# NEW DECOMPOSITIONS OF THE DISPLACEMENT GRADIENT FOR INFINITESIMAL STRAIN

By

P. BOULANGER
Département de Mathématique, Université Libre de Bruxelles, Belgium

and

M. HAYES*, M.R.I.A.
Department of Mechanical Engineering, University College, Dublin

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ABSTRACT

We consider, within the context of infinitesimal strain theory, the shears of pairs and triads of material line elements emanating from a point $P$ in a body. The set of elements along the principal axes of strain at $P$ is the only orthogonal set of unsheared elements. There is an infinite set of other unsheared triads. In general, if an unsheared pair is known, a third element completing an unsheared triad may be found.

## 1. Introduction

We consider the three-dimensional infinitesimal strain at a point $P$ in a body. If the displacement gradient at $P$ is denoted by $\mathbf{H}$, then the strain tensor $\mathbf{e}$ is defined by $2\mathbf{e} = \mathbf{H} + \mathbf{H}^T$. We assume that $\mathbf{e}$ has eigenvalues $e_\alpha$, the principal strains, ordered $e_3 > e_2 > e_1$. In general $\mathbf{e}$ is not positive definite, but a positive definite tensor $\mathbf{E}$, defined by $\mathbf{E} = \mathbf{e} + \lambda\mathbf{1}$ (where $\lambda$ is suitably large: $\lambda + e_1 > 0$), may be associated with $\mathbf{e}$ [7]. The associated quadric, $\mathbf{x} \cdot \mathbf{E}\mathbf{x} = 1$, is an ellipsoid $\mathscr{E}$ (say), whose principal axes are the principal axes of strain. Because we assume that $e_3 > e_2 > e_1$, it follows that $\mathscr{E}$ has two planes of central circular section (see, for instance, [2]). These planes are denoted by $\mathscr{C}^+$ and $\mathscr{C}^-$.

Previously it was shown [4] that, apart from two exceptions, for any infinitesimal material line element $\mathscr{L}$ lying in a plane $\Pi$ at the point $P$, there is just one other material line element $\mathscr{L}'$ in $\Pi$ such that the angle between the elements along $\mathscr{L}$ and $\mathscr{L}'$ is unchanged in the deformation, so that they are unsheared (in this case the element along $\mathscr{L}'$ is said to be *conjugate to* the element along $\mathscr{L}$—they form a *conjugate* or unsheared pair). The two exceptions to this general result are as follows. If the plane $\Pi$ coincides with either $\mathscr{C}^+$ or $\mathscr{C}^-$, then all pairs of infinitesimal material line elements in $\Pi$ are unsheared [6], so that corresponding to any material line element in $\Pi$ there is an infinity of conjugate line elements. Otherwise, if the plane $\Pi$ does not coincide with either $\mathscr{C}^+$ or $\mathscr{C}^-$, then there are two particular elements in $\Pi$ such that neither has a conjugate element forming an unsheared pair with it. The directions of these particular material line elements in $\Pi$ are called *limiting directions* [4].

*Corresponding author, e-mail: michael.hayes@ucd.ie

 

Here we consider the possibility of having unsheared triads, that is, three (non-coplanar) material line elements at a point in the material such that they form three unsheared pairs. The best-known such triad, indeed the only one we have come across in the literature, is the triad of mutually orthogonal elements along the principal axes of the strain ellipsoid at the point. It is seen that generally there is only one such orthogonal triad.

In considering unsheared triads, we adopt two different approaches. In the first (§3) we introduce the new decomposition of the displacement gradient $\mathbf{H}$, as $\mathbf{H} = \mathbf{P} + \boldsymbol{\Omega}$ where $\boldsymbol{\Omega}$ is skew-symmetric, and show that unsheared triads exist—they are along the right eigenvectors of $\mathbf{P}$. This approach is developed in §§4 and 5. It is shown (§4) how, if the displacement gradient $\mathbf{H}$ is known at a point $P$, and if an unsheared triad of material line elements along three linearly independent vectors at $P$ is also known, then $\mathbf{P}$ and $\boldsymbol{\Omega}$ are determined explicitly at the point $P$. In the second approach (§6) we develop the result of the previous paper [4], in which it was shown how to construct unsheared pairs of elements. Assuming that we have an unsheared pair, we show how to determine a third element (not in the plane of the unsheared pair) to form an unsheared triad. There is an infinity of such triads (*genuine unsheared triads*) in any deformation. Because no pair of material line elements lying entirely either in $\mathscr{C}^+$ or in $\mathscr{C}^-$ is sheared, it follows that any three infinitesimal material line elements all of which lie entirely either in $\mathscr{C}^+$ or in $\mathscr{C}^-$ form an unsheared triad. Such a triad is called an unsheared *coplanar* triad to distinguish it from a *genuine* unsheared triad, in which the directions of the three line elements are linearly independent.

