

# SPECTRA OF TENSOR PRODUCT ELEMENTS I: BASIC THEORY

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## ABSTRACT

This article extends the Waelbroeck vector-valued spectrum by considering arbitrary uniform tensor products and non-commutative Banach algebras. The main result is a vector-valued Gelfand invertibility result.

## 1. Introduction

Our initial interest in the topic of this and two further articles was in contributing to the further development of the infinite-dimensional functional calculus defined, using projective tensor products, by L. Waelbroeck [30; 31]. As our involvement grew, so did the realisation that the subject matter had connections with a number of other topics, for example the Harte joint spectrum, vector-valued Gelfand transforms, invertibility theorems, tensor products of Banach spaces, polynomial estimates and extensions and monomial expansions. We also found that we were ‘discovering’ known results, sometimes with different proofs. Nevertheless, we feel that the setting in which we operate leads to a unified treatment of a number of different topics and an environment in which further development is possible.

### 1.1. Commutative spectra

When  $\mathcal{A}$  is a commutative Banach algebra and  $X$  is a Banach space Waelbroeck defined an  $X$ -valued spectrum,  $\sigma_W(\mathbf{a})$ , of an element  $\mathbf{a}$  of the projective tensor product  $\mathcal{A} \hat{\otimes}_\pi X$ . This theory was extended by M.C. Matos in two articles, [21] and [22]. In [21] Matos considered the spectrum as lying in  $\mathcal{C}(\mathcal{M}(\mathcal{A}), X)$ , where  $\mathcal{A}$  is a uniform Banach algebra with maximal ideal space  $\mathcal{M}(\mathcal{A})$  and  $X$  is a locally convex space with the approximation property; in [22] he considered the spectrum when  $X$  has an unconditional basis and  $\mathcal{A}$  is an arbitrary commutative unital Banach algebra. In all these articles an infinite-dimensional holomorphic functional calculus

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is developed, and in [21] non-trivial applications to infinite-dimensional holomorphy are presented. A commutative theory in algebras with a bounded structure is also developed in [10] and [32].

### 1.2. Non-commutative spectra

In [14] R.E. Harte defined, using left and right invertibility, the joint spectrum of a system of elements in an arbitrary Banach algebra and, subject to certain commutativity hypotheses, obtained spectral mapping theorems. In a different setting G.R. Allan [1, theorem 1] obtained a Gelfand invertibility theorem, in an arbitrary unital Banach algebra  $\mathcal{A}$ , by imposing internal commutativity conditions on a sufficiently large subalgebra.

### 1.3. Tensor products of Banach algebras

The spaces  $L^1(G)$ ,  $G$  a locally compact group, and  $\mathcal{C}(K)$ ,  $K$  a compact Hausdorff topological space, have been at the centre of many investigations in analysis and have been examined from many different perspectives. Their vector-valued extensions  $L^1(G, E) \cong L^1(G) \hat{\otimes}_\pi E$  and  $\mathcal{C}(K, E) \cong \mathcal{C}(K) \hat{\otimes}_e E$ ,  $E$  a Banach space, provided motivation for the introduction of topological tensor products and, when  $E = \mathcal{A}$ , a Banach algebra, for the investigation of tensor products of Banach algebras. That the projective tensor product  $\mathcal{A} \hat{\otimes}_\pi \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  is an algebra and the relationship  $\mathcal{M}(\mathcal{A} \hat{\otimes}_\pi \mathcal{B}) = \mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{B})$  for commutative Banach algebras was established by B.R. Gelbaum [12; 11] and J. Tomiyama [26], and their results were extended to certain locally  $m$ -convex algebras by A. Mallios [19; 20]. N.Th. Varopoulos [27; 28; 29], motivated by harmonic analysis considerations, investigated the injective tensor product and completely characterised the commutative Banach algebras  $\mathcal{A}$  such that  $\mathcal{A} \hat{\otimes}_e \mathcal{A}$  is again a Banach algebra (see Example 19).

We now describe the approach and content of this paper. The Waelbroeck approach suggests the setting—tensor products—in which we operate, while the joint spectrum provides the tools we require.

In §2 we recall (Definition 1) the definition of the joint spectrum and using the dual space  $X'$  define intrinsically (that is, without reference to the different representations of elements in the tensor product) spectra of elements of  $\mathcal{A} \hat{\otimes}_\gamma X$  where  $\mathcal{A}$  is a unital Banach algebra,  $X$  is a Banach space and  $\gamma$  is a uniform crossnorm.

We next define, using multiplicative linear functionals on  $\mathcal{A}$  ( $\mathcal{A}$  commutative), a Waelbroeck spectrum of elements in the same space (the projective tensor product case,  $\pi$ , is considered by Waelbroeck [30; 31], and various other special cases are considered by Matos [21; 22]). In Proposition 7 we show that these definitions coincide (modulo a canonical bidual identification) when  $\mathcal{A}$  is commutative. Taken together these results extend the now-classical Waelbroeck spectrum to the non-commutative setting, to arbitrary uniform tensor norms and to arbitrary Banach spaces, give an intrinsically functorial definition of the spectra and link together the Harte and Waelbroeck approaches. By specialising to a commutative Banach algebra  $\mathcal{A}$  and letting  $X = \mathcal{B}$ , a Banach algebra, we introduce (Definition 8) a vector-valued Gelfand transform on  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ .

In §3 we prove, using the joint spectrum, a number of technical spectral mapping theorems, which leads to a new proof of the result of Allan [1] mentioned above. In §4 we give a number of examples and applications. In subsequent papers we discuss the functional calculus.

As we deal with concepts from a number of different areas we have found it convenient to introduce these as required and to outline their basic properties by providing details in examples. We refer to A. Defant and K. Floret [8] for the theory of tensor products of Banach spaces, and to R.E. Harte [15] and R.G. Douglas [9] for spectral theory and Banach algebra theory.

### 2. Spectra of infinite systems

**Definition 1.** The joint left spectrum  $\sigma^{\text{left}}(\mathbf{a})$  of an arbitrarily indexed collection  $\mathbf{a} := (\mathbf{a}_i)_{i \in I}$  of elements of a Banach algebra  $\mathcal{A}$  with identity  $1_{\mathcal{A}}$  is the set of all  $\lambda := (\lambda_i)_{i \in I}$  such that  $1_{\mathcal{A}}$  does not belong to the left ideal of  $\mathcal{A}$  generated by  $(\mathbf{a}_i - \lambda_i 1_{\mathcal{A}})_{i \in I}$ , that is,

$$\sigma^{\text{left}}(\mathbf{a}) := \left\{ (\lambda_i)_{i \in I} : 1_{\mathcal{A}} \notin \left\{ \sum_{\substack{i \in F \subset I \\ F \text{ finite}}} b_i (\mathbf{a}_i - \lambda_i 1_{\mathcal{A}}) : b_i \in \mathcal{A} \text{ all } i \right\} \right\}. \quad (2.1)$$

*Remark 2.* Since we only use finite sums in the above definition it follows that  $(\lambda_i)_{i \in I} \in \sigma^{\text{left}}((\mathbf{a}_i)_{i \in I})$  if and only if  $(\lambda_i)_{i \in J} \in \sigma^{\text{left}}((\mathbf{a}_i)_{i \in J})$  for every finite subset  $J$  of  $I$ . Moreover, since  $\mathcal{A}$  is unital we may replace the left ideal in Definition 1 by its closure.

The joint right spectrum  $\sigma^{\text{right}}(\mathbf{a})$  is defined to be the joint left spectrum of  $\mathbf{a}$  in  $\mathcal{A}^{\text{op}}$  where  $\mathcal{A}^{\text{op}}$  is the algebra derived from  $\mathcal{A}$  by reversal of products, that is,  $\mathcal{A} = \mathcal{A}^{\text{op}}$  as a set and a Banach space, while the multiplication  $*$  in  $\mathcal{A}^{\text{op}}$  is defined to be  $a * b = b \cdot a$  where  $\cdot$  is the usual multiplication in  $\mathcal{A}$ . The Harte spectrum is defined to be  $\sigma(\mathbf{a}) := \sigma^{\text{left}}(\mathbf{a}) \cup \sigma^{\text{right}}(\mathbf{a})$ . We use  $\sigma_{\mathcal{A}}^{\text{left}}$  etc. if it is necessary to indicate the algebra with respect to which we are taking the spectrum. If  $\mathcal{A}$  is commutative the left and right joint spectra coincide and can be expressed in terms of Gelfand characters:

$$\sigma(\mathbf{a}) := \{h(\mathbf{a}) : h \in \mathcal{M}(\mathcal{A})\}. \quad (2.2)$$

If  $I$  is finite and  $\mathcal{A}$  is commutative this is well known (see, for instance, [23, p. 220], [17, p. 66] or [24]) and

$$\sigma(\mathbf{a}_1, \dots, \mathbf{a}_n) = \{(h(\mathbf{a}_1), \dots, h(\mathbf{a}_n)) : h \in \mathcal{M}(\mathcal{A})\}.$$

If the indexing set  $I$  is a Banach space  $X$  then  $(\mathbf{a}_x)_{x \in X}$  can be identified with the mapping  $\varphi : X \rightarrow \mathcal{A}$  by letting  $\varphi(x) = \mathbf{a}_x$  for all  $x \in X$ , and  $(\lambda_x)_{x \in X}$  can be identified with the mapping  $\lambda : X \rightarrow \mathbb{C}$  where  $\lambda(x) = \lambda_x$  for all  $x \in X$ . Many structural properties of  $\varphi$  are inherited by  $\lambda$ . We collect in the following lemma those

required in this paper but the method of proof can be developed in an obvious way (see [14] for further details).

