SPECTRA OF TENSOR PRODUCT ELEMENTS II: POLYNOMIAL EXTENSIONS

By

Seán Dineen
Department of Mathematics, University College Dublin

Robin E. Harte
School of Mathematics, Trinity College, Dublin

and

Ciaran Taylor*
School of Science, Institute of Technology, Tallaght, Dublin

[Received 10 April 2000. Read 30 November 2000. Published 28 December 2001.]

Abstract

This article investigates properties of the vector-valued spectrum defined in Part I. The main results are a two-way polynomial spectral mapping theorem for commutative elements of $\mathscr{A}\ \hat{\otimes}_\gamma X$, $\mathscr{A}$ a unital Banach algebra, X a Banach space and γ a uniform crossnorm, and a one-way theorem for arbitrary elements when X has the bounded approximation property. Preliminary results on polynomial extensions are obtained, and bounds on the norms of extensions are shown to be related to polarisation constants of the Banach space X. Examples using injective Banach algebras, Q-algebras (as defined by J. Wermer) and nuclear polynomials are given.

1. Introduction

In [7] we defined a non-commutative version of the Waelbroeck spectrum for tensor product elements in $\mathscr{A}\ \hat{\otimes}_\gamma X$, $\mathscr{A}$ a unital Banach algebra, X a Banach space and γ a uniform crossnorm, and by specialising to the case where X was itself a unital Banach algebra obtained a number of applications. In this paper we continue this investigation and examine the behaviour of the spectrum under polynomial mappings between Banach spaces.

In Section 2 we consider the general problem of extending $P \in \mathscr{P}(X : Y)$, a polynomial mapping from the Banach space X to the Banach space Y, to $P_{\mathscr{A}} \in \mathscr{P}(\mathscr{A}\ \hat{\otimes}_\gamma X : \mathscr{A}\ \hat{\otimes}_\gamma Y)$ where $\mathscr{A}$ is a unital Banach algebra and γ is a suitable uniform crossnorm so that

$$P_{\mathscr{A}}(\mathsf{a} \otimes x) = \mathsf{a}^n \otimes P(x)$$

whenever P is n-homogeneous. Under certain conditions we obtain a continuous

*Corresponding author, e-mail: ciaran.taylor@it-tallaght.ie

extension which respects algebra homomorphisms. We discuss the norm of the extension and show by example that it is not always an isometry. Furthermore, we prove that our extension agrees with the classical method of substituting scalars by elements of an algebra to obtain vector-valued polynomials. Various examples are given in this section and these include an infinite-dimensional polynomial characterisation of Q-algebras based on finite-dimensional results of I.G. Craw, A.M. Davie and N.Th. Varopoulos (see [1; 3; 12; 15]).

In Section 3 we use the results of Section 2 to obtain a two-way polynomial spectral mapping theorem for commutative elements of the tensor product and a one-way polynomial spectral mapping theorem for arbitrary elements. In Section 4 we show that the spectrum is polynomially convex for elements which are dense generators. In a subsequent paper [8] we consider a holomorphic functional calculus.

We refer to S. Dineen [6, chapter 1] for background information on polynomial mappings between Banach spaces, to R.G. Douglas [9] and R.E. Harte [11] for spectral theory of Banach algebras, and to A. Defant and K. Floret [4] for tensor products. This paper is self-contained, but we also refer to [7] for further details on the topics discussed.

2. Polynomial extensions

If X and Y are vector spaces over $\mathbb{C}$ and n is a positive integer we let $\mathscr{L}_a({}^nX;Y)$ denote the set of n-linear mappings from X to Y. An n-homogeneous polynomial from X to Y is the composition of $\Delta_n : X \longrightarrow X^n, x \mapsto (x,\ldots,x)$ and $L \in \mathscr{L}_a({}^nX;Y)$. A polynomial of degree zero is a constant mapping from X to Y. We denote by $\mathscr{P}_a({}^nX;Y)$ the space of n-homogeneous polynomials from X to Y. A polynomial is a finite sum of homogeneous polynomials and we let $\mathscr{P}_a(X;Y)$ denote the space of all polynomials from X to Y.

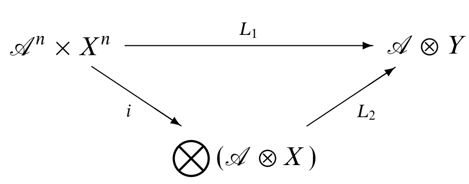
If $L \in \mathscr{L}_a({}^nX;Y)$ and $\mathscr{A}$ is an algebra over $\mathbb{C}$ then the mapping

$$L_1 : \mathscr{A}^n \times X^n \longrightarrow \mathscr{A} \otimes Y,\ L_1(\mathsf{a}_1,\ldots,\mathsf{a}_n,x_1,\ldots,x_n) := \mathsf{a}_1 \ldots \mathsf{a}_n \otimes L(x_1,\ldots,x_n)$$

is $2n$-linear. The definition of tensor products and associativity imply that there exists a unique linear mapping

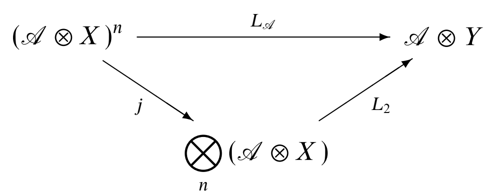
$$L_2 : \bigotimes_n (\mathscr{A} \otimes X) \longrightarrow \mathscr{A} \otimes Y$$

such that the following diagram commutes:



where $i(\mathsf{a}_1,\ldots,\mathsf{a}_n,x_1,\ldots,x_n) = \mathsf{a}_1 \otimes x_1 \otimes \mathsf{a}_2 \otimes x_2 \otimes \cdots \otimes \mathsf{a}_n \otimes x_n$. A further application of the definition of tensor products implies the existence of a unique n-linear mapping

$L_{\mathscr{A}} : (\mathscr{A} \otimes X)^n \longrightarrow \mathscr{A} \otimes Y$ such that the following diagram commutes:



where $j(\mathsf{a}_1 \otimes x_1, \ldots, \mathsf{a}_n \otimes x_n) = \mathsf{a}_1 \otimes x_1 \otimes \mathsf{a}_2 \otimes x_2 \otimes \cdots \otimes \mathsf{a}_n \otimes x_n$.

Let $\mathscr{L}_a^s(^nX; Y)$ denote the set of all symmetric n-linear mappings from X to Y. The mapping

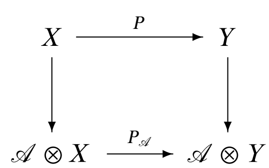
$$L \in \mathscr{L}_a^s(^nX; Y) \longrightarrow \hat{L} \in \mathscr{P}_a(^nX; Y)$$

defined by $\hat{L}(x) = L(x, \ldots, x)$ is a linear isomorphism. We denote its inverse by $\check{}$. If $P \in \mathscr{P}_a(^nX; Y)$ let $P_{\mathscr{A}} := \widehat{\left(\check{P}\right)_{\mathscr{A}}}$. The above shows that $P_{\mathscr{A}} \in \mathscr{P}_a(^n(\mathscr{A} \otimes X); \mathscr{A} \otimes Y)$ and

$$P_{\mathscr{A}}(\mathsf{a} \otimes x) = \mathsf{a}^n \otimes P(x)$$

for all $\mathsf{a} \in \mathscr{A}$ and $x \in X$.

If $\mathscr{A}$ has an identity $1_{\mathscr{A}}$ we may identify X with a subspace of $\mathscr{A} \otimes X$ by means of the embedding $x \in X \longrightarrow 1_{\mathscr{A}} \otimes x \in \mathscr{A} \otimes X$. We obtain the commutative diagram



and may thus regard $P_{\mathscr{A}}$ as an extension of P. In this setting we may also extend polynomials of degree zero by letting $P_{\mathscr{A}}(\theta) = 1_{\mathscr{A}} \otimes P(0)$ for all $\theta \in \mathscr{A} \otimes X$. We only consider polynomials without constant term (that is, $P(0) = 0$) when $\mathscr{A}$ is not unital.

The n-linear form $(P_{\mathscr{A}})^{\vee}$ is not in general symmetric but we do have the following expression for $\check{P}_{\mathscr{A}} := (P_{\mathscr{A}})^{\vee}$:

$$\check{P}_{\mathscr{A}}\left(\sum_{i=1}^{k} \mathsf{a}_{1,i} \otimes x_{1,i}, \sum_{i=1}^{k} \mathsf{a}_{2,i} \otimes x_{2,i}, \ldots, \sum_{i=1}^{k} \mathsf{a}_{n,i} \otimes x_{n,i}\right)$$

$$:= \sum_{i_1, i_2, \ldots, i_n = 1}^{k} s(\mathsf{a}_{1,i_1}, \mathsf{a}_{2,i_2}, \ldots, \mathsf{a}_{n,i_n}) \otimes \check{P}\left(x_{1,i_1}, \ldots, x_{n,i_n}\right)$$

where s is the symmetrisation operator

$$s(\mathsf{a}_1, \ldots, \mathsf{a}_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathsf{a}_{\sigma(1)} \ldots \mathsf{a}_{\sigma(n)}$$

and S_n is the set of permutations of $\{1, \ldots, n\}$.

If $m = (m_1, \dots, m_l)$ is an l-tuple of non-negative integers we let $|m| := \sum_{j=1}^{l} m_j$ and $m! := m_1! \dots m_l!$. If $x = (x_1, \dots, x_k)$ is a k-tuple of vectors and if $m = (m_1, \dots, m_l)$, $l \geq k$, we let

$$x^m := x_1^{m_1} x_2^{m_2} \dots x_k^{m_k} = (x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_k, \dots, x_k)$$

where x_j is repeated m_j times. To prove that the mapping $P \longrightarrow P_{\mathscr{A}}$ is an algebra homomorphism we require the following lemma. We let $\mathbb{N}_0$ denote the set of non-negative integers.

