in \tilde{\sigma}_{x_1}\}$$

with

$$\Omega_\tau = \{(x_1, x') \in \Omega : \; |x_1| < \tau\}.$$

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