**Lemma 3.** (a) If  $\varphi \in \mathcal{L}(X; \mathcal{A})$  and  $\lambda \in \sigma^{\text{left}}(\varphi)$  then  $\lambda \in X'$  and  $\|\lambda\| \leq \|\varphi\|$ .

(b) If  $X$  is a Banach algebra,  $\varphi$  is an algebra homomorphism and  $\lambda \in \sigma^{\text{left}}(\varphi)$  then  $\lambda$  is an algebra homomorphism.

PROOF. (a) Since

$$\begin{aligned} \varphi(x_1 + x_2) - \lambda(x_1 + x_2)1_{\mathcal{A}} &= \sum_{i=1}^2 (\varphi(x_i) - \lambda(x_i)1_{\mathcal{A}}) \\ &= (\varphi(x_1 + x_2) - \varphi(x_1) - \varphi(x_2)) - (\lambda(x_1 + x_2) - \lambda(x_1) - \lambda(x_2))1_{\mathcal{A}} \\ &= (\lambda(x_1) + \lambda(x_2) - \lambda(x_1 + x_2))1_{\mathcal{A}} \end{aligned}$$

this is a scalar times the identity, and by the definition of the joint left spectrum if the left-hand side is not invertible the right-hand side must be zero. Hence  $\lambda(x_1 + x_2) = \lambda(x_1) + \lambda(x_2)$ . Similarly  $\lambda(\beta x) = \beta\lambda(x)$  for  $x \in X$  and  $\beta \in \mathbb{C}$ . If  $|\lambda(x)| > \|\varphi(x)\|$  then  $\left\| \frac{\varphi(x)}{\lambda(x)} \right\| < 1$  and  $\varphi(x) - \lambda(x)1_{\mathcal{A}} = \lambda(x) \left( \frac{\varphi(x)}{\lambda(x)} - 1_{\mathcal{A}} \right)$  is invertible. Hence  $|\lambda(x)| \leq \|\varphi(x)\|$  for all  $x \in X$ . This shows that  $\lambda \in X'$  and  $\|\lambda\| \leq \|\varphi\|$ .

(b) Use the method in (a) and the identity

$$\begin{aligned} (\varphi(xy) - \lambda(xy)1_{\mathcal{A}}) - \varphi(x)(\varphi(y) - \lambda(y)1_{\mathcal{A}}) - \lambda(y)(\varphi(x) - \lambda(x)1_{\mathcal{A}}) \\ = (\lambda(x)\lambda(y) - \lambda(xy))1_{\mathcal{A}}. \quad \blacksquare \end{aligned}$$

We let  $\gamma$  denote a uniform crossnorm [8] defined on all pairs of Banach spaces  $X$  and  $Y$ . Hence if  $x \in X$  and  $y \in Y$  then

$$\gamma(x \otimes y) = \|x\| \|y\|,$$

and if  $X_1$  and  $Y_1$  are another pair of Banach spaces,  $T \in \mathcal{L}(X, X_1)$  and  $S \in \mathcal{L}(Y, Y_1)$ , then  $T \otimes S \in \mathcal{L}(X \hat{\otimes}_{\gamma} Y, X_1 \hat{\otimes}_{\gamma} Y_1)$  and

$$\|T \otimes S : X \otimes_{\gamma} Y \longrightarrow X_1 \otimes_{\gamma} Y_1\| \leq \|T\| \|S\|,$$

where  $[T \otimes S](x \otimes y) = T(x) \otimes S(y)$  for all  $x \in X$  and  $y \in Y$ . We let  $X \hat{\otimes}_{\gamma} Y$  denote the completion of  $X \otimes Y$  with respect to the norm  $\gamma$ . We denote by  $\varepsilon$  the injective tensor norm and by  $\pi$  the projective tensor norm. Both  $\varepsilon$  and  $\pi$  are uniform crossnorms and any uniform crossnorm  $\gamma$  satisfies  $\varepsilon \leq \gamma \leq \pi$ .

**Definition 4.** If  $\mathcal{A}$  is a unital Banach algebra,  $X$  is a Banach space and  $\gamma$  is a uniform crossnorm, the left joint tensor spectrum  $\sigma^{\text{left}}(\mathbf{a})$  of  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma} X$  is defined to be the set  $\sigma^{\text{left}}\left(\left([I_{\mathcal{A}} \otimes x'](\mathbf{a})\right)_{x' \in X'}\right)$ .

From our identification above we see that  $\sigma^{\text{left}}(\mathbf{a}) \subseteq X''$  and consists of all  $x'' \in X''$  such that the closed left ideal in  $\mathcal{A}$  generated by

$$\{ [I_{\mathcal{A}} \otimes x'](\mathbf{a}) - x''(x')1_{\mathcal{A}} \}_{x' \in X'}$$

does not contain the identity. By Lemma 3(a), if  $x'' \in \sigma^{\text{left}}(\mathbf{a})$  then

$$\|x''\| \leq \sup \{ \| [I_{\mathcal{A}} \otimes x'](\mathbf{a}) \|, x' \in X', \|x'\| \leq 1 \} \leq \|\mathbf{a}\|.$$

Our next result shows that  $\sigma^{\text{left}}(\mathbf{a})$  can be identified via the canonical mapping  $J_X : X \rightarrow X''$  with a subset of  $X$ . The special case  $X = c_0$  was obtained in a different fashion in [16].

**Proposition 5.** *If  $\mathcal{A}$  is a unital Banach algebra,  $X$  is a Banach space,  $\gamma$  is a uniform crossnorm and  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma} X$ , then if  $x'' \in \sigma^{\text{left}}(\mathbf{a})$  there exists  $x \in X$  such that  $x'' = J_X(x)$ . Consequently we have the identification*

$$\sigma^{\text{left}}(\mathbf{a}) = \left\{ x \in X : 1_{\mathcal{A}} \notin \left\{ \sum_i \mathbf{b}_i ([I_{\mathcal{A}} \otimes x'_i](\mathbf{a} - (1_{\mathcal{A}} \otimes x))) \right\} \right\}$$

where  $(\mathbf{b}_i)_i$  and  $(x'_i)_i$  are finite subsets of  $\mathcal{A}$  and  $X'$  respectively.

PROOF. We first note that if  $\mathbf{b} := \sum_{i=1}^n \mathbf{b}_i \otimes x_i \in \mathcal{A} \otimes X$  then the mapping

$$x' \in (X', \sigma(X', X)) \rightarrow [I_{\mathcal{A}} \otimes x'](\mathbf{b}) = \sum_{i=1}^n x'(x_i)\mathbf{b}_i \in (\mathcal{A}, \|\cdot\|)$$

is continuous (it is possible to show by example that this mapping is not in general continuous when  $\mathbf{b}$  is replaced by an arbitrary element of the tensor product).

If  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma} X$  is arbitrary then for any  $\varepsilon > 0$  we can choose  $\mathbf{b} \in \mathcal{A} \otimes X$  such that  $\|\mathbf{a} - \mathbf{b}\| < \varepsilon$ . Let  $(x'_\alpha)_\alpha$  denote a bounded net in  $X'$  which converges  $\sigma(X', X)$  to  $x' \in X'$ . Since  $\gamma$  is a uniform crossnorm,

$$\begin{aligned} \| [I_{\mathcal{A}} \otimes x'_\alpha](\mathbf{a}) - [I_{\mathcal{A}} \otimes x'](\mathbf{a}) \| &\leq \| [I_{\mathcal{A}} \otimes x'_\alpha](\mathbf{b} - \mathbf{a}) \| + \| [I_{\mathcal{A}} \otimes x'](\mathbf{b} - \mathbf{a}) \| \\ &+ \| [I_{\mathcal{A}} \otimes x'_\alpha](\mathbf{b}) - [I_{\mathcal{A}} \otimes x'](\mathbf{b}) \| \leq 2\varepsilon + \| [I_{\mathcal{A}} \otimes (x'_\alpha - x')](\mathbf{b}) \|. \end{aligned}$$

Hence for  $\mathbf{a}$  arbitrary in  $\mathcal{A} \hat{\otimes}_{\gamma} X$  the mapping

$$x' \in X' \rightarrow [I_{\mathcal{A}} \otimes x'](\mathbf{a}) \in (\mathcal{A}, \|\cdot\|) \tag{2.3}$$

is continuous on each bounded subset of  $X'$  endowed with the  $\sigma(X', X)$  topology.