Lemma 1. *Let P and Q denote homogeneous polynomials of degree n and m respectively on the vector space E. If $s = (s_1, \dots, s_t)$, $s_i \in \mathbb{N}_0$, $|s| = m+n$ and $x = (x_1, \dots, x_t) \in E^t$, then*

$$\sum_{\substack{k,l \in (\mathbb{N}_0)^t \\ k+l=s \\ |k|=n, |l|=m}} \frac{\check{P}(x^k)\check{Q}(x^l)}{k!\,l!} = \frac{(m+n)!}{m!n!s!} [PQ]^{\vee} (x^s). \tag{2.1}$$

Proof. The proof is by induction on t. If $t = 1$ then $x = x_1$, $k = n$, $l = m$, $s = m+n$ and the left-hand side of (2.1) contains precisely one term

$$\frac{\check{P}(x^n)\check{Q}(x^m)}{m!n!} = \frac{P(x)Q(x)}{m!n!} = \frac{PQ(x)}{m!n!}.$$

The right side of (2.1) equals

$$\frac{(m+n)!}{m!n!} \frac{1}{(m+n)!} [PQ]^{\vee} (x^{n+m}) = \frac{1}{m!n!} PQ(x)$$

and hence (2.1) holds when $t = 1$. Now suppose that the result holds for the positive integer t and for all homogeneous polynomials P and Q. Let $S := (s, s') := (s_1, \dots, s_t, s')$, $X := (x, x') := (x_1, \dots, x_t, x')$. The left-hand side of (2.1) for X^S now becomes

$$\sum_{\substack{k,l \in (\mathbb{N}_0)^t \\ k',l' \in \mathbb{N}_0 \\ k+l=s \\ k'+l'=s' \\ |k|+k'=n, |l|+l'=m}} \frac{\check{P}(x^k, (x')^{k'})\check{Q}(x^l, (x')^{l'})}{k!\,k'!\,l!\,l'!}$$

$$= \sum_{\substack{k'+l'=s' \\ 0 \leq k', l' \leq s'}} \frac{1}{k'!\,l'!} \Bigg\{ \sum_{\substack{k,l \in (\mathbb{N}_0)^t \\ k+l=s \\ |k|=n-k', |l|=m-l'}} \frac{\check{P}(x^k, (x')^{k'})\check{Q}(x^l, (x')^{l'})}{k!\,l!} \Bigg\}.$$

Now the mapping

$$(y_1, \dots, y_{|k|}) \in E^{|k|} \longrightarrow \check{P}(y_1, \dots, y_{|k|}, (x')^{k'})$$

is a symmetric $|k|$-linear form on $E^{|k|}$ with associated $|k|$-homogeneous polynomial

$$y \in E \longrightarrow \check{P}(y^{|k|}, (x')^{k'}).$$

Since a similar argument applies to Q and $[PQ]^\vee\left(\,\cdot\,, (x')^{s'}\right)$ is the symmetric $(n+m-k'-l')$-linear form associated with the polynomial

$$y \in E \longrightarrow \check{P}(y^{|k|}, (x')^{k'})\check{Q}(y^{|l|}, (x')^{l'}),$$

our induction hypothesis implies that

$$\sum_{\substack{k,l\in(\mathbb{N}_0)^t \\ k+l=s \\ |k|=n-k',|l|=m-l'}} \frac{\check{P}(x^k, (x')^{k'})\check{Q}(x^l, (x')^{l'})}{k!\,l!} = \frac{(m+n-k'-l')!}{(m-k')!(n-l')!}\frac{1}{s!}\,[PQ]^\vee\left(x^s, (x')^{s'}\right).$$

Now $X^S = \left(x^s, (x')^{s'}\right)$ and $S! = s!(s')!$ so the left-hand side of (2.1) equals

$$\begin{aligned}
&\left(\sum_{\substack{k'+l'=s' \\ 0\le k',l'\le s'}} \frac{1}{k'!\,l'!}\frac{(m+n-k'-l')!}{(m-k')!(n-l')!}\frac{1}{s!}\right)[PQ]^\vee\left(x^s, (x')^{s'}\right) \\
&\qquad = \left(\sum_{\substack{k'+l'=s' \\ 0\le k',l'\le s'}} \frac{s'!}{k'!\,l'!}\frac{(m+n-k'-l')!}{(m-k')!(n-l')!}\right)\frac{[PQ]^\vee\left(X^S\right)}{S!} \\
&\qquad\qquad = \left(\sum_{0\le k'\le s'} \binom{s'}{k'}\binom{m+n-s'}{m-k'}\right)\frac{[PQ]^\vee\left(X^S\right)}{S!}.
\end{aligned}$$

Now if we are given $m+n$ objects and fix s' of them then the number of ways of choosing m objects from the $(m+n)$ objects,

$$\binom{m+n}{m} = \frac{(m+n)!}{m!\,n!},$$

is the sum over all k' of the ways of choosing k' from s' objects and $m-k'$ from the remaining $n+m-k'$ objects. Hence

$$\sum_{0\le k'\le s'} \binom{s'}{k'}\binom{m+n-s'}{m-k'} = \frac{(m+n)!}{m!\,n!}.$$

Hence the left-hand side equals the right-hand side and by induction this completes the proof of the lemma. ■

Proposition 2. *If $\mathscr{A}$ and $\mathscr{B}$ are commutative algebras, X is a vector space and $P, Q \in \mathscr{P}_a(X;\mathscr{B})$, then*

$$(P\cdot Q)_{\mathscr{A}} = P_{\mathscr{A}}\cdot Q_{\mathscr{A}}.$$

PROOF. It is easily verified that $(P+Q)_{\mathscr{A}} = P_{\mathscr{A}} + Q_{\mathscr{A}}$ for any polynomials P and Q. Hence we can suppose without loss of generality that $P \in \mathscr{P}_a({}^nX;\mathscr{B})$ and $Q \in \mathscr{P}_a({}^mX;\mathscr{B})$ and hence $PQ \in \mathscr{P}_a\left({}^{n+m}X;\mathscr{B}\right)$. If $\sum_{i=1}^k \mathtt{a}_i \otimes x_i \in \mathscr{A} \otimes X$ then, by Lemma 1,

$$\begin{aligned}
(P.Q)_{\mathscr{A}}\left(\sum_{i=1}^k \mathtt{a}_i \otimes x_i\right) &= \sum_{j,|j|=m+n} \frac{(m+n)!}{j!}\boldsymbol{a}^j \otimes [PQ]^\vee\left(x^j\right)\\
&= \sum_{j,|j|=m+n} \frac{(m+n)!}{j!}\boldsymbol{a}^j \otimes \left(\frac{m!n!j!}{(m+n)!}\sum_{\substack{u,v\in(\mathbb{N}_0)^k\\ u+v=j\\ |u|=n,|v|=m}} \frac{\check{P}(x^u)\check{Q}(x^v)}{u!v!}\right)\\
&= \sum_{\substack{u,v\\ |u|=n,|v|=m}} \frac{m!n!}{u!v!}\boldsymbol{a}^u\boldsymbol{a}^v \otimes \check{P}(x^u)\check{Q}(x^v)\\
&= \left(\sum_{\substack{u\\ |u|=n}} \frac{n!}{u!}\boldsymbol{a}^u \otimes \check{P}(x^u)\right).\left(\sum_{\substack{v\\ |v|=m}} \frac{m!}{v!}\boldsymbol{a}^v \otimes \check{Q}(x^v)\right)\\
&= P_{\mathscr{A}}\left(\sum_{i=1}^k \mathtt{a}_i \otimes x_i\right) Q_{\mathscr{A}}\left(\sum_{i=1}^k \mathtt{a}_i \otimes x_i\right). \qquad \blacksquare
\end{aligned}$$

If $(\mathtt{a}_i)_{i=1}^t$ is a commuting set of elements in $\mathscr{A}$ and $P \in \mathscr{P}_a({}^nX;Y)$ then

$$P_{\mathscr{A}}\left(\sum_{i=1}^t \mathtt{a}_i \otimes x_i\right) = \sum_{|m|=n} \frac{n!}{m!}\boldsymbol{a}^m \otimes \check{P}(x^m).$$

If X is a finite-dimensional vector space with normalised unit vector basis $\left(e_j\right)_{j=1}^t$ then we denote by z^m, $m=(m_1,\ldots,m_t)$, the monomial which maps $\sum_{i=1}^t z_ie_i$ to $z_1^{m_1}\ldots z_t^{m_t}$. If $\mathbf{a} := \sum_{i=1}^l \in \mathscr{A}\otimes X$ then we can rewrite $\mathbf{a}$ as $\sum_{i=1}^t \mathtt{b}_i \otimes e_i$ and we obtain

$$(z^m)_{\mathscr{A}}\left(\sum_{i=1}^t \mathtt{b}_i \otimes e_i\right) = \sum_{i_1,\ldots,i_n=1}^t \mathtt{b}_{i_1}\ldots\mathtt{b}_{i_n}(z^m)^\vee\left(e_{i_1},\ldots,e_{i_n}\right)$$

where $n=|m|$. Since $(z^m)^\vee$ is a symmetric $|n|$-linear form,

$$(z^m)^\vee\left(e_{i_1},\ldots,e_{i_n}\right) = (z^m)^\vee\left(e^s\right)$$

where $s=s_{i_1},\ldots,s_{i_t}$, $s_j = |\{i_l : i_l = j\}|$ and $e^s = e_1^{s_1}\ldots e_t^{s_t}$.

Lemma 3. *If* $m \in \mathbb{N}_0^t$ *then*

$$(z^m)^\vee(e^s) = \begin{cases} \dfrac{m!}{|m|!} & \text{if } s = m, \\ 0 & \text{if } s \neq m. \end{cases}$$

PROOF. We prove this by induction on $|m|$.

If $|m| = 1$ then z^m is a linear functional and equals the jth-coordinate functional for some positive integer j. Hence $(z^m)^\vee = z^m$. This implies that $|s| = 1$ and $e^s = e_l$ for some positive integer l and

$$(z^m)^\vee(e^s) = \delta_{lj} = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \neq j. \end{cases}$$

Hence $m = s$. Since $m! = |m|! = 1$ this completes the proof when $|m| = 1$.