Let  $(x'_\alpha)_{\alpha \in \Gamma}$  denote a bounded net in  $X'$  which converges  $\sigma(X', X)$  to  $x' \in X'$ . Suppose, by taking a subnet if necessary, that there exists  $\delta > 0$  such that

$|x''(x'_\alpha) - x''(x')| \geq \delta$  for all  $\alpha$ . By (2.3) there exists  $\alpha_0$  such that

$$\| [I_{\mathcal{A}} \otimes (x'_\alpha - x')] (\mathbf{a}) \| \leq \frac{\delta}{2}$$

for all  $\alpha \geq \alpha_0$ . Hence, for  $\alpha \geq \alpha_0$ , we have

$$[I_{\mathcal{A}} \otimes (x'_\alpha - x')] (\mathbf{a}) - x''(x'_\alpha - x')1_{\mathcal{A}} = -x''(x'_\alpha - x') \left[ 1_{\mathcal{A}} - \frac{[I_{\mathcal{A}} \otimes (x'_\alpha - x')] (\mathbf{a})}{x''(x'_\alpha - x')} \right].$$

Since

$$\left\| \frac{[I_{\mathcal{A}} \otimes (x'_\alpha - x')] (\mathbf{a})}{x''(x'_\alpha - x')} \right\| \leq \frac{(\delta/2)}{\delta} = \frac{1}{2}$$

$[I_{\mathcal{A}} \otimes (x'_\alpha - x')] (\mathbf{a}) - x''(x'_\alpha - x')1_{\mathcal{A}}$  is invertible. This contradicts the fact that  $x''$  belongs to  $\sigma^{\text{left}}(\mathbf{a})$ . Hence the linear functional  $x''$  is  $\sigma(X'', X')$ -continuous on the bounded subsets of  $X'$ . By a result of Banach [18, corollary 4, p. 250]  $x''$  is  $\sigma(X', X)$ -continuous, that is, there exists  $x \in X$  such that  $x'' = J_X(x)$ . This completes the proof. ■

We now define the Waelbroeck tensor spectrum (see [30; 31; 21; 22]).

**Definition 6.** If  $\mathcal{A}$  is a commutative Banach algebra,  $X$  is a Banach space and  $\gamma$  is a uniform crossnorm on  $\mathcal{A} \otimes X$ , then the Waelbroeck tensor spectrum  $\sigma_W(\mathbf{a})$  of  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma X$  is the set

$$\{ [h \otimes I_X] (\mathbf{a}) : h \in \mathcal{M}(\mathcal{A}) \}.$$

The relationship between the joint tensor and Waelbroeck spectra is outlined in the next proposition and we may thus regard Definition 4 as an extension to arbitrary unital Banach algebras of the Waelbroeck spectrum. We write  $\sigma$  in place of  $\sigma^{\text{left}}$  when dealing with elements of the tensor product of a commutative unital algebra and a Banach space.

**Proposition 7.** If  $\mathcal{A}$  is a commutative unital Banach algebra,  $X$  is a Banach space,  $\gamma$  is a uniform crossnorm on  $\mathcal{A} \otimes X$  and  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma X$ , then

$$J_X(\sigma_W(\mathbf{a})) = \sigma(\mathbf{a}).$$

PROOF. Let  $\sum_{i=1}^n \mathbf{a}_i \otimes x_i$  denote an arbitrary element in  $\mathcal{A} \otimes X$ . If  $h \in \mathcal{M}(\mathcal{A})$  and

$x' \in X'$  then

$$\begin{aligned}
 h \left( [I_{\mathcal{A}} \otimes x'] \left( \sum_{i=1}^n \mathbf{a}_i \otimes x_i \right) \right) &= h \left( \sum_{i=1}^n x'(x_i) \mathbf{a}_i \right) \\
 &= \sum_{i=1}^n h(\mathbf{a}_i) x'(x_i) \\
 &= [h \otimes x'] \left( \sum_{i=1}^n \mathbf{a}_i \otimes x_i \right) \\
 &= x' \left( \sum_{i=1}^n h(\mathbf{a}_i) x_i \right) \\
 &= x' \left( [h \otimes I_X] \left( \sum_{i=1}^n \mathbf{a}_i \otimes x_i \right) \right).
 \end{aligned}$$

Since  $\gamma$  is a uniform crossnorm and  $\mathcal{A} \otimes X$  is dense in  $\mathcal{A} \hat{\otimes}_{\gamma} X$  we have

$$h \circ [I_{\mathcal{A}} \otimes x'] = h \otimes x' = x' \circ [h \otimes I_X]$$

for all  $h \in \mathcal{M}(\mathcal{A})$  and  $x' \in X'$ .

If  $x'' \in \sigma(\mathbf{a}) \subset X''$  then

$$1_{\mathcal{A}} \notin \left\{ \sum_{\substack{x' \in F \subset X' \\ F \text{ finite}}} \mathbf{a}_i ([I_{\mathcal{A}} \otimes x'](\mathbf{a}) - x''(x')1_{\mathcal{A}}) : \mathbf{a}_i \in \mathcal{A}, \text{ all } i \in F \right\}. \quad (2.4)$$

The set of elements in (2.4) form a proper ideal in  $\mathcal{A}$  which is contained in a closed maximal ideal and hence in the kernel of some  $h \in \mathcal{M}(\mathcal{A})$ . Hence  $h([I_{\mathcal{A}} \otimes x'](\mathbf{a})) = x''(x')$  for all  $x' \in X'$ . This implies  $x'([h \otimes I_X](\mathbf{a})) = h([I_{\mathcal{A}} \otimes x'](\mathbf{a})) = x''(x')$  for all  $x' \in X'$  and hence  $J_X([h \otimes I_X](\mathbf{a})) = x''$ . We thus have  $\sigma(\mathbf{a}) \subset J_X(\sigma_W(\mathbf{a})) \subset J_X(X)$ .

If  $x \in X$  and  $J_X(x) \notin \sigma(\mathbf{a})$  then there exist  $(\mathbf{a}_i)_{i=1}^n \subset \mathcal{A}$  and  $(x'_i)_{i=1}^n \subset X'$  such that

$$1_{\mathcal{A}} = \sum_{i=1}^n \mathbf{a}_i ([I_{\mathcal{A}} \otimes x'_i](\mathbf{a}) - x'_i(x)1_{\mathcal{A}}).$$

Hence, since  $h(1_{\mathcal{A}}) = 1$ , we have

$$\begin{aligned}
 1 &= \sum_{i=1}^n h(\mathbf{a}_i) x'_i([h \otimes I_X](\mathbf{a}) - x) \\
 &= \left[ \sum_{i=1}^n h(\mathbf{a}_i) x'_i \right] ([h \otimes I_X](\mathbf{a}) - x)
 \end{aligned}$$

and  $x \neq [h \otimes I_X](\mathbf{a})$  for any  $h \in \mathcal{M}(\mathcal{A})$ . This implies that  $x \notin \sigma_W(\mathbf{a})$ . Hence  $J_X(\sigma_W(\mathbf{a})) \subset \sigma(\mathbf{a})$ . ■

If  $\mathcal{A}$  and  $\mathcal{B}$  are algebras then the mapping

$$M_1 : (\mathbf{a}_1, b_1, \mathbf{a}_2, b_2) \in \mathcal{A} \times \mathcal{B} \times \mathcal{A} \times \mathcal{B} \longrightarrow \mathbf{a}_1 \mathbf{a}_2 \otimes b_1 b_2 \in \mathcal{A} \otimes \mathcal{B}$$

is 4-linear and the fundamental existence theorem for tensor products implies that there exists a unique linear mapping  $M_2 : \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B}$  such that

$$M_2(\mathbf{a}_1 \otimes b_1 \otimes \mathbf{a}_2 \otimes b_2) = M_1(\mathbf{a}_1, b_1, \mathbf{a}_2, b_2).$$

Associativity of tensor products and the fundamental existence theorem imply that there exists a unique bilinear mapping  $M_3 : (\mathcal{A} \otimes \mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B}) \longrightarrow \mathcal{A} \otimes \mathcal{B}$  such that

$$M_3(\mathbf{a}_1 \otimes b_1, \mathbf{a}_2 \otimes b_2) = \mathbf{a}_1 \mathbf{a}_2 \otimes b_1 b_2.$$

Hence  $\mathcal{A} \otimes \mathcal{B}$  is an algebra with multiplication defined by  $M_3$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are Banach algebras and  $\mathcal{A} \otimes \mathcal{B}$  is endowed with the projective tensor norm then  $\|M_1\| \leq 1$ . Hence  $\|M_2\| \leq 1$  and  $\|M_3\| \leq 1$  and  $\mathcal{A} \otimes_\pi \mathcal{B}$  is a normed algebra. A density argument shows that  $\mathcal{A} \hat{\otimes}_\pi \mathcal{B}$  is a Banach algebra. This, as we have noted in our introduction, has been known for many years (see, for example, [11; 12; 19; 20; 26]).

If  $\mathcal{A}$  and  $\mathcal{B}$  are unital then  $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$  is the identity of  $\mathcal{A} \otimes \mathcal{B}$ . We say that the uniform crossnorm  $\gamma$  on  $\mathcal{A} \otimes \mathcal{B}$  is suitable for  $\mathcal{A} \otimes \mathcal{B}$  if  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  is a Banach algebra when endowed with the product derived from  $M_3$ . Further examples are given in Section 4.

If  $K$  is a compact topological space and  $\mathcal{B}$  is a Banach algebra we let  $\mathcal{C}(K; \mathcal{B})$  denote the algebra of continuous  $\mathcal{B}$ -valued mappings on  $K$ . When endowed with the norm  $\|\cdot\|_K$ ,  $\mathcal{C}(K; \mathcal{B})$  is a Banach algebra and if  $\mathcal{B}$  is unital the mapping  $1_{\mathcal{B}}(x) := 1_{\mathcal{B}}$  is the identity.

We maintain the above notation. Let  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma X$  and let  $(h_\alpha)_{\alpha \in \Gamma}$  denote a net in  $\mathcal{M}(\mathcal{A})$  which converges in the weak\* topology on  $\mathcal{M}(\mathcal{A})$  to  $h$ . Let  $\varepsilon > 0$  be arbitrary. Choose  $\mathbf{b} := \sum_{i=1}^k \mathbf{a}_i \otimes x_i \in \mathcal{A} \otimes X$  so that  $\|\mathbf{b} - \mathbf{a}\| < \varepsilon$ . Now choose  $\alpha_0$  such that

$$\|[h_\alpha \otimes I_X - h \otimes I_X](\mathbf{b})\| \leq \sum_{i=1}^k |[h_\alpha - h](\mathbf{a}_i)| \cdot \|x_i\| \leq \varepsilon$$

for all  $\alpha \geq \alpha_0$ . We have, for  $\alpha \geq \alpha_0$ ,

$$\begin{aligned} \|[h_\alpha \otimes I_X](\mathbf{a}) - [h \otimes I_X](\mathbf{a})\| &\leq \|[h_\alpha \otimes I_X](\mathbf{a}) - [h_\alpha \otimes I_X](\mathbf{b})\| \\ &\quad + \|[h_\alpha \otimes I_X](\mathbf{b}) - [h \otimes I_X](\mathbf{b})\| \\ &\quad + \|[h \otimes I_X](\mathbf{b}) - [h \otimes I_X](\mathbf{a})\| \\ &\leq 3\varepsilon. \end{aligned}$$

This implies that  $[h_\alpha \otimes I_X](\mathbf{a}) \rightarrow [h \otimes I_X](\mathbf{a})$  as  $\alpha \rightarrow \infty$  in  $X$  and, in the notation of the following definition, that  $\hat{\mathbf{a}} \in \mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$ . The next definition is implicit in the work of Waelbroeck [30; 31] and Matos [21; 22].

**Definition 8.** If  $\mathcal{A}$  and  $\mathcal{B}$  are Banach algebras with  $\mathcal{A}$  commutative and  $\gamma$  is a suitable tensor norm on  $\mathcal{A} \otimes \mathcal{B}$  then the Gelfand transform  $\hat{\cdot}$  from  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  into  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$  is defined by letting

$$\hat{\mathbf{a}}(h) = [h \otimes I_{\mathcal{B}}](\mathbf{a})$$

for all  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  and all  $h \in \mathcal{M}(\mathcal{A})$ .

*Remark 9.* Notice that  $I_{\mathcal{B}}$  above is the identity in the algebra  $\mathcal{L}(\mathcal{B}, \mathcal{B})$ . Note also that the transform does not depend on the crossnorm; that is, if  $\mathbf{a}$  lies in two tensor products (see Corollary 23) it will have the same Gelfand transform in each case.

**Proposition 10.** *The Gelfand transform is an algebra homomorphism and a linear contraction.*

PROOF. Since  $h \otimes I_{\mathcal{B}} \in \mathcal{L}(\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}; \mathcal{B})$  the transform is linear. Moreover, since  $\gamma$  is a uniform crossnorm,

$$\|\hat{\mathbf{a}}(h)\| = \|[h \otimes I_{\mathcal{B}}](\mathbf{a})\| \leq \|h\| \|I_{\mathcal{B}}\| \|\mathbf{a}\|$$

and  $\|\hat{\mathbf{a}}(h)\| \leq \|\mathbf{a}\|$ . This shows that  $\hat{\cdot}$  is a contraction.

If  $\mathbf{a} = \sum_{i=1}^k a_i \otimes c_i$  and  $\mathbf{b} = \sum_{j=1}^n b_j \otimes d_j$  and  $h \in \mathcal{M}(\mathcal{A})$  then

$$\begin{aligned} \widehat{\mathbf{a} \cdot \mathbf{b}}(h) &= [h \otimes I_{\mathcal{B}}](\mathbf{a} \cdot \mathbf{b}) \\ &= [h \otimes I_{\mathcal{B}}] \left( \sum_{i,j=1}^{k,n} a_i b_j \otimes c_i d_j \right) \\ &= \sum_{i,j=1}^{k,n} h(a_i b_j) c_i d_j \\ &= \sum_{i,j=1}^{k,n} h(a_i) h(b_j) c_i d_j \\ &= \sum_{i=1}^k h(a_i) c_i \cdot \sum_{j=1}^n b_j h(b_j) d_j \\ &= [h \otimes I_{\mathcal{B}}](\mathbf{a}) \cdot [h \otimes I_{\mathcal{B}}](\mathbf{b}) \\ &= \hat{\mathbf{a}}(h) \hat{\mathbf{b}}(h). \end{aligned}$$

Hence  $\widehat{\mathbf{a}\mathbf{b}} = \widehat{\mathbf{a}}\widehat{\mathbf{b}}$ . Since  $\gamma$  is a suitable crossnorm, a density argument shows that the Gelfand transform is an algebra homomorphism. ■

### 3. Stability of the spectrum under projections

In this section we prove a number of technical results concerning the behaviour of the spectrum under certain transformations. A number of these results have appeared in [14] and [16]. We take the opportunity to expand on their proofs here.

**Proposition 11.** *If  $\mathcal{A}$  is a unital Banach algebra and if  $\mathbf{a} = (a_i)_{i \in I}$  and  $\mathbf{b} = (b_j)_{j \in J}$  then*

$$\sigma^{\text{left}}(\mathbf{a}, \mathbf{b}) \subset \sigma^{\text{left}}(\mathbf{a}) \times \sigma^{\text{left}}(\mathbf{b}).$$

*If the commutativity relations*

$$a_i a_k = a_k a_i \text{ for } i, k \in I \text{ and } a_i b_j = b_j a_i \text{ for } i \in I \text{ and } j \in J$$

*are satisfied, then*

$$\pi_J(\sigma^{\text{left}}(\{\mathbf{a}, \mathbf{b}\})) = \sigma^{\text{left}}(\mathbf{b})$$

*where  $\pi_J$  is the canonical projection from  $\mathbb{C}^I \times \mathbb{C}^J$  onto  $\mathbb{C}^J$ .*

PROOF. By Remark 2 we have

$$\pi_J(\sigma^{\text{left}}(\{\mathbf{a}, \mathbf{b}\})) \subset \sigma^{\text{left}}(\mathbf{b})$$

and

$$\pi_I(\sigma^{\text{left}}(\{\mathbf{a}, \mathbf{b}\})) \subset \sigma^{\text{left}}(\mathbf{a}).$$