We now suppose that the result is true for $|m| = n \geq 1$. If $|\overline{m}| = n + 1 = |\overline{s}|$ then, without loss of generality, we can let $\overline{m} = (1, 0, \dots) + m$ where $m = (m_1, \dots, m_t)$. By Lemma 1

$$\frac{|\overline{m}|!}{|m|!1!\overline{s}!}(z_1 z^m)^\vee\left(e^{\overline{s}}\right) = \sum_{\substack{k,l \in (\mathbb{N}_0)^t \\ k+l=\overline{s} \\ |k|=1, |l|=|m|}} \frac{z_1(e^k)(z^m)^\vee(e^l)}{k!\,l!}.$$

Since $|k| = 1$, $e^k = e_j$ for some positive integer j and hence

$$z_1(e^k) = \delta_{1j} = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{if } j > 1. \end{cases}$$

Hence

$$\frac{(z^m)^\vee(e^l)}{l!}$$

is the only non-zero form in the above sum. By our induction hypothesis $l = m$ and

$$(z_1 z^{\overline{m}})^\vee\left(e^{\overline{s}}\right) = \begin{cases} \dfrac{|m|!\overline{m}!}{|\overline{m}|!}\dfrac{m!}{|m|!}\dfrac{1}{m!} = \dfrac{\overline{m}!}{|\overline{m}|!} & \text{if } \overline{m} = \overline{s}, \\ 0 & \text{if } \overline{m} \neq \overline{s}. \end{cases}$$

This completes the proof. ■

If $\mathbf{b} = \sum_{i=1}^k b_i \otimes e_i$, $\boldsymbol{b} = (b_i)_{i=1}^k \subset \mathscr{A}$, a commutative unital algebra, then, by the previous lemma,

$$
\begin{aligned}
(z^m)_{\mathscr{A}}\left(\sum_{i=1}^{k} \mathsf{b}_i \otimes e_i\right) &= \sum_{|k|=|s|} \frac{|m|!}{s!}\boldsymbol{b}^s \otimes \check{P}(e^s) \\
&= \frac{|m|!}{m!}\boldsymbol{b}^m \frac{m!}{|m|!} \\
&= \boldsymbol{b}^m.
\end{aligned}
$$

Hence if

$$
P\left(\sum_{i=1}^{k} z_i e_i\right) = \sum_{|m|\leq l} \alpha_m z^m \tag{2.2}
$$

then

$$
P_{\mathscr{A}}(\mathbf{b}) = P_{\mathscr{A}}\left(\sum_{i=1}^{k} \mathsf{b}_i \otimes e_i\right) = \sum_{|m|\leq l} \alpha_m \boldsymbol{b}^m.
$$

If we identify $\mathscr{A} \otimes \mathbb{C}^n$ and $\mathscr{A}^n$ by the usual method,

$$
\sum_{i=1}^{n} \mathsf{a}_i \otimes e_i \longleftrightarrow (\mathsf{a}_1, \ldots, \mathsf{a}_n)
$$

then we obtain

$$
P_{\mathscr{A}}(\mathsf{a}_1, \ldots, \mathsf{a}_n) = \sum_{|m|\leq l} \alpha_m \boldsymbol{a}^m \tag{2.3}
$$

where P has the form (2.2). This agrees with the classical usage of substituting elements of a commutative algebra for variables in a polynomial and places our approach in perspective. Note that we assume that $P(0) = 0$ in the non-unital case.

We obtain similar but more complicated formulas in the non-commuting case. As a particular example, if $P\left(\sum_{i=1}^{n} z_i e_i\right) = z_1 z_2$ then

$$
P_{\mathscr{A}}(\mathsf{a}_1 \otimes e_1 + \mathsf{a}_2 \otimes e_2) = \frac{\mathsf{a}_1\mathsf{a}_2 + \mathsf{a}_2\mathsf{a}_1}{2}
$$

(since $\check{P}\left(\sum_{i=1}^{n} x_i e_i, \sum_{j=1}^{n} y_j e_j\right) = \dfrac{x_1 y_2 + x_2 y_1}{2}$). The extension $P \longrightarrow P_{\mathscr{A}}$ respects algebra homomorphisms. If $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$ is a homomorphism between the algebras

$\mathscr{A}$ and $\mathscr{B}$ and $P \in \mathscr{P}_a(^nX;Y)$ then for $\sum_{i=1}^k \mathsf{a}_i \otimes x_i \in \mathscr{A} \otimes X$ we have

$$\begin{aligned}
[\varphi \otimes I_Y] \circ P_{\mathscr{A}} &\left(\sum_{i=1}^k \mathsf{a}_i \otimes x_i \right) \\
&= [\varphi \otimes I_Y] \left(\sum_{1 \le i_1 \le \cdots \le i_n \le k} s(\mathsf{a}_{i_1}, \ldots, \mathsf{a}_{i_n}) \otimes \check{P}\left(x_{i_1}, \ldots, x_{i_n}\right) \right) \\
&= \sum_{1 \le i_1 \le \cdots \le i_n \le k} s(\varphi(\mathsf{a}_{i_1}), \ldots, \varphi(\mathsf{a}_{i_n})) \otimes \check{P}\left(x_{i_1}, \ldots, x_{i_n}\right) \\
&= P_{\mathscr{B}} \left(\sum_{i=1}^k \varphi(\mathsf{a}_i) \otimes x_i \right) \\
&= P_{\mathscr{B}} \circ [\varphi \otimes I_X] \left(\sum_{i=1}^k \mathsf{a}_i \otimes x_i \right).
\end{aligned}$$

Hence

$$[\varphi \otimes I_Y] \circ P_{\mathscr{A}} = P_{\mathscr{B}} \circ [\varphi \otimes I_X] \tag{2.4}$$

on $\mathscr{A} \otimes X$.

We now suppose that X and Y are Banach spaces, that $\mathscr{A}$ is a Banach algebra and that γ is a uniform crossnorm. We let $\mathscr{P}(^nX;Y)$, $\mathscr{L}(^nX;Y)$ and $\mathscr{L}^s(^nX;Y)$ denote the spaces of *continuous* n-homogeneous polynomials from X to Y, the *continuous* n-linear mappings from X^n to Y and the *continuous* symmetric n-linear mappings from X to Y respectively. If $P \in \mathscr{P}(^nX;Y)$ then our algebraic introduction shows that there exists a unique $P_{\mathscr{A}} \in \mathscr{P}_a(^n(\mathscr{A} \otimes X);\mathscr{A} \otimes Y)$ satisfying

$$\check{P}_{\mathscr{A}}(\mathsf{a}_1 \otimes x_1, \ldots, \mathsf{a}_n \otimes x_n) = s(\mathsf{a}_1, \ldots, \mathsf{a}_n) \otimes \check{P}\,(x_1, \ldots, x_n)$$

for all $(\mathsf{a}_1, \ldots, \mathsf{a}_n) \in \mathscr{A}^n$ and $(x_1, \ldots, x_n) \in X^n$. Since $\mathscr{A} \otimes X$ is dense in $\mathscr{A}\ \hat{\otimes}_\gamma X$, and polynomials on normed linear spaces are continuous if and only if they are locally bounded and, moreover, continuous Banach-valued polynomials can always be extended from a normed linear space to its completion, it follows that $P_{\mathscr{A}}$ admits an extension (which is necessarily unique) to $\mathscr{A}\ \hat{\otimes}_\gamma X$ to define an element of $\mathscr{P}\left(^n(\mathscr{A}\ \hat{\otimes}_\gamma X);\mathscr{A}\ \hat{\otimes}_\gamma Y\right)$ if and only if

$$\sup_{\substack{\mathbf{a} \in \mathscr{A} \otimes X \\ \|\mathbf{a}\|_\gamma \le 1}} \|P_{\mathscr{A}}(\mathbf{a})\| < \infty.$$

The existence of extensions will depend on $\mathscr{A}$, γ, X, Y and P and the interaction between them. When this extension exists we say that P can be adapted to $\mathscr{A}\ \hat{\otimes}_\gamma X$ and denote the extension[1] by $P_{\mathscr{A}}$. We thus see that the existence of a continuous

[1]Since the tensor norm γ is clear from the context we use for simplicity the notation $P_{\mathscr{A}}$ in place of the more formally correct $P_{\mathscr{A},\gamma}$.

extension to the completion can be verified on the algebraic tensor product. In fact we may replace X by a dense subspace X_1 and confine our attention to the subspace $\mathscr{A} \otimes X_1$ (see Example 8).

We have $\check{P}_{\mathscr{A}}(\mathbf{a}, \ldots, \mathbf{a}) = P_{\mathscr{A}}(\mathbf{a})$ for $\mathbf{a} \in \mathscr{A} \otimes X$, and by the Polarisation Formula [6, p. 9] $\check{P}_{\mathscr{A}}$ can be extended to $\left(\mathscr{A} \hat{\otimes}_\gamma X\right)^n$ if and only if $P_{\mathscr{A}}$ can be extended to $\mathscr{A} \hat{\otimes}_\gamma X$. In this case we have

$$\|P_{\mathscr{A}}\| \leq \left\|\check{P}_{\mathscr{A}}\right\| \leq \frac{n^n}{n!} \|P_{\mathscr{A}}\| .$$

Our algebraic results extend to topological results and we obtain the following proposition.

Proposition 4. (a) *If $\mathscr{A}$ and $\mathscr{B}$ are commutative Banach algebras, γ is a uniform crossnorm, $\mathscr{A} \hat{\otimes}_\gamma \mathscr{B}$ is a Banach algebra and $P, Q \in \mathscr{P}(X; \mathscr{B})$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$, then $P \cdot Q$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ and*

$$(P \cdot Q)_{\mathscr{A}} = P_{\mathscr{A}} \cdot Q_{\mathscr{A}}.$$

In particular, if all $P \in \mathscr{P}(X; \mathscr{B})$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ then the mapping

$$P \in \mathscr{P}(X; \mathscr{B}) \longrightarrow P_{\mathscr{A}} \in \mathscr{P}(\mathscr{A} \hat{\otimes}_\gamma X ; \mathscr{A} \hat{\otimes}_\gamma \mathscr{B})$$

is an algebraic and continuous homomorphism and

$$\|P_{\mathscr{A}} \cdot Q_{\mathscr{A}}\| \leq \|P_{\mathscr{A}}\| \, \|Q_{\mathscr{A}}\| .$$

(b) *If $\mathscr{A}$ and $\mathscr{B}$ are Banach algebras and $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$ is a homomorphism, γ is a uniform crossnorm, P is a continuous polynomial mapping between the Banach spaces X and Y which can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ and $\mathscr{B} \hat{\otimes}_\gamma X$ then $[\varphi \otimes I_Y] \circ P_{\mathscr{A}} = P_{\mathscr{B}} \circ [\varphi \otimes I_X]$, that is, we obtain the following commutative diagram:*

$$\begin{array}{ccc} \mathscr{A} \hat{\otimes}_\gamma X & \xrightarrow{P_{\mathscr{A}}} & \mathscr{A} \hat{\otimes}_\gamma Y \\ {\scriptstyle \varphi\otimes I_X}\downarrow & & \downarrow{\scriptstyle \varphi\otimes I_Y} \\ \mathscr{B} \hat{\otimes}_\gamma X & \xrightarrow{P_{\mathscr{B}}} & \mathscr{B} \hat{\otimes}_\gamma Y \end{array}$$

If $P = \sum_{k=0}^{l} P_k$ where P_k is k-homogeneous from X to Y we call P a polynomial and denote by $\mathscr{P}(X; Y)$ the vector space of all polynomials from X to Y. We define $P_{\mathscr{A}}$ to be $\sum_{k=0}^{l} (P_k)_{\mathscr{A}}$ for all $P \in \mathscr{P}(X; Y)$. If Y is a commutative algebra then $\mathscr{P}(X; Y)$ is also an algebra.