This implies  $\sigma^{\text{left}}(\{\mathbf{a}, \mathbf{b}\}) \subset \sigma^{\text{left}}(\mathbf{a}) \times \sigma^{\text{left}}(\mathbf{b})$ . We prove the reverse inclusion by transfinite induction on  $I$ . First suppose  $|I| = 1$ . In this case  $(a_i)_{i \in I}$  can be identified with an element  $\mathbf{a} \in \mathcal{A}$ . Let  $\mu = (\mu_j)_{j \in J} \in \sigma^{\text{left}}(\mathbf{b})$  and let  $N$  denote the closed left ideal of  $\mathcal{A}$  generated by  $(b_j - \mu_j 1_{\mathcal{A}})_{j \in J}$ . Let  $M = \{y \in \mathcal{A} : Ny \subset N\}$ . Clearly  $1_{\mathcal{A}} \in M$  and

$$N(y_1 + y_2) \subseteq Ny_1 + Ny_2 \subseteq N + N = N$$

$$N(\theta y_1) = \theta Ny_1 \subseteq Ny_1 \subseteq N$$

$$N(y_1 y_2) \subseteq (Ny_1)y_2 \subseteq Ny_2 \subseteq N$$

show that  $M$  is a unital subalgebra of  $\mathcal{A}$ . If  $x \in \mathcal{A}$  and  $j \in J$  then, by our commutativity assumptions,

$$x(b_j - \mu_j 1_{\mathcal{A}}) \cdot \mathbf{a} = x \cdot \mathbf{a}(b_j - \mu_j 1_{\mathcal{A}}) \in N$$

and  $\mathbf{a} \in M$ . If  $n \in N$  then

$$Nn \subseteq N.N \subseteq \mathcal{A}.N = N,$$

so  $N \subseteq M$  and, by construction of  $M$ ,  $N$  is a two-sided closed ideal in  $M$ . Now consider the unital Banach algebra  $M/N$  and choose  $\lambda \in \partial(\sigma_{M/N}(\mathbf{a} + N))$ , the boundary of the spectrum of  $\mathbf{a} + N$  in  $M/N$ . Our choice implies that  $\mathbf{a} - \lambda 1_{\mathcal{A}} + N$  is a topological divisor of zero in  $M/N$ . We claim that  $(\lambda, \mu) \in \sigma^{\text{left}}(\{\mathbf{a}, \mathbf{b}\})$ . If not, there exists  $\mathbf{a}' \in \mathcal{A}$ ,  $(b'_j)_{j \in J} \in \mathcal{A}^J$ , with  $b'_j = 0$  for all but a finite number of  $j$  such that

$$1_{\mathcal{A}} = \mathbf{a}'(\mathbf{a} - \lambda 1_{\mathcal{A}}) + \sum_j b'_j (b_j - \mu_j 1_{\mathcal{A}}).$$

If  $y \in M$  then

$$y - \mathbf{a}'(\mathbf{a} - \lambda 1_{\mathcal{A}})y = \sum_j b'_j (b_j - \mu_j 1_{\mathcal{A}})y \in N$$

and

$$y + N = \mathbf{a}'(\mathbf{a} - \lambda 1_{\mathcal{A}})y + N. \tag{3.1}$$

Since  $N$  is a left ideal in  $\mathcal{A}$ ,  $\mathbf{a}'N \subseteq N$ . Hence, if  $\mathbf{z} \in N$ , (3.1) implies that we can find  $\mathbf{w} \in N$  such that

$$y + \mathbf{w} = \mathbf{a}'(\mathbf{a} - \lambda 1_{\mathcal{A}})y + \mathbf{a}'\mathbf{z} = \mathbf{a}'[(\mathbf{a} - \lambda 1_{\mathcal{A}})y + \mathbf{z}]$$

and

$$\|y + \mathbf{w}\| \leq \|\mathbf{a}'\| \|(\mathbf{a} - \lambda 1_{\mathcal{A}})y + \mathbf{z}\|.$$

Since  $\mathbf{z}$  was arbitrary we have

$$\|y + N\|_{M/N} \leq \|\mathbf{a}'\| \|(\mathbf{a} - \lambda 1_{\mathcal{A}})y + N\|_{M/N} \tag{3.2}$$

for all  $y \in M$ . As  $\mathbf{a} - \lambda 1_{\mathcal{A}} + N$  is a topological divisor of zero in  $M/N$  there exists  $(y_n)_n \subset M$  such that  $\|y_n + N\|_{M/N} = 1$  and  $\|(\mathbf{a} - \lambda 1_{\mathcal{A}})y_n + N\|_{M/N} \rightarrow 0$ . This contradicts (3.2) and establishes our claim. Since  $\pi_J(\{\lambda, \mu\}) = \mu$  this completes the proof when  $|I| = 1$ .

For the general case we use transfinite induction and for this purpose we may suppose that  $I$  is the set of all ordinals less than or equal to the ordinal  $\alpha$ . Let  $\mu \in \sigma^{\text{left}}(\mathbf{b})$ . By the  $|I| = 1$  case we can find  $\lambda_1$  such that  $(\lambda_1, \mu) \in \sigma^{\text{left}}(\{\mathbf{a}_1, \mathbf{b}\})$ . Suppose that  $\beta \leq \alpha$  and that we have found  $(\lambda_\omega)_{\omega < \beta}$  such that for all  $\beta_0 < \beta$

$$((\lambda_\omega)_{\omega \leq \beta_0}, \mu) \in \sigma^{\text{left}}(\{(\mathbf{a}_\omega)_{\omega \leq \beta_0}, \mathbf{b}\}).$$

If  $\beta$  has an immediate predecessor  $\beta_1$  then the first part of the proof shows that we can find  $\lambda_\beta$  such that

$$(\lambda_\beta, \{(\lambda_\omega)_{\omega \leq \beta_1}, \mu\}) \in \sigma^{\text{left}}(\mathbf{a}_\beta, \{(\mathbf{a}_\omega)_{\omega \leq \beta_1}, \mathbf{b}\}),$$

that is,

$$\{(\lambda_\omega)_{\omega \leq \beta}, \mu\} \in \sigma^{\text{left}}((\mathbf{a}_\omega)_{\omega \leq \beta}, \mathbf{b}).$$

If  $\beta$  does not have an immediate predecessor then  $\beta$  is infinite and by Remark 2

$$\{(\lambda_\omega)_{\omega < \beta}, \mu\} \in \sigma^{\text{left}}((\mathbf{a}_\omega)_{\omega < \beta}, \mathbf{b}).$$

The addition of a single element  $\mathbf{a}_\beta$  and the first part of the proof show that we can find  $\lambda_\beta$  such that

$$\{(\lambda_\omega)_{\omega \leq \beta}, \mu\} \in \sigma^{\text{left}}((\mathbf{a}_\omega)_{\omega \leq \beta}, \mathbf{b}).$$

By transfinite induction we can thus find  $\lambda := (\lambda_\omega)_{\omega \leq \alpha}$  such that  $(\lambda, \mu) \in \sigma^{\text{left}}(\mathbf{a}, \mathbf{b})$ . Since  $\pi_J(\lambda, \mu) = \mu$ , this completes the proof. ■

**Corollary 12.** *If  $(\mathbf{a}_i)_{i \in I}$  is a commuting system of elements in a Banach algebra, then for any  $J \subset I$*

$$\pi_J(\sigma^{\text{left}}((\mathbf{a}_i)_{i \in I})) = \sigma^{\text{left}}((\mathbf{a}_j)_{j \in J}).$$

**Proposition 13.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be unital Banach algebras and suppose that  $\gamma$  is a suitable crossnorm. If  $\mathbf{a} := (\mathbf{a}_i)_{i \in I} \subset \mathcal{A}$ ,  $\mathbf{b} := (\mathbf{b}_j)_{j \in J} \subset \mathcal{B}$ ,  $\mathbf{a} \otimes 1_{\mathcal{B}} := (\mathbf{a}_i \otimes 1_{\mathcal{B}})_{i \in I}$  and  $1_{\mathcal{A}} \otimes \mathbf{b} := (1_{\mathcal{A}} \otimes \mathbf{b}_j)_{j \in J}$  then*

$$\sigma_{\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}}^{\text{left}}(\{\mathbf{a} \otimes 1_{\mathcal{B}}, 1_{\mathcal{A}} \otimes \mathbf{b}\}) = \sigma_{\mathcal{A}}^{\text{left}}(\mathbf{a}) \times \sigma_{\mathcal{B}}^{\text{left}}(\mathbf{b}).$$

**PROOF.** If  $\lambda \notin \sigma^{\text{left}}(\mathbf{a})$  or  $\mu \notin \sigma^{\text{left}}(\mathbf{b})$  it is easily seen that  $(\lambda, \mu)$  is not an element of  $\sigma_{\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}}^{\text{left}}(\{\mathbf{a} \otimes 1_{\mathcal{B}}, 1_{\mathcal{A}} \otimes \mathbf{b}\})$ . Hence

$$\sigma_{\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}}^{\text{left}}(\{\mathbf{a} \otimes 1_{\mathcal{B}}, 1_{\mathcal{A}} \otimes \mathbf{b}\}) \subseteq \sigma_{\mathcal{A}}^{\text{left}}(\mathbf{a}) \times \sigma_{\mathcal{B}}^{\text{left}}(\mathbf{b}).$$