Example 5. Let $\mathscr{A}$ denote a Banach algebra, let X and Y be arbitrary Banach spaces and let π denote the projective tensor norm. If $\mathbf{a} \in \mathscr{A} \otimes X$, $\mathbf{a} = \sum_{i=1}^{k} \mathsf{a}_i \otimes x_i$ and

$P \in \mathscr{P}\left({}^nX\,;Y\right)$ (the case $P \in \mathscr{P}(X;Y)$ is handled in the obvious way), then

$$\begin{aligned} \|P_{\mathscr{A}}(\mathbf{a})\| &= \left\| \sum_{s_1,s_2,\ldots,s_n=1}^{k} \mathsf{a}_{s_1}\ldots\mathsf{a}_{s_n} \otimes \check{P}\left(x_{s_1},\ldots,x_{s_n}\right) \right\| \\ &\le \sum_{s_1,s_2,\ldots,s_n=1}^{k} \|\mathsf{a}_{s_1}\|\ldots\|\mathsf{a}_{s_n}\| \left\|\check{P}\right\| \|x_{s_1}\|\ldots\|x_{s_n}\| \\ &\le \left\|\check{P}\right\| \left(\sum_{i=1}^{k} \|\mathsf{a}_i\|\,\|x_i\| \right)^n . \end{aligned} \tag{2.5}$$

Hence

$$\|P_{\mathscr{A}}\| := \sup_{\substack{\mathbf{a}\in\mathscr{A}\ \hat{\otimes}_\pi X \\ \|\mathbf{a}\|<1}} \|P_{\mathscr{A}}(\mathbf{a})\| \le \left\|\check{P}\right\| < \infty \tag{2.6}$$

and $P_{\mathscr{A}} \in \mathscr{P}\left({}^n(\mathscr{A}\ \hat{\otimes}_\pi X);\mathscr{A}\ \hat{\otimes}_\pi Y\right)$.

For every Banach space X and every positive integer n there exists a smallest real number $c(n,X)$, called the nth Polarisation Constant of X, such that

$$\|P_n\| \le \left\|\check{P}_n\right\| \le c(n,X)\,\|P_n\|$$

for all $P_n \in \mathscr{P}\left({}^nX\,;Y\right)$ where $\|P\| = \sup_{\|x\|\le 1}\|P(x)\|$ and

$$\left\|\check{P}_n\right\| = \sup_{\substack{\|x_i\|\le 1 \\ i=1,\ldots,n}} \left\|\check{P}\left(x_1,\ldots,x_n\right)\right\| .$$

We always have $1 \le c(n,X) \le n^n/n!$ and we refer to [6, chapter 1] for details.

If

$$\mathscr{E}_n : P \in \mathscr{P}\left({}^nX\,;Y\right) \longrightarrow \mathscr{P}_{\mathscr{A}} \in \mathscr{P}\left({}^n(\mathscr{A}\ \hat{\otimes}_\pi X);\mathscr{A}\ \hat{\otimes}_\pi Y\right)$$

then (2.6) implies that $\|\mathscr{E}_n\| \le c(n,X)$. Since

$$\|P_{\mathscr{A}}(\mathsf{a}\otimes x)\| = \|\mathsf{a}^n \otimes P(x)\| = \|\mathsf{a}^n\|\,.\,\|P(x)\|$$

and $\|\mathsf{a}\otimes x\| = \|\mathsf{a}\|\,.\,\|x\|$ we have $\|\mathscr{E}_n\| \ge 1$ if $\mathscr{A}$ is a unital Banach algebra. This also holds if $\{\mathsf{a}^n : \|\mathsf{a}\| \le 1\}$ is dense in the unit ball of $\mathscr{A}$, for example if $\mathscr{A} = \mathscr{C}_0(K)$, where K is locally compact and $\mathscr{C}_0(K)$ is the Banach algebra of continuous functions which vanish at infinity. Thus when the mapping $\mathscr{E}_n$ satisfies $1 \le \|\mathscr{E}_n\| \le c(n,X)$ then $\mathscr{E}_n$ is a linear isomorphism from $\mathscr{P}\left({}^nX\,;Y\right)$ onto a closed subspace of $\mathscr{P}\left({}^n(\mathscr{A}\ \hat{\otimes}_\pi X);\mathscr{A}\ \hat{\otimes}_\pi Y\right)$ consisting of polynomials Q satisfying $Q(\mathsf{a}.\mathsf{b}\otimes x) = \mathsf{a}^n \cdot Q(\mathsf{b}\otimes x)$, for a, $\mathsf{b} \in \mathscr{A}$ and $x \in X$ and, by continuity and the open mapping theorem, these two spaces are isomorphic as Banach spaces.

Two special cases are worthy of mention. If X is a Hilbert space then $c(n, X) = 1$ for all n and $\mathscr{E}_n$ is an isometric extension. If $X = \ell_1$ then the monomials $(z_n)_{n\in\mathbb{N}^{(\mathbb{N})}}$ form an unconditional (but not absolute) basis for $\mathscr{P}(^n\ell_1)$ (see [13, proposition 3.4] and [14]) and, moreover, the monomial expansion converges absolutely at each point. Hence if $P \in \mathscr{P}(^n\ell_1)$ then there exists a countable set of scalars $(\alpha_m)_{\substack{m\in\mathbb{N}^{(\mathbb{N})}\\|m|=n}}$ and $\sum_{\substack{m\in\mathbb{N}^{(\mathbb{N})}\\|m|=n}} \alpha_m z^m$ converges absolutely and has sum $P(z)$ for all $z \in \ell_1$. If $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\pi \ell_1$, $\mathscr{A}$ a commutative Banach algebra, then $\mathbf{a}$ has a unique representation in the form $\sum_{i=1}^{\infty} \mathsf{a}_i \otimes e_1$ where $\mathsf{a}_i \in \mathscr{A}$, $\sum_{i=1}^{\infty} \|\mathsf{a}_i\| < \infty$, $(e_i)_{i=1}^{\infty}$ is the standard unit vector basis for ℓ_1 and $\|\mathbf{a}\|_\pi = \sum_{i=1}^{\infty} \|\mathsf{a}_i\|$. Let $\boldsymbol{a} = (\mathsf{a}_n)_n$ and $|\boldsymbol{a}| := \left(\|\mathsf{a}_n\|\right)_{n=1}^{\infty}$. Then $|\boldsymbol{a}| \in \ell_1$ and

$$\sum_{\substack{m\in\mathbb{N}^{(\mathbb{N})}\\|m|=n}} |\alpha_m|\, \|\boldsymbol{a}^m\| \le \sum_{\substack{m\in\mathbb{N}^{(\mathbb{N})}\\|m|=n}} |\alpha_m| \,.\, |\boldsymbol{a}|^m < \infty.$$

Hence we may sum the following series in any order and we obtain

$$P_{\mathscr{A}}(\mathbf{a}) = \sum_{\substack{m\in\mathbb{N}^{(\mathbb{N})}\\|m|=n}} \alpha_m \boldsymbol{a}^m$$

for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\pi X$.

Example 6. Let $\mathscr{A}$ denote a uniform Banach algebra (that is, $\mathscr{A}$ is isometrically isomorphic to a closed subalgebra of $\mathscr{C}(K)$ for some compact Hausdorff space K). If X and Y are Banach spaces, $P = \sum_{n=0}^{l} P_n \in \mathscr{P}(X; Y)$, $P_n \in \mathscr{P}\left(^nX; Y\right)$ (and $P(0) = 0$ if $\mathscr{A}$ is non-unital) and $\mathbf{a} = \sum_{i=1}^{k} \mathsf{a}_i \otimes x_i \in \mathscr{A} \otimes X$, then

$$\begin{aligned}
\|P_{\mathscr{A}}(\mathbf{a})\|_\varepsilon &= \sup_{\substack{\psi\in Y'\\ \|\psi\|\le 1}} \left\| \sum_{n=0}^{l} \sum_{s_1,\dots,s_n=1}^{k} \left(\mathsf{a}_{s_1}\dots\mathsf{a}_{s_n}\right) \psi \circ \check{P}_n\left(x_{s_1},\dots,x_{s_n}\right) \right\|_{\mathscr{A}} \\
&= \sup_{\substack{\psi\in Y'\\ \|\psi\|\le 1}} \left\{ \sup_{h\in\mathscr{M}(\mathscr{A})} \left| \sum_{n=0}^{l} \sum_{s_1,\dots,s_n=1}^{k} \left(h(\mathsf{a}_{s_1})\dots h(\mathsf{a}_{s_n})\right) \psi \circ \check{P}_n\left(x_{s_1},\dots,x_{s_n}\right) \right| \right\} \\
&= \sup_{\substack{\psi\in Y'\\ \|\psi\|\le 1}} \left\{ \sup_{h\in\mathscr{M}(\mathscr{A})} \left| \psi \circ P\left(\sum_{i=1}^{k} h(\mathsf{a}_i)x_i\right) \right| \right\} \\
&= \sup_{h\in\mathscr{M}(\mathscr{A})} \left\| P\left(\sum_{i=1}^{k} h(\mathsf{a}_i)x_i\right) \right\|_Y \\
&= \sup_{h\in\mathscr{M}(\mathscr{A})} \|P \circ [h \otimes I_X](\mathbf{a})\|_Y
\end{aligned}$$

and since P is continuous $P_{\mathscr{A}} \in \mathscr{P}\left(\mathscr{A} \hat{\otimes}_\varepsilon X; \mathscr{A} \hat{\otimes}_\varepsilon Y\right)$. Since ε is a uniform crossnorm

$\|h \otimes I_X\| \leq 1$ for all $h \in \mathcal{M}(\mathscr{A})$ and we obtain

$$\|P_{\mathscr{A}}\| := \sup_{\|\mathsf{a}\|_\varepsilon \leq 1} \|P_{\mathscr{A}}(\mathbf{a})\| \leq \|P\|.$$

Hence $P_{\mathscr{A}}$ extends to define an element of $\mathscr{P}\left(\mathscr{A}\ \hat{\otimes}_\varepsilon X\,;\mathscr{A}\ \hat{\otimes}_\varepsilon Y\right)$ and, moreover, $\|P_{\mathscr{A}}\| = \|P\|$ when $\mathscr{A}$ is unital.

A Banach algebra $\mathscr{A}$ is injective if $\mathscr{A}\ \hat{\otimes}_\varepsilon \mathscr{B}$ is a Banach algebra for any Banach algebra $\mathscr{B}$. These algebras were introduced and characterised by Varopoulos [17; 15] as being the complemented subalgebras of uniform algebras, that is, $\mathscr{A}$ is injective if and only if there exists a uniform algebra $\mathscr{U}$, a continuous linear mapping $\varphi : \mathscr{U} \longrightarrow \mathscr{A}$ and a continuous surjective homomorphism $l : \mathscr{A} \longrightarrow \mathscr{U}$ such that $\varphi \circ l = I_{\mathscr{A}}$. If $P \in \mathscr{P}\left({}^nX\,;Y\right)$ and $\mathscr{A}$ is an injective Banach algebra let

$$Q = [\varphi \otimes I_Y] \circ P_{\mathscr{U}} \circ [l \otimes I_X].$$

Clearly $Q \in \mathscr{P}\left(\mathscr{A}\ \hat{\otimes}_\varepsilon X\ :\mathscr{A}\ \hat{\otimes}_\varepsilon Y\right)$. Since $\check{Q} = [\varphi \otimes I_Y] \circ (P_{\mathscr{U}})^\vee \circ [l \otimes I_X]$ we have for $(\mathsf{a}_i \otimes x_i)_{i=1}^n \subset \mathscr{A} \otimes X$

$$\begin{aligned}
\check{Q}(\mathsf{a}_1 \otimes x_1, \ldots, \mathsf{a}_n \otimes x_n) &= [\varphi \otimes I_Y] \circ (P_{\mathscr{U}})^\vee \left(l(\mathsf{a}_1) \otimes x_1, \ldots, l(\mathsf{a}_n) \otimes x_n\right) \\
&= [\varphi \otimes I_Y] \left(s\left(l(\mathsf{a}_1), \ldots, l(\mathsf{a}_n)\right) \otimes \check{P}(x_1, \ldots, x_n)\right) \\
&= [\varphi \otimes I_Y] \left(l\left(s(\mathsf{a}_1, \ldots, \mathsf{a}_n)\right) \otimes \check{P}(x_1, \ldots, x_n)\right) \\
&= [\varphi \otimes I_Y] \circ [l \otimes I_Y] \left(s(\mathsf{a}_1, \ldots, \mathsf{a}_n) \otimes \check{P}(x_1, \ldots, x_n)\right) \\
&= [(\varphi \circ l) \otimes I_Y] \left((P_{\mathscr{A}})^\vee (\mathsf{a}_1 \otimes x_1, \ldots, \mathsf{a}_n \otimes x_n)\right) \\
&= (P_{\mathscr{A}})^\vee (\mathsf{a}_1 \otimes x_1, \ldots, \mathsf{a}_n \otimes x_n).
\end{aligned}$$

Hence $Q = P_{\mathscr{A}}$ and $\|P_{\mathscr{A}}\| \leq \|\varphi^n\|\,\|P\|$. Since we do not know whether $\mathscr{A}$ is 1-complemented in $\mathscr{U}$, we may not have a norm-preserving extension. The above methods can be applied to complemented subalgebras whenever all the other conditions hold.

Example 7. Since ε is the smallest uniform crossnorm, the method used in the previous example can be extended to arbitrary uniform crossnorms when we consider scalar-valued polynomials and we obtain the following result: if γ is a uniform crossnorm, $\mathscr{A}$ is an injective Banach algebra and $P \in \mathscr{P}(X)$, then $P_{\mathscr{A}}$ extends to define an element of $\mathscr{P}(\mathscr{A}\ \hat{\otimes}_\gamma X\,;\mathscr{A})$.

Example 8. A commutative Banach algebra $\mathscr{A}$ which is a quotient of a uniform algebra is called a Q-algebra. These algebras were introduced in [21] and studied in [1; 3; 12; 19; 16; 18; 15]. The space ℓ_p, $1 \leq p \leq \infty$, with pointwise multiplication and $\mathscr{C}^k[0,1]$ are Q-algebras. The collection of Q-algebras is closed under quotients, subspaces and taking biduals (with Arens' multiplication) but not under the ε tensor product. We require the following characterisation (due to I.G. Craw) of Q-algebras (see [1; 3; 12; 15]).

A commutative Banach algebra $\mathscr{A}$ is isometric to a Q-algebra if and only if for every positive integer N, for every $(\mathsf{x}_i)_{i=1}^N \subset \mathscr{A}$, $\|\mathsf{x}_i\| \leq 1$ all i, and every complex polynomial P in N variables, without constant term,

$$\|P(\mathsf{x}_1, \ldots, \mathsf{x}_N)\| \leq \|P\|_\infty \tag{2.7}$$

where $\|P\|_\infty = \sup\{|P(\lambda_1, \ldots, \lambda_N)| : \lambda_i \in \mathbb{C}, |\lambda_i| \leq 1\}$ and the left-hand side in (2.7) is obtained by substituting Banach algebra elements for variables in the polynomial P in the usual way.

Now let $P \in \mathscr{P}(c_0)$, $P = \sum_{j=1}^l P_j$ where $P_j \in \mathscr{P}(^j c_0)$ for $1 \leq j \leq l$. By [5] and [6], the monomials $(z^m)_{m \in \mathbb{N}^{(\mathbb{N})}}$ with the square ordering form a *conditional* basis for $\mathscr{P}(^j c_0)$. Since l is finite we can thus write $P = \sum_{k=1}^\infty Q_k$ (uniformly over the unit ball of c_0) and, moreover, each Q_k only depends on a finite number of coordinates, that is, for all k there exists a positive integer $n(k)$ such that

$$Q_k\left((z_n)_{n=1}^\infty\right) = Q_k\left((z_n)_{n=1}^{n(k)}\right).$$

Let $\mathbf{a} := \sum_{i=1}^s \mathsf{a}_i \otimes x_i \in \mathscr{A} \otimes c_0$ and suppose that each x_i lies in the vector space span of the standard unit vector basis for c_0. Then $\mathbf{a} := \sum_{i=1}^{s'} \mathsf{a}'_i \otimes e_i$ for some finite integer s' and some finite subset $\left(\mathsf{a}'_i\right)_{i=1}^{s'} \subset \mathscr{A}$. Hence $\|\mathbf{a}\|_\varepsilon = \sup_i \|\mathsf{a}'_i\|$, and substituting as in (2.3) we obtain

$$(Q_k)_{\mathscr{A}}(\mathbf{a}) = Q_k\left(\left(\mathsf{a}'_i\right)_{i=1}^{s'}\right).$$

Hence, by (2.7),

$$\|(Q_k)_{\mathscr{A}} - (Q_{k'})_{\mathscr{A}}\| \leq \|Q_k - Q_{k'}\| \longrightarrow 0 \text{ as } k, k' \longrightarrow \infty.$$

Thus each $(Q_k)_{\mathscr{A}}$ can be extended to define a continuous polynomial from $\mathscr{A} \hat{\otimes}_\varepsilon c_0$ into $\mathscr{A}$ and $((Q_k)_{\mathscr{A}})_{k=1}^\infty$ is a Cauchy sequence in the Banach space $\bigoplus_{1 \leq j \leq l} \mathscr{P}(^j c_0)$.

If $\tilde{Q} := \lim_k (Q_k)_{\mathscr{A}}$ and $P\left((z_n)_{n=1}^\infty\right) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \alpha_m z^m$ then, by restricting to finite-dimensional subspaces, one sees easily that $\tilde{Q} = P_{\mathscr{A}}$, $\|P_{\mathscr{A}}\| \leq \|P\|$ and

$$P_{\mathscr{A}}\left(\sum_{n=1}^\infty \mathsf{a}_n \otimes e_n\right) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \alpha_m \mathsf{a}^m \tag{2.8}$$

for $(\mathsf{a}_n)_{n=1}^\infty \in c_0(\mathscr{A}) := \{(\mathsf{a}_n)_{n=1}^\infty \subset \mathscr{A} : \lim_{n \longrightarrow \infty} \|\mathsf{a}_n\| = 0\}$. In particular, we may substitute any sequence from $c_0(\mathscr{A})$ into the monomial expansion (2.8) (the series converges but, in general, only conditionally so some care has to be taken in the order of summation; see [5] and [6, §4.1]). In view of (2.7) we obtain the following characterisation of Q-algebras: a commutative Banach algebra is a Q-algebra if and only if for all

$$P(z) = \sum_{\substack{m \in \mathbb{N}^{(\mathbb{N})} \\ |m| \neq 0}} \alpha_m z^m \in \mathscr{P}(c_0)$$

we have $P_{\mathscr{A}} \in \mathscr{P}\left(\mathscr{A}\ \hat{\otimes}_{\varepsilon} c_0; \mathscr{A}\right)$ and

$$\left\| \sum_{\substack{m \in \mathbb{N}^{(\mathbb{N})} \\ |m| \neq 0}} \alpha_m \mathsf{a}^m \right\| \leq \|P\|$$

for all $(\mathsf{a}_n)_{n=1}^{\infty} \in c_0(\mathscr{A})$, $\sup_i \|\mathsf{a}_i\| \leq 1$.