Suppose  $\lambda := (\lambda_i)_{i \in I} \in \sigma_{\mathcal{A}}^{\text{left}}(\mathbf{a})$  and  $\mu := (\mu_j)_{j \in J} \in \sigma_{\mathcal{B}}^{\text{left}}(\mathbf{b})$ . The closed left ideals  $M$  and  $N$  in  $\mathcal{A}$  and  $\mathcal{B}$  generated by  $(\mathbf{a}_i - \lambda_i 1_{\mathcal{A}})_{i \in I}$  and  $(\mathbf{b}_j - \mu_j 1_{\mathcal{B}})_{j \in J}$  respectively are proper. By the Hahn–Banach Theorem there exist  $x' \in \mathcal{A}'$  and  $y' \in \mathcal{B}'$  such that  $x'(1_{\mathcal{A}}) = y'(1_{\mathcal{B}}) = 1$  and  $x'(M) = y'(N) = 0$ . Since  $\gamma$  is a uniform crossnorm,  $x' \otimes y' \in (\mathcal{A} \hat{\otimes}_\gamma \mathcal{B})'$ . We have  $[x' \otimes y'](1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = 1$  and  $[x' \otimes y'](M \otimes \mathcal{B} + \mathcal{A} \otimes N) = 0$ . Hence  $M \otimes \mathcal{B} + \mathcal{A} \otimes N$  generates a proper closed left ideal in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ . Since this ideal contains  $(\mathbf{a}_i - \lambda_i 1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}$  for all  $i \in I$  and  $1_{\mathcal{A}} \otimes (\mathbf{b}_j - \mu_j 1_{\mathcal{B}})$  for all  $j \in J$  it

follows that

$$\left\{ (\lambda_i)_{i \in I}, (\mu_j)_{j \in J} \right\} = (\lambda, \mu) \in \sigma_{\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}}^{\text{left}}(\{\mathbf{a} \otimes 1_{\mathcal{B}}, 1_{\mathcal{A}} \otimes \mathbf{b}\}). \quad \blacksquare$$

We will also require the following result.

**Lemma 14.** *If  $\mathcal{A}$  is a commutative unital Banach algebra,  $\mathcal{B}$  is a unital Banach algebra,  $\gamma$  is a suitable crossnorm for  $\mathcal{A} \otimes \mathcal{B}$ ,  $\mathbf{b} \in \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ ,  $\mu \in \mathbb{C}$  and  $h \in \mathcal{M}(\mathcal{A})$ , then the closed ideals generated by*

$$\{[(\mathbf{a} - h(\mathbf{a})1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{\mathbf{a} \in \mathcal{A}}, \mathbf{b} - \mu(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})\}$$

and

$$\{[(\mathbf{a} - h(\mathbf{a})1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{\mathbf{a} \in \mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{b}) - \mu 1_{\mathcal{B}})\}$$

coincide.

PROOF. It suffices to show that

$$(\mathbf{b} - \mu(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})) - (1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{b}) - \mu 1_{\mathcal{B}})) = \mathbf{b} - 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{b}))$$

belongs to the closed left ideal  $J$  generated by  $\{(\mathbf{a} - h(\mathbf{a})1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}\}_{\mathbf{a} \in \mathcal{A}}$ . If  $\mathbf{c} = \sum_{i=1}^n \mathbf{a}_i \otimes \mathbf{b}_i \in \mathcal{A} \otimes \mathcal{B}$  then

$$\begin{aligned} \mathbf{c} - 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{c})) &= \sum_{i=1}^n \mathbf{a}_i \otimes \mathbf{b}_i - 1_{\mathcal{A}} \otimes \sum_{i=1}^n h(\mathbf{a}_i) \mathbf{b}_i \\ &= \sum_{i=1}^n \mathbf{a}_i \otimes \mathbf{b}_i - \sum_{i=1}^n h(\mathbf{a}_i) 1_{\mathcal{A}} \otimes \mathbf{b}_i \\ &= \sum_{i=1}^n (\mathbf{a}_i - h(\mathbf{a}_i) 1_{\mathcal{A}}) \otimes \mathbf{b}_i. \end{aligned}$$

Hence  $\mathbf{c} - (1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{c}))) \in J$  for all  $\mathbf{c} \in \mathcal{A} \otimes \mathcal{B}$ . Since  $\gamma$  is a uniform crossnorm and  $\mathcal{A} \otimes \mathcal{B}$  is dense in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  a standard density argument completes the proof.  $\blacksquare$

**Proposition 15.** *If  $\mathcal{A}$  is a commutative unital Banach algebra,  $\mathcal{B}$  is a unital Banach algebra,  $\gamma$  is a suitable crossnorm for  $\mathcal{A} \otimes \mathcal{B}$  and  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ , then*

$$\sigma_{\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}}^{\text{left}}(\mathbf{a}) = \bigcup_{h \in \mathcal{M}(\mathcal{A})} \{ \sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathbf{a})) \}.$$

PROOF. Since  $h \otimes I_{\mathcal{B}}$  is an algebra homomorphism and  $[h \otimes I_{\mathcal{B}}](1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = 1_{\mathcal{B}}$ ,  $h \otimes I_{\mathcal{B}}$  maps left invertible elements to left invertibles. Hence

$$\sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathbf{a})) \subset \sigma_{\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}}^{\text{left}}(\mathbf{a})$$

for all  $h \in \mathcal{M}(\mathcal{A})$ .

Suppose  $\mu \in \sigma_{\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}}^{\text{left}}(\mathbf{a})$ . Since the system  $\{(\mathbf{a} \otimes 1_{\mathcal{B}})_{\mathbf{a} \in \mathcal{A}}, \mathbf{a}\}$  is commutative, Lemma 3 and Proposition 11 show that there exists  $h \in \mathcal{M}(\mathcal{A})$  such that

$$((h(\mathbf{a}))_{\mathbf{a} \in \mathcal{A}}, \mu) \in \sigma_{\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}}^{\text{left}}((\mathbf{a} \otimes 1_{\mathcal{B}})_{\mathbf{a} \in \mathcal{A}}, \mathbf{a}).$$

By Lemma 14

$$\{[(\mathbf{a} - h(\mathbf{a})1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{\mathbf{a} \in \mathcal{A}}, \mathbf{a} - \mu(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})\}$$

and

$$\{[(\mathbf{a} - h(\mathbf{a})1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{\mathbf{a} \in \mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{a}) - \mu 1_{\mathcal{B}})\}$$

generate the same proper left ideal in  $\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}$ . Hence

$$((h(\mathbf{a}))_{\mathbf{a} \in \mathcal{A}}, \mu) \in \sigma_{\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}}^{\text{left}}((\mathbf{a} \otimes 1_{\mathcal{B}})_{\mathbf{a} \in \mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathbf{a}))).$$

By Proposition 13 this coincides with

$$\sigma_{\mathcal{A}}((\mathbf{a})_{\mathbf{a} \in \mathcal{A}}) \times \sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathbf{a})).$$

Hence  $\mu \in \sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathbf{a}))$ . This completes the proof. ■

The above methods lead to an alternative proof of a result of G.R. Allan [1, theorem 1]. In passing we note that the conditions imposed to obtain this result appear very natural when described in a tensor product setting.

**Proposition 16.** *Let  $Z_1$  and  $\mathcal{A}$  denote closed unital subalgebras of the unital Banach algebra  $\mathcal{B}$ . We suppose that*

- (i)  $Z_1$  is contained in the centre of  $\mathcal{B}$ ,
- (ii)  $Z_1$  and  $\mathcal{A}$  generate a dense subalgebra of  $\mathcal{B}$ ,
- (iii) for each  $\varphi \in \mathcal{M}(Z_1)$  there exists  $M_{\varphi} > 0$  such that

$$\left\| \sum_i \varphi(z_i) \mathbf{a}_i \right\| \leq M_{\varphi} \left\| \sum_i z_i \mathbf{a}_i \right\| \quad (3.3)$$

for all finite sums  $\sum_i z_i \mathbf{a}_i$ ,  $z_i \in Z_1$  and  $\mathbf{a}_i \in \mathcal{A}$ ;  
then

$$\sigma_{\mathcal{B}}^{\text{left}}(\mathbf{b}) = \bigcup_{\varphi \in \mathcal{M}(Z_1)} \sigma_{\mathcal{A}}^{\text{left}}(\tilde{\varphi}(\mathbf{b})) \quad (3.4)$$

for all  $b \in \mathcal{B}$  where, for  $\varphi \in \mathcal{M}(Z_1)$ , we denote by  $\tilde{\varphi}$  the unique extension of  $\varphi$  to an  $\mathcal{A}$ -valued homomorphism on  $\mathcal{B}$  satisfying  $\tilde{\varphi}|_{\mathcal{A}} = I_{\mathcal{A}}$ .