In general we note that taking the ε tensor product of a space with itself leads to a loss of structure. A good example of this process is provided by the space ℓ_2 with Banach algebra structure provided by pointwise multiplication. The following results can be found in [8]:
(a) ℓ_2 is a Q-algebra,
(b) $\ell_2 \hat{\otimes}_{\varepsilon} \ell_2$ is a Banach algebra but is not injective,
(c) $\ell_2 \hat{\otimes}_{\varepsilon} \ell_2 \hat{\otimes}_{\varepsilon} \ell_2$ is not a Banach algebra.
This also shows that the result in Example 8 is not covered by that given in Example 6.

In Examples 6 and 8 we proved that the extension $P \longrightarrow P_{\mathscr{A}}$ was norm-preserving but it is unclear whether this happens in Example 5. What we do know is that $\|P\| \leq \|P_{\mathscr{A}}\| \leq c(n, X)\|P\|$ for all $P \in \mathscr{P}(^n X; Y)$ and $c(n, X) \leq n^n/n!$ for all n whenever $\mathscr{A}$ is a unital Banach algebra. An isometric extension is desirable since it implies that the radius of convergence of holomorphic extensions is preserved (see [8]). Our next example shows that extensions are not always isometric. The fact that $c(n, \ell_1) = n^n/n!$ suggested this example (see [6, example 1.39]). The extremal example in this case is $P\left(\sum_{i=1}^{n} z_i e_i\right) = z_1 \ldots z_n$ but this polynomial extends with the same norm since (as the next example makes clear) the coefficients (in this case just 1) all have the same argument.

Let $X = \ell_1^2$ (two-dimensional ℓ_1 space) and let $Y = \mathbb{C}$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the usual basis in ℓ_1^2. If $P \in \mathscr{P}\left(^2 \ell_1^2\right)$ then there exist scalars a, b and c such that

$$P(xe_1 + ye_2) = ax^2 + by^2 + cxy.$$

For this example we require the following result from Choi *et al.* [2].

Lemma 9. (a) *Let a, b, $c \in \mathbb{R}$, $|a| < 1$, $|b| < 1$ and $2 < |c| \leq 4$. Then*

$$\|P\| = 1 \text{ if and only if } 4|c| - c^2 = 4(|a + b| - ab). \tag{2.9}$$

(b) *Let a, $b \in \mathbb{R}$, $|a| < 1$, $|b| < 1$ and let c be a purely imaginary number with $2 < |c| \leq 4$. Then*

$$\|P\| = 1 \text{ if and only if } 4|c| - |c|^2 = 4(|a - b| + ab). \tag{2.10}$$

Example 10. Let $\mathsf{AC}(\mathbb{T})$ denote the set of all functions on the unit circle $\mathbb{T}$ with absolutely convergent Fourier series, normed by

$$\left\| \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \right\| = \sum_{n=-\infty}^{\infty} |\alpha_n|$$

and with multiplication $\cdot$ given by

$$\left\{ \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \right\} \cdot \left\{ \sum_{m=-\infty}^{\infty} \beta_m e^{im\theta} \right\} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} \alpha_m \beta_{n-m} \right) e^{in\theta}.$$

The space $(\mathsf{AC}(\mathbb{T}), \cdot)$ is a unital commutative Banach algebra which is isometrically isomorphic to ℓ_1 as a Banach space. The maximal ideal space of $(\mathsf{AC}(\mathbb{T}), \cdot)$ can be identified with $\mathbb{T}$ and multiplicative linear functionals identified with point evaluations at points of $\mathbb{T}$.

Let $P \in \mathscr{P}\left({}^2\ell_1^2\right) := \mathscr{P}\left({}^2\ell_1^2; \mathbb{C}\right)$ be given by

$$P(x,y) = \frac{1}{\sqrt{2}} x^2 + \frac{1}{\sqrt{2}} y^2 + (2+\sqrt{2}) i\, xy,$$

for x, $y \in \mathbb{C}$. By Lemma 9(b), $\|P\| = 1$. Let $P_{\mathscr{A}}$ denote the extension of P to $\mathscr{P}\left({}^2\left(\mathsf{AC}(\mathbb{T})\ \hat{\otimes}_\pi\ \ell_1^2\right); \mathsf{AC}(\mathbb{T})\right)$. We have

$$P_{\mathscr{A}}\left(u \otimes xe_1 + v \otimes ye_2\right) = \frac{1}{\sqrt{2}} u^2 x^2 + \frac{1}{\sqrt{2}} v^2 y^2 + (2+\sqrt{2}) i\, uvxy$$

for all u, $v \in \mathsf{AC}(\mathbb{T})$ and x, $y \in \mathbb{C}$. If $u = e^{i\theta}$, $v = e^{-i\theta}$, $|x| + |y| = 1$ and $\mathbf{a} = u \otimes xe_1 + v \otimes ye_2$ then $\|\mathbf{a}\|_\pi = \|u\|\,|x| + \|v\|\,|y| = 1$. Since

$$P_{\mathscr{A}}(\mathbf{a}) = \frac{1}{\sqrt{2}} e^{2i\theta} x^2 + \frac{1}{\sqrt{2}} e^{-2i\theta} y^2 + (2+\sqrt{2}) i\, 1_{\mathsf{AC}(\mathbb{T})} xy$$

we have

$$\|P_{\mathscr{A}}\| \geq \sup_{|x|+|y|=1} \left(\frac{1}{\sqrt{2}} \left|x^2\right| + \frac{1}{\sqrt{2}} \left|y^2\right| + (2+\sqrt{2})\,|xy| \right).$$

Applying Lemma 9(a) to

$$Q(x,y) := \frac{2}{1+\sqrt{2}} \left(\frac{1}{\sqrt{2}} x^2 + \frac{1}{\sqrt{2}} y^2 + (2+\sqrt{2}) xy \right)$$

we see that

$$\|P_{\mathscr{A}}\| \geq \frac{1+\sqrt{2}}{2}.$$

Hence $\|P_{\mathscr{A}}\| > \|P\|$ and the extension is not norm-preserving.

Example 11. A polynomial P in $\mathscr{P}(X;Y)$ is said to be of *nuclear type* if there exist

$(\varphi_j)_j \subset X'$ and $(b_j)_j \subset Y$ such that $\sum_{j=1}^{\infty} \|\varphi_j\|^n \|b_j\| < \infty$ and $P(x) = \sum_{j=1}^{\infty} \varphi_j^n(x) b_j$ for all $x \in X$. Matos [13, proposition 3.3] has characterised nuclear polynomials on c_0 as precisely those continuous polynomials with pointwise absolutely convergent monomial expansions. For further information on nuclear polynomials we refer to [10]. If $\mathscr{A}$ is a Banach algebra, P is a nuclear n-homogeneous polynomial and $\sum_{i=1}^{k} \mathsf{a}_i \otimes x_i \in \mathscr{A} \otimes X$, then

$$P_{\mathscr{A}}\left(\sum_{i=1}^{k} \mathsf{a}_i \otimes x_i\right) = \sum_{j=1}^{\infty} \left\{ \sum_{s_1,\ldots,s_n=1}^{k} \mathsf{a}_{s_1} \ldots \mathsf{a}_{s_n} \otimes \varphi_j(x_{s_1}) \ldots \varphi_j(x_{s_n}) \right\} b_j$$
$$= \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k} \varphi_j(x_i) \mathsf{a}_i \right)^n b_j.$$

Hence

$$\sup_{\substack{\mathbf{a} \in \mathscr{A} \otimes X \\ \|\mathbf{a}\|_\varepsilon < 1}} \|P_{\mathscr{A}}(\mathbf{a})\| \le \sum_{j=1}^{\infty} \left\| \sum_{i=1}^{k} \varphi_j(x_i) \mathsf{a}_i \right\|_{\mathscr{A}}^n . \|b_j\|$$
$$= \sum_{j=1}^{\infty} \left\| \left[I_{\mathscr{A}} \otimes \varphi_j \right] (\mathbf{a}) \right\|^n . \|b_j\|$$
$$\le \sum_{j=1}^{\infty} \|\varphi_j\|^n . \|\mathbf{a}\|_\varepsilon^n . \|b_j\| .$$

Hence $P_{\mathscr{A}}$ extends to define an element of $\mathscr{P}\left({}^n(\mathscr{A} \hat{\otimes}_\varepsilon X); \mathscr{A} \hat{\otimes}_\varepsilon Y\right)$ and $\|P_{\mathscr{A}}\| \le \|P\|_N$.

Definition 12. If X and Y are Banach spaces, γ is a uniform crossnorm and $\mathscr{A}$ is a Banach algebra then we say that $P \in \mathscr{P}(X;Y)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ if $P_{\mathscr{A}} \in \mathscr{P}\left(\mathscr{A} \hat{\otimes}_\gamma X ; \mathscr{A} \hat{\otimes}_\gamma Y\right)$.

3. Polynomial spectral mapping theorems

In this section we prove a number of polynomial spectral mapping theorems. We first recall the definition from [7] of the left spectrum of an element of the tensor product $\mathscr{A} \hat{\otimes}_\gamma X$, where $\mathscr{A}$ is a unital Banach algebra, X is a Banach space and γ is a uniform crossnorm.

Definition 13. If $\mathscr{A}$ is a unital Banach algebra, X is a Banach space and γ is a uniform crossnorm, $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$, then

$$\sigma^{\text{left}}(\mathbf{a}) = \left\{ x \in X : 1_{\mathscr{A}} \notin \left\{ \sum_i \mathsf{b}_i \left(\left[I_{\mathscr{A}} \otimes x_i' \right] (\mathbf{a} - (1_{\mathscr{A}} \otimes x)) \right) \right\} \right\}$$

where $(\mathsf{b}_i)_i$ and $(x_i')_i$ are finite subsets of $\mathscr{A}$ and X' respectively.

Remark 14. The spectrum $\sigma^{\text{left}}(\mathbf{a})$ is defined in [7] to be the joint left spectrum of the collection $\left\{\left[I_{\mathscr{A}} \otimes x'\right]\right\}_{x' \in X'}$. By [7, proposition 5] the two definitions coincide modulo the canonical mapping $J_X : X \longrightarrow X''$. When $\mathscr{A}$ is commutative we have a further identification with the classical definition due to L. Waelbroeck [20] for elements of the projective tensor product $\mathscr{A} \hat{\otimes}_\pi X$. If $\mathscr{A}$ is commutative we use the notation $\sigma(\mathbf{a})$ in place of $\sigma^{\text{left}}(\mathbf{a})$.