*Remark 17.* To see the relationships between this result and our approach note that conditions (i) and (ii) together are our hypothesis in Proposition 11 if we identify  $(a_i)_{i \in I}$  with  $Z_1$  and  $(b_j)_{j \in J}$  with  $\mathcal{A}$ . Since  $Z_1$  and  $\mathcal{A}$  are subalgebras of  $\mathcal{B}$  and  $Z_1$  commutes with all elements of  $\mathcal{A}$ , the subalgebra generated by  $Z_1$  and  $\mathcal{A}$  consists of all finite sums  $\sum_i z_i a_i$ ,  $z_i \in Z_1$ ,  $a_i \in \mathcal{A}$ . Condition (iii) implies that for each  $\varphi \in \mathcal{M}(Z_1)$ ,  $\varphi \otimes I_{\mathcal{A}}$  is well defined on the tensor product  $Z_1 \otimes \mathcal{A}$  and continuous when the tensor product has the topology induced by  $\mathcal{B}$ . Moreover, condition (3.3) implies that for each  $\varphi \in \mathcal{M}(Z_1)$  the mappings  $\tilde{\varphi}$  in (3.4) are well defined, continuous, and uniquely determined by  $\varphi$ . Hence this condition is analogous to our condition that the tensor crossnorm is uniform.

PROOF OF PROPOSITION 16. Since  $\tilde{\varphi} : \mathcal{B} \rightarrow \mathcal{A}$  is an algebra homomorphism

$$\bigcup_{\varphi \in \mathcal{M}(Z_1)} \sigma_{\mathcal{A}}^{\text{left}}(\tilde{\varphi}(b)) \subset \sigma_{\mathcal{B}}^{\text{left}}(b).$$

Let  $\mu \in \sigma_{\mathcal{B}}^{\text{left}}(b)$ . By (i) and Proposition 11 there exists  $(\lambda_z)_{z \in Z_1} \in \sigma((z)_{z \in Z_1})$  such that  $((\lambda_z)_{z \in Z_1}, \mu) \in \sigma^{\text{left}}((z)_{z \in Z_1}, b)$ . By Lemma 3(b) there exists  $\varphi \in \mathcal{M}(Z_1)$  such that  $\lambda_z = \varphi(z)$  for all  $z \in Z_1$ . If  $c := \sum_{i=1}^n z_i a_i$  then, by (iii),

$$\tilde{\varphi}\left(\sum_{i=1}^n z_i a_i\right) = \sum_{i=1}^n \varphi(z_i) a_i.$$

By (i)

$$c - \varphi(c) = \sum_{i=1}^n (z_i - \varphi(z_i)1_{\mathcal{B}}) a_i = \sum_{i=1}^n a_i (z_i - \varphi(z_i)1_{\mathcal{B}}). \tag{3.5}$$

Since  $(b - \mu 1_{\mathcal{B}}) - (\tilde{\varphi}(b) - \mu 1_{\mathcal{B}}) = b - \tilde{\varphi}(b)$  it follows by a density argument and (3.5) (see Lemma 14) that the closed left ideals in  $\mathcal{B}$  generated by

$$((z - \varphi(z)1_{\mathcal{B}})_{z \in Z_1}, \tilde{\varphi}(b) - \mu 1_{\mathcal{B}})$$

and

$$((z - \varphi(z)1_{\mathcal{B}})_{z \in Z_1}, b - \mu 1_{\mathcal{B}})$$

coincide. Since  $((\varphi(z))_{z \in Z_1}, \mu) \in \sigma^{\text{left}}((\varphi(z))_{z \in Z_1}, b)$  both of these ideals are proper and

$$((\varphi(z))_{z \in Z_1}, \mu) \in \sigma_{\mathcal{B}}^{\text{left}}((z)_{z \in Z_1}, \tilde{\varphi}(b)).$$

A further application of Proposition 11 implies that  $\mu \in \sigma_{\mathcal{B}}^{\text{left}}(\tilde{\varphi}(b))$  and since  $\mathcal{A} \subset \mathcal{B}$  we have  $\mu \in \sigma_{\mathcal{A}}^{\text{left}}(\tilde{\varphi}(b))$ . Hence

$$\sigma_{\mathcal{B}}^{\text{left}}(b) = \bigcup_{\varphi \in \mathcal{H}(Z_1)} \sigma_{\mathcal{A}}^{\text{left}}(\tilde{\varphi}(b))$$

and this completes the proof. ■

*Remark 18.* A similar argument leads to the analogous result for the right spectrum and combining the two cases we obtain a result for the (two-sided) spectrum.

Allan gave a number of refinements and applications in [2] and [3] and, in particular, noted that his extension property is satisfied by the projective tensor product of a commutative unital Banach algebra and a Banach algebra. The  $C^*$ -algebra version of Allan's Theorem was rediscovered by Douglas [9, theorem 7.47] (see also [25] and [7]).

#### 4. Examples

*Example 19.* Varopoulos [28; 29] defined injective algebras (condition (1) below) and obtained the following characterisation.

The following are equivalent conditions on the commutative Banach algebra  $\mathcal{A}$ :

- (1)  $\mathcal{A} \hat{\otimes}_{\varepsilon} \mathcal{A}$  is a Banach algebra,
- (2)  $\mathcal{A}$  is a complemented subalgebra of a uniform algebra (see below),
- (3)  $\mathcal{A} \hat{\otimes}_{\varepsilon} \mathcal{B}$  is a Banach algebra for every (commutative) Banach algebra  $\mathcal{B}$ .

In a further paper [27] he provided a number of non-trivial examples of injective non-uniform Banach algebras. For instance, if  $A_{\alpha}$ ,  $\alpha \geq 0$ , is the algebra of functions  $f$  on the unit circle  $\mathbb{T}$ ,

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{in\theta}$$

normed by

$$\|f\|_{\alpha} := \sum_{n=-\infty}^{\infty} |\hat{f}_n| (1 + |n|^{\alpha})$$

then  $A_{\alpha}$  is injective if and only if  $\alpha > 1/2$ . Note that  $A_{\alpha} \cong \ell_1$  as a Banach space for any  $\alpha$  and that  $A_0$  is the classical Wiener space of absolutely convergent Fourier series (see Example 22).

We illustrate some properties of the  $\varepsilon$ -norm by showing that uniform algebras are injective. Let  $\mathcal{A}$  denote a uniform algebra; that is,  $\mathcal{A}$  is a Banach algebra satisfying  $\|x^2\| = \|x\|^2$  for all  $x \in \mathcal{A}$ . Uniform algebras have been characterised as the closed subalgebras of  $\mathcal{C}(K)$  for  $K$  compact Hausdorff and in particular are always commutative. Let  $\mathcal{B}$  denote a Banach algebra. Using the fact that the  $\varepsilon$ -tensor product respects subspaces it is possible to show that  $\mathcal{A} \hat{\otimes}_{\varepsilon} \mathcal{B}$  is a Banach

subalgebra of  $\mathcal{C}(K; \mathcal{B})$ , but we prefer to prove this directly. If  $\mathbf{a} := \sum_{i=1}^k a_i \otimes b_i$  and  $\mathbf{b} := \sum_{j=1}^n a'_j \otimes b'_j$  belong to  $\mathcal{A} \otimes \mathcal{B}$  then

$$\begin{aligned} \|\mathbf{ab}\|_\varepsilon &= \left\| \sum_{i,j=1}^{k,n} a_i a'_j \otimes b_i b'_j \right\|_\varepsilon \\ &= \sup_{\substack{\psi \in \mathcal{A}' \\ \varphi \in \mathcal{B}' \\ \|\psi\| \leq 1, \|\varphi\| \leq 1}} \left| \sum_{i,j=1}^{k,n} \psi(a_i a'_j) \varphi(b_i b'_j) \right| \\ &= \sup_{\varphi \in \mathcal{B}', \|\varphi\| \leq 1} \left\| \sum_{i,j=1}^{k,n} \varphi(b_i b'_j) a_i a'_j \right\|_{\mathcal{A}} \\ &= \sup_{\varphi \in \mathcal{B}', \|\varphi\| \leq 1} \sup_{h \in \mathcal{M}(\mathcal{A})} \left| \sum_{i,j=1}^{k,n} \varphi(b_i b'_j) h(a_i) h(a'_j) \right| \end{aligned}$$

(since  $\mathcal{A}$  is a uniform algebra the classical Gelfand transform is an isometry)

$$\begin{aligned} &= \sup_{h \in \mathcal{M}(\mathcal{A})} \left\| \sum_{i,j=1}^{k,n} h(a_i) h(a'_j) b_i b'_j \right\|_{\mathcal{B}} \\ &= \sup_{h \in \mathcal{M}(\mathcal{A})} \left\| \left( \sum_{i=1}^k h(a_i) b_i \right) \cdot \left( \sum_{j=1}^n h(a'_j) b'_j \right) \right\|_{\mathcal{B}} \\ &\leq \sup_{h \in \mathcal{M}(\mathcal{A})} \left\| \sum_{i=1}^k h(a_i) b_i \right\|_{\mathcal{B}} \cdot \sup_{h \in \mathcal{M}(\mathcal{A})} \left\| \sum_{j=1}^n h(a'_j) b'_j \right\|_{\mathcal{B}} \\ &= \sup_{h \in \mathcal{M}(\mathcal{A})} \|[h \otimes I_{\mathcal{B}}](\mathbf{a})\|_{\mathcal{B}} \cdot \sup_{h \in \mathcal{M}(\mathcal{A})} \|[h \otimes I_{\mathcal{B}}](\mathbf{b})\|_{\mathcal{B}} \\ &= \|\hat{\mathbf{a}}\| \cdot \|\hat{\mathbf{b}}\| \\ &\leq \|\mathbf{a}\|_\varepsilon \cdot \|\mathbf{b}\|_\varepsilon \end{aligned}$$

since  $\varepsilon$  is a uniform crossnorm. A density argument shows that the result holds for all  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathcal{A} \hat{\otimes}_\varepsilon \mathcal{B}$ . Hence the  $\varepsilon$ -norm is a suitable tensor norm on  $\mathcal{A} \otimes \mathcal{B}$ .

The following result is given for the projective tensor product of Banach algebras by R.A. Bonic [6] and appears in [4, p. 132].

**Proposition 20.** *If  $\mathcal{A}$  is a commutative unital Banach algebra,  $\mathcal{B}$  is a unital Banach algebra and  $\gamma$  is a suitable tensor norm on  $\mathcal{A} \otimes \mathcal{B}$  then the following are equivalent for  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ :*

- (a)  $\mathbf{a}$  is left invertible in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ ,
- (b)  $\hat{\mathbf{a}}$  is left invertible in  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$ ,
- (c)  $\hat{\mathbf{a}}(h)$  is left invertible in  $\mathcal{B}$  for all  $h \in \mathcal{M}(\mathcal{A})$ .

PROOF. Since  $\mathbf{a} \longrightarrow \hat{\mathbf{a}}$  is a non-zero algebra homomorphism, (a)  $\Rightarrow$  (b) and clearly (b)  $\Rightarrow$  (c). By Proposition 15, (a)  $\Rightarrow$  (c). ■

**Proposition 21.** *If  $\mathcal{A}$  is a commutative unital Banach algebra,  $\mathcal{B}$  is a unital Banach algebra and  $\gamma$  is a suitable tensor norm on  $\mathcal{A} \otimes \mathcal{B}$ , then  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  is invertible if and only if  $\hat{\mathbf{a}}$  is invertible in  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$ .*

PROOF. Since the Gelfand transform is a non-zero algebra homomorphism from  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  into  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$  it follows that  $\hat{\mathbf{a}}$  is invertible if  $\mathbf{a}$  is invertible.

Conversely suppose that  $\hat{\mathbf{a}}$  is invertible. Clearly  $\gamma$  is a suitable tensor norm on  $\mathcal{A} \otimes \mathcal{B}^{\text{op}}$  and  $(\mathcal{A} \hat{\otimes}_\gamma \mathcal{B})^{\text{op}} = \mathcal{A} \hat{\otimes}_\gamma \mathcal{B}^{\text{op}}$ . Moreover,  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  and  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}^{\text{op}}$  coincide as sets and  $\hat{\mathbf{a}}(h)$  has the same value when  $\mathbf{a}$  is considered in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  or  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}^{\text{op}}$ . Since  $\hat{\mathbf{a}}$  is invertible in  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$  it is right invertible in  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B})$  and hence left invertible in  $\mathcal{C}(\mathcal{M}(\mathcal{A}); \mathcal{B}^{\text{op}})$ . By Proposition 20,  $\mathbf{a}$  is left invertible in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}^{\text{op}}$  and hence right invertible in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$ . Hence  $\mathbf{a}$  has a left and a right inverse in  $\mathcal{A} \hat{\otimes}_\gamma \mathcal{B}$  and thus is invertible. ■

*Example 22* (a non-commutative Wiener Lemma). One of the early successes of Gelfand's theory was a proof of Wiener's Lemma on absolutely convergent Fourier series. Shortly afterwards S. Bochner and R.S. Phillips [5] obtained a vector-valued generalisation both by adapting Wiener's original proof [33] and by using Gelfand's methods [13]. More recently Matos [22] obtained a further generalisation by using the Waelbroeck spectrum and the functional calculus. Here we present another extension which, on replacing  $\mathbb{T}^\infty$  by  $\mathbb{T}$ , gives another proof of the result in [5].

If  $m := (m_j)_j \in \mathbb{Z}^{(\mathbb{N})}$ , the set of integer-valued sequences which are eventually zero, and  $\theta := (\theta_j)_j \in \mathbb{R}^{\mathbb{N}}$  let  $m \cdot \theta = \sum_j m_j \theta_j$ . Let

$$\text{AC}(\mathbb{T}^\infty; \mathcal{A}) = \left\{ (\mathbf{a}_m)_{m \in \mathbb{Z}^{(\mathbb{N})}} : \mathbf{a}_m \in \mathcal{A}, \sum_{m \in \mathbb{Z}^{(\mathbb{N})}} \|\mathbf{a}_m\| < \infty \right\}$$

where  $\mathcal{A}$  is a unital Banach algebra and  $\mathbb{T}$  is the unit circle in  $\mathbb{C}$ . We identify  $\text{AC}(\mathbb{T}^\infty; \mathcal{A})$  with the set of functions  $f : \mathbb{T}^\infty \longrightarrow \mathcal{A}$  with absolutely convergent Fourier series by letting  $f(\theta) = (\sum_{m \in \mathbb{Z}^{(\mathbb{N})}} \mathbf{a}_m e^{im \cdot \theta})$  for  $\theta \in \mathbb{R}^{\mathbb{N}}$ . Under pointwise multiplication  $\text{AC}(\mathbb{T}^\infty; \mathcal{A})$  becomes a unital Banach algebra. The space  $\text{AC}(\mathbb{T}^\infty) := \text{AC}(\mathbb{T}^\infty; \mathbb{C})$  is a commutative unital Banach algebra with maximal ideal space  $\mathbb{T}^\infty$  where the multiplicative linear functionals are the point evaluations at points of  $\mathbb{T}^\infty$ . In addition, we have  $\text{AC}(\mathbb{T}^\infty; \mathcal{A}) = \text{AC}(\mathbb{T}^\infty) \hat{\otimes}_\pi \mathcal{A}$ . An application of Proposition 21 yields the following result: if  $(\mathbf{a}_m)_{m \in \mathbb{Z}^{(\mathbb{N})}} \subset \mathcal{A}$ ,  $\sum_{m \in \mathbb{Z}^{(\mathbb{N})}} \|\mathbf{a}_m\| < \infty$  and  $\sum_{m \in \mathbb{Z}^{(\mathbb{N})}} \mathbf{a}_m e^{im \cdot \theta}$  is left invertible in  $\mathcal{A}$  for all  $\theta \in \mathbb{T}^\infty$ , then there exists  $(\mathbf{b}_m)_{m \in \mathbb{Z}^{(\mathbb{N})}} \subset \mathcal{A}$ ,

$\sum_{m \in \mathbb{Z}(\mathbb{N})} \|b_m\| < \infty$  such that

$$\left( \sum_{m \in \mathbb{Z}(\mathbb{N})} b_m e^{im \cdot \theta} \right) \cdot \left( \sum_{m \in \mathbb{Z}(\mathbb{N})} a_m e^{im \cdot \theta} \right) = 1_{\mathcal{A}}$$

for all  $\theta \in \mathbb{R}^{\mathbb{N}}$ . Similar results hold for invertibility and right invertibility.

**Corollary 23.** *Let  $\mathcal{A}$  denote a commutative Banach algebra with identity, let  $\mathcal{B}$  denote a Banach algebra with identity and suppose that either  $\mathcal{A}$  or  $\mathcal{B}$  has the approximation property and that  $\gamma_1$  and  $\gamma_2$  are two suitable tensor norms on  $\mathcal{A} \otimes \mathcal{B}$ . If  $\gamma_1 \leq \gamma_2$  and  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma_2} \mathcal{B}$  then  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma_1} \mathcal{B}$  and  $\mathbf{a}$  is invertible in  $\mathcal{A} \hat{\otimes}_{\gamma_2} \mathcal{B}$  if and only if  $\mathbf{a}$  is invertible in  $\mathcal{A} \hat{\otimes}_{\gamma_1} \mathcal{B}$ .*

*Remark 24.* If  $\mathcal{A}$  and  $\mathcal{B}$  are both commutative Banach algebras with identity and  $\gamma$  is a suitable tensor norm on  $\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}$ , then  $\mathcal{M}(\mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}) = \mathcal{M}(\mathcal{A}) \times \mathcal{M}(\mathcal{B})$  and from the classical Gelfand result we see that  $\mathbf{a} \in \mathcal{A} \hat{\otimes}_{\gamma} \mathcal{B}$  is invertible if and only if  $[h \otimes k](\mathbf{a}) \neq 0$  for all  $h \in \mathcal{M}(\mathcal{A})$  and  $k \in \mathcal{M}(\mathcal{B})$ .

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