Proposition 15 [7, proposition 7]. *If $\mathscr{A}$ is a commutative unital Banach algebra, X is a Banach space and γ is a uniform crossnorm, $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$, then*

$$\sigma(\mathbf{a}) = \{[h \otimes I_X](\mathbf{a}) : h \in \mathscr{M}(\mathscr{A})\}.$$

If P can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ then (2.4) implies that

$$P([h \otimes I_X](\mathbf{a})) = [h \otimes I_Y](P_{\mathscr{A}}(\mathbf{a}))$$

for all $h \in \mathscr{M}(\mathscr{A})$ and all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$. This proves the following proposition.

Proposition 16. *If X and Y are Banach spaces, $P \in \mathscr{P}(X; Y)$, $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform crossnorm on $\mathscr{A} \otimes X$ and P can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$, then*

$$\sigma(P_{\mathscr{A}}(\mathbf{a})) = P(\sigma(\mathbf{a})).$$

We now consider the non-commutative case and this is much more technical.

Lemma 17. *Let $\mathscr{A}$ denote a unital Banach algebra, X and Y denote Banach spaces, $P \in \mathscr{P}\left({}^n X; Y\right)$, let γ be a uniform crossnorm and suppose that P can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$. If $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ and $x \in X$ then*

$$P_{\mathscr{A}}(\mathbf{a}) - 1_{\mathscr{A}} \otimes P(x) = \sum_{j=0}^{n-1} \check{P}_{\mathscr{A}}\left(\mathbf{a}^j, \mathbf{a} - 1_{\mathscr{A}} \otimes x, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right).$$

Proof. This is immediate since

$$\begin{aligned} &P_{\mathscr{A}}(\mathbf{a}) - 1_{\mathscr{A}} \otimes P(x) \\ &\quad = \sum_{j=0}^{n-1} \left\{\check{P}_{\mathscr{A}}\left(\mathbf{a}^j, \mathbf{a}, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right) - \check{P}_{\mathscr{A}}\left(\mathbf{a}^j, 1_{\mathscr{A}} \otimes x, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right)\right\} \\ &\quad = \sum_{j=0}^{n-1} \check{P}_{\mathscr{A}}\left(\mathbf{a}^j, \mathbf{a} - 1_{\mathscr{A}} \otimes x, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right). \end{aligned}$$ ■

In the following proposition we frequently use the fact that $\mathscr{A} \otimes X$ is a left $\mathscr{A}$-module by means of the mapping $(c, a \otimes x) \longrightarrow (c \cdot a) \otimes x$.

Proposition 18 (one-way spectral mapping theorem). *If $P \in \mathscr{P}(X;Y)$, where X and Y are Banach spaces, P can be adapted to $\mathscr{A}\ \hat{\otimes}_\gamma X$, γ is a uniform crossnorm and $\mathscr{A}$ is a unital Banach algebra, then*

$$P\left(\sigma^{\text{left}}(\mathbf{a})\right) \subset \sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a}))$$

for all $\mathbf{a} \in \mathscr{A}\ \hat{\otimes}_\gamma X$.

PROOF. We suppose $P \in \mathscr{P}\left({}^nX;Y\right)$. The modifications necessary for the general case are immediate from our proof. Let $x \in \sigma^{\text{left}}(\mathbf{a})$ and suppose $P(x) \notin \sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a}))$. By Lemma 17 this implies that there exist finite sets $(\mathsf{b}_k)_{k=1}^l \subset \mathscr{A}$ and $\left(y'_k\right)_{k=1}^l \subset Y'$ such that

$$\begin{aligned} 1_{\mathscr{A}} &= \sum_{k=1}^{l} \mathsf{b}_k \left(\left[I_{\mathscr{A}} \otimes y'_k\right](P_{\mathscr{A}}(\mathbf{a}) - 1_{\mathscr{A}} \otimes P(x))\right) \\ &= \sum_{k=1}^{l}\sum_{j=0}^{n-1} \mathsf{b}_k \left(\left[I_{\mathscr{A}} \otimes y'_k\right]\left(\check{P}_{\mathscr{A}}\left(\mathbf{a}^j, \mathbf{a} - 1_{\mathscr{A}} \otimes x, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right)\right)\right). \end{aligned} \tag{3.1}$$

Since $\check{P}_{\mathscr{A}}$, y' and $I_{\mathscr{A}} \otimes y'_k$ are all continuous we can choose $\mathbf{c} := \sum_{r=1}^{t} \mathsf{c}_r \otimes x_r \in \mathscr{A} \otimes X$ such that $\|\mathsf{q} - 1_{\mathscr{A}}\| < 1/2$ where

$$\mathsf{q} := \sum_{k=1}^{l}\sum_{j=0}^{n-1} \mathsf{b}_k \left(\left[I_{\mathscr{A}} \otimes y'_k\right]\left(\check{P}_{\mathscr{A}}\left(\mathbf{c}^j, \mathbf{a} - 1_{\mathscr{A}} \otimes x, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right)\right)\right). \tag{3.2}$$

If $\mathbf{z} \in \mathscr{A}\ \hat{\otimes}_\gamma X$ then

$$\begin{aligned} &\left[I_{\mathscr{A}} \otimes y'_k\right]\left(\check{P}_{\mathscr{A}}\left(\mathbf{c}^j, \mathbf{z}, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right)\right) = \\ &\sum_{r_1,\ldots,r_j=1}^{t} \left[I_{\mathscr{A}} \otimes y'_k\right]\left(\mathsf{c}_{r_1}\ldots\mathsf{c}_{r_j}\left(\check{P}_{\mathscr{A}}\left(1_{\mathscr{A}} \otimes x_{r_1}, \ldots, 1_{\mathscr{A}} \otimes x_{r_j}, \mathbf{z}, (1_{\mathscr{A}} \otimes x)^{n-j-1}\right)\right)\right). \end{aligned} \tag{3.3}$$

For k, $1 \le k \le l$; j, $0 \le j \le n-1$ and $r_1, \ldots, r_j$, $1 \le r_i \le t$, let

$$x'_{(k,j,r_1,\ldots,r_j)}(w) = y'_k\left(\check{P}(x_{r_1}, \ldots, x_{r_j}, w, x^{n-j-1})\right)$$

for all $w \in X$. Since y'_k and P are continuous $x'_{(k,j,r_1,\dots,r_j)} \in X'$. If $\mathbf{d} = \sum_{s=1}^{s_0} d_s \otimes w_s \in \mathscr{A} \otimes X$ then

$$\begin{aligned} & c_{r_1} \dots c_{r_j} \left[I_{\mathscr{A}} \otimes x'_{(k,j,r_1,\dots,r_j)} \right] (\mathbf{d}) \\ & \quad = \sum_{s=1}^{s_0} c_{r_1} \dots c_{r_j} d_s . y'_k \left(\check{P}(x_{r_1}, \dots, x_{r_j}, w_s, x^{n-j-1}) \right) \\ & \quad = \sum_{s=1}^{s_0} \left[I_{\mathscr{A}} \otimes y'_k \right] \left(c_{r_1} \dots c_{r_j} d_s \otimes \check{P}(x_{r_1}, \dots, x_{r_j}, w_s, x^{n-j-1}) \right) \\ & \quad = \left[I_{\mathscr{A}} \otimes y'_k \right] c_{r_1} \dots c_{r_j} \check{P}_{\mathscr{A}} \left(1_{\mathscr{A}} \otimes x_{r_1}, \dots, 1_{\mathscr{A}} \otimes x_{r_j}, \mathbf{d}, (1_{\mathscr{A}} \otimes x)^{n-j-1} \right). \qquad (3.4) \end{aligned}$$

By continuity this holds for all $\mathbf{d} \in \mathscr{A} \hat{\otimes}_\gamma X$. Since $\|q - 1_{\mathscr{A}}\| < 1/2$, q is invertible, and combining (3.2), (3.3) and (3.4) we obtain, on letting $\mathbf{z} = \mathbf{d} = \mathbf{a} - 1_{\mathscr{A}} \otimes x$,

$$1_{\mathscr{A}} = \sum_{k=1}^{l} \sum_{j=0}^{n-1} \sum_{r_1,\dots,r_j=1}^{t} q^{-1} b_k c_{r_1} \dots c_{r_j} . \left[I_{\mathscr{A}} \otimes x'_{(k,j,r_1,\dots,r_j)} \right] (\mathbf{a} - 1_{\mathscr{A}} \otimes x)$$

and hence $x \notin \sigma^{\text{left}}(\mathbf{a})$. This contradicts our hypothesis and completes the proof. ∎

An element $\mathbf{a}$ in $\mathscr{A} \hat{\otimes}_\gamma X$ is said to be commutative if $\left\{ \left[I_{\mathscr{A}} \otimes x' \right] (\mathbf{a}) \right\}_{x' \in X'}$ is a commutative subset of $\mathscr{A}$. Clearly when $\mathscr{A}$ itself is commutative every element of the tensor product is commutative. Thus our next proposition gives a further proof of Proposition 16.

A Banach space X has the bounded approximation property if there exists a bounded net of finite-rank operators $(T_\alpha)_{\alpha \in \Gamma}$ in $\mathscr{L}(X; X)$ which converges uniformly on compact sets to the identity mapping on X. By using elements of $\mathscr{A} \otimes X$ and a density argument one easily sees that $[I_{\mathscr{A}} \otimes T_\alpha] (\mathbf{a}) \longrightarrow \mathbf{a}$ as $\alpha \longrightarrow \infty$ for any uniform crossnorm γ and any $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$. If $P \in \mathscr{P}\left({}^n X ; Y\right)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ then $P \circ T_\alpha$ can also be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ and $(P \circ T_\alpha)_{\mathscr{A}} = P_{\mathscr{A}} \circ [I_{\mathscr{A}} \otimes T_\alpha]$. For any $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$

$$\|P_{\mathscr{A}}(\mathbf{a}) - (P \circ T_\alpha)_{\mathscr{A}} (\mathbf{a})\| = \|P_{\mathscr{A}} \circ [I_{\mathscr{A}} \otimes (I_X - T_\alpha)] (\mathbf{a})\|$$

and $[P \circ T_\alpha] (\mathbf{a}) \longrightarrow P_{\mathscr{A}}(\mathbf{a})$ as $\alpha \longrightarrow \infty$.

Proposition 19 (two-way spectral mapping theorem). *If γ is a uniform crossnorm, $\mathscr{A}$ is a unital Banach algebra, X is a Banach space with the bounded approximation property and $P \in \mathscr{P}(X; Y)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$, then*

$$\sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a})) = P(\sigma^{\text{left}}(\mathbf{a}))$$

for any $\mathbf{a}$ commutative in $\mathscr{A} \hat{\otimes}_\gamma X$.

Proof. By Proposition 18 we are required to show

$$\sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a})) \subset P(\sigma^{\text{left}}(\mathbf{a})).$$

We may suppose without loss of generality that P is n-homogeneous. Let $y \in \sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a}))$. If $y' \in Y'$ then $\left[I_{\mathscr{A}} \otimes y'\right](P_{\mathscr{A}}(\mathbf{a})) = \left(y' \circ P\right)_{\mathscr{A}}(\mathbf{a})$. Hence, since $\mathbf{a}$ is commutative, we can apply [7, proposition 11] to

$$(\mathbf{a}, P_{\mathscr{A}}(\mathbf{a})) \cong \left(\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}), \left[I_{\mathscr{A}} \otimes y'\right](P_{\mathscr{A}}(\mathbf{a}))\right)_{x' \in X', y' \in Y'}$$

once we have verified that

$$\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}) \cdot Q_{\mathscr{A}}(\mathbf{a}) = Q_{\mathscr{A}}(\mathbf{a}) \cdot \left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}) \tag{3.5}$$

for any $Q_{\mathscr{A}} := \left(y' \circ P\right)_{\mathscr{A}}, y' \in Y'$.

Let $\varepsilon > 0$ be arbitrary. Since X has the bounded approximation property there exists, by our discussion preceding the proposition, $\left(x_i'\right)_{i=1}^{k} \subset X'$ such that

$$\|Q_{\mathscr{A}}(\mathbf{a}) - R_{\mathscr{A}}(\mathbf{a})\| \leq \varepsilon$$

where $R = \sum_{i=1}^{k}(x_i')^n$. Since

$$R_{\mathscr{A}}(\mathbf{a}) = \sum_{i=1}^{k} \left[(x_i')^n\right]_{\mathscr{A}}(\mathbf{a}) = \sum_{i=1}^{k} \left[(x_i')_{\mathscr{A}}\right]^n(\mathbf{a}) = \sum_{i=1}^{k} \left[\left[I_{\mathscr{A}} \otimes x_i'\right](\mathbf{a})\right]^n$$

and $\mathbf{a}$ is commutative we have

$$\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}) \cdot R_{\mathscr{A}}(\mathbf{a}) = R_{\mathscr{A}}(\mathbf{a}) \cdot \left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}).$$

Hence

$$\begin{aligned}
\| \left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a}) \cdot Q_{\mathscr{A}}(\mathbf{a}) - Q_{\mathscr{A}}(\mathbf{a}) \cdot \left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a})\| \\
\leq 2 \left\|\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a})\right\| \cdot \|Q_{\mathscr{A}}(\mathbf{a}) - R_{\mathscr{A}}(\mathbf{a})\| \\
\leq 2 \|x'\| \cdot \|\mathbf{a}\| \cdot \varepsilon
\end{aligned}$$

and as ε was arbitrary we have verified (3.5). By [7, proposition 11] there exists $x \in \sigma^{\text{left}}(\mathbf{a})$ such that $(x, y) \in \sigma^{\text{left}}(\mathbf{a}, P_{\mathscr{A}}(\mathbf{a}))$. Let $Q(z, w) = w - P(z)$ for $z \in X$ and $w \in Y$. Then $Q \in \mathscr{P}(X \times Y ; Y)$ and Q can be adapted to $\mathscr{A} \hat{\otimes}_\gamma (X \times Y) \cong (\mathscr{A} \hat{\otimes}_\gamma X) \times (\mathscr{A} \hat{\otimes}_\gamma Y)$. A density argument and continuity show that for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ and $\mathbf{b} \in \mathscr{A} \hat{\otimes}_\gamma Y$

$$Q_{\mathscr{A}}(\mathbf{a}, \mathbf{b}) = \mathbf{b} - P_{\mathscr{A}}(\mathbf{a}).$$

By Proposition 18

$$Q(x, y) \in \sigma^{\text{left}}(Q_{\mathscr{A}}(\mathbf{a}, P_{\mathscr{A}}(\mathbf{a}))).$$

Since $Q_{\mathscr{A}}(\mathbf{a}, P_{\mathscr{A}}(\mathbf{a})) = P_{\mathscr{A}}(\mathbf{a}) - P_{\mathscr{A}}(\mathbf{a}) = 0$ and $\sigma^{\text{left}}(\{0\}) = 0$ we have $Q(x, y) = y - P(x) = 0$. Hence $y = P(x) \in P\left(\sigma^{\text{left}}(\mathbf{a})\right)$ and

$$\sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a})) \subset P\left(\sigma^{\text{left}}(\mathbf{a})\right).$$

This completes the proof. ■

4. Polynomial convexity of the spectrum

In this section we show, when $\mathscr{A}$ is a commutative unital Banach algebra and $\mathbf{a}$ is a dense generator, that $\sigma(\mathbf{a})$ is a polynomially convex compact subset of X.

Definition 20 (polynomially convex). (a) The *polynomially convex hull* $\tilde{S}$ of a subset S of a Banach space X is defined by

$$\tilde{S} = \{x \in X : |P(x)| \leq \|P\|_S \text{ for all } P \in \mathscr{P}(X)\}.$$

(b) A subset K of X is *polynomially convex* if it coincides with its polynomially convex hull.

Definition 21 (dense generator). Let γ denote a uniform crossnorm, $\mathscr{A}$ a Banach algebra and X a Banach space. An element $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ is a *dense generator* if $\left\{\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a})\right\}_{x' \in X'}$ generates a dense subalgebra of $\mathscr{A}$.

Proposition 22. *If $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform crossnorm, X is a Banach space and $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ is a dense generator and all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$, then $\sigma(\mathbf{a})$ is polynomially convex. Moreover, $x \in \sigma(\mathbf{a})$ if and only if*

$$|P(x)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$$

for all $P \in \mathscr{P}(X)$.

Proof. Suppose $x \in \sigma(\mathbf{a})$, so that $x = [h \otimes I_X](\mathbf{a})$ for some $h \in \mathscr{M}(\mathscr{A})$. If $P \in \mathscr{P}(X)$ then

$$P(x) = P([h \otimes I_X](\mathbf{a})) = h(P_{\mathscr{A}}(\mathbf{a}))$$

by (2.4) and the accompanying arguments. Then $|P(x)| = |h(P_{\mathscr{A}}(\mathbf{a}))| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$ so that

$$x \in \sigma(\mathbf{a}) \Rightarrow |P(x)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$$

for all $P \in \mathscr{P}(X)$.

Suppose $x \in X$ and $|P(x)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$ for all polynomials $P \in \mathscr{P}(X)$. Let $\mathscr{B} = \{P_{\mathscr{A}}(\mathbf{a}) : P \in \mathscr{P}(X)\}$. If $P = x' \in X'$ then $P_{\mathscr{A}} = I_{\mathscr{A}} \otimes x'$, and since $\mathbf{a}$ is a dense generator it follows that $\mathscr{B}$ is a dense subalgebra of $\mathscr{A}$.

Let $h : \mathscr{B} \longrightarrow \mathbb{C}$ be defined by

$$h(P_{\mathscr{A}}(\mathbf{a})) = P(x) \tag{4.1}$$

for all $P \in \mathscr{P}(X)$. If $P_{\mathscr{A}}(\mathbf{a}) = Q_{\mathscr{A}}(\mathbf{a})$ then $[P_{\mathscr{A}} - Q_{\mathscr{A}}](\mathbf{a}) = 0$ and hence

$$|P(x) - Q(x)| = |[P - Q](x)| \leq \|[P_{\mathscr{A}} - Q_{\mathscr{A}}](\mathbf{a})\| = 0.$$

This shows that h is well defined. By Proposition 2, h is a multiplicative linear functional and since

$$|h(P_{\mathscr{A}}(\mathbf{a}))| = |P(x)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$$

h is continuous. Hence h can be extended to the closure of $\mathscr{B}$, that is, to $\mathscr{A}$, and defines an element of $\mathscr{M}(\mathscr{A})$. Hence $[h \otimes I_X](\mathbf{a}) \in \sigma(\mathbf{a})$. By (2.4) we have

$$x'([h \otimes I_X](\mathbf{a})) = h\left(\left[I_{\mathscr{A}} \otimes x'\right](\mathbf{a})\right)$$

and by (4.1) this coincides with $x'(x)$ since $(x')_{\mathscr{A}} = I_{\mathscr{A}} \otimes x'$. By the Hahn–Banach Theorem $[h \otimes I_X](\mathbf{a}) = x$ and $x \in \sigma(\mathbf{a})$. We have thus shown that

$$x \in \sigma(\mathbf{a}) \Leftrightarrow |P(x)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$$

for all $P \in \mathscr{P}(X)$.

Now suppose $x \in \widetilde{\sigma}(\mathbf{a})$, the polynomially convex hull of $\sigma(\mathbf{a})$. If $P \in \mathscr{P}(X)$ then, by (2.4),

$$\begin{aligned}
|P(x)| &\leq \sup\{|P(\xi)| : \xi \in \sigma(\mathbf{a})\} \\
&= \|P\|_{\sigma(\mathbf{a})} \\
&= \sup\{|P([h \otimes I_X](\mathbf{a}))| : h \in \mathscr{M}(\mathscr{A})\} \\
&= \sup\{|h(P_{\mathscr{A}}(\mathbf{a}))| : h \in \mathscr{M}(\mathscr{A})\} \\
&\leq \|P_{\mathscr{A}}(\mathbf{a})\|.
\end{aligned}$$

But we have just seen that this means that $x \in \sigma(\mathbf{a})$. Hence $\sigma(\mathbf{a})$ coincides with its polynomially convex hull and so is polynomially convex. ∎

ACKNOWLEDGEMENT

Ciaran Taylor acknowledges partial support from an Enterprise Ireland basic research scholarship.

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