The method of determining the third element depends upon a vector product involving two vectors that depend upon the directions of the elements of the unsheared pair. However, this method fails in certain circumstances. These circumstances are explored completely (§§7–8). First, the method does not yield the unsheared triads consisting of three coplanar line elements (*coplanar* unsheared triads). In §7 it is shown that all coplanar unsheared triads consist of three line elements in one of the planes $\mathscr{C}^\pm$ of central circular section of the ellipsoid $\mathscr{E}$. Next (§8) it is seen that, for special choices of the unsheared pair, there is no third element completing an unsheared triad. Such pairs are called *singular pairs*. Also, in general ($e_3 + e_1 \neq 2e_2$), when one element of the unsheared pair is along the normal to the plane $\mathscr{C}^+$ (or the plane $\mathscr{C}^-$), then there is an infinity of genuine unsheared triads with a common edge along this normal and the other two edges in the plane $\mathscr{C}^-$ (or the plane $\mathscr{C}^+$). It is also shown that to such an infinite family of unsheared triads corresponds a special case of the decomposition $\mathbf{H} = \mathbf{P} + \boldsymbol{\Omega}$ (§9) that we call a *CCS* (*central circular section*) *decomposition*. Because there are two planes of central circular section of the ellipsoid $\mathscr{E}$, there are two such CCS decompositions. Also, decompositions $\mathbf{H} = \mathbf{P} + \boldsymbol{\Omega}$, in which $\mathbf{P}$ has three zero off-diagonal elements, are considered. These are called *triangular decompositions*.

Finally (§10), after introducing the concept of areal shear [3], we consider unsheared triads of material areal elements. It is seen that line elements along three directions form an unsheared triad of material line elements if and only if areal elements normal to these directions form an unsheared triad of material areal elements. This result is typical of the infinitesimal theory and is not valid for finite strain [5].

*Notation*

The summation convention applies on repeated subscripts. For any set of three linearly independent vectors $\{\mathbf{a}_\alpha\}$, $(\alpha = 1, 2, 3)$, its reciprocal set is denoted by $\{\mathbf{a}^*_\alpha\}$ and defined by $\mathbf{a}_\alpha \cdot \mathbf{a}^*_\beta = \delta_{\alpha\beta}$.

## 2. Basic equations: unsheared pairs

Here we introduce the basic equations and definitions. We recall results [4] on unsheared pairs of material line elements.

The displacement components $u_i$ with respect to a rectangular Cartesian coordinate system $Ox_1x_2x_3$ are given by

$$u_i = x_i - X_i, \tag{2.1}$$

where $x_i$ are the current coordinates of a material point initially with coordinates $X_i$. A material line element $d\mathbf{X}$ at $\mathbf{X}$ is assumed to be deformed into $d\mathbf{x}$ at $\mathbf{x}$, where

$$d\mathbf{x} = d\mathbf{X} + \mathbf{H}d\mathbf{X}, \quad dx_i = dX_i + H_{ij}dX_j, \quad H_{ij} = \frac{\partial u_i}{\partial X_j}. \tag{2.2}$$

The components $H_{ij}$ of the displacement gradient tensor $\mathbf{H}$ are assumed to be sufficiently small [10] that the classical infinitesimal strain theory is valid. The strain tensor $\mathbf{e}$ is given by

$$2\mathbf{e} = \mathbf{H} + \mathbf{H}^T, \quad 2e_{ij} = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}. \tag{2.3}$$

The displacement gradient tensor $\mathbf{H}$ may be decomposed uniquely into the sum of the symmetric strain tensor $\mathbf{e}$ and a skew-symmetric tensor $\boldsymbol{\omega}$, the rotation tensor,

$$\mathbf{H} = \mathbf{e} + \boldsymbol{\omega}, \quad 2\boldsymbol{\omega} = \mathbf{H} - \mathbf{H}^T, \quad 2\omega_{ij} = \frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}. \tag{2.4}$$

Thus, equation $(2.2)_2$ may be written

$$dx_i = dX_i + e_{ij}dX_j + \omega_{ij}dX_j, \tag{2.5}$$

from which it follows that any infinitesimal deformation may be regarded as the superposition of a rigid-body translation upon a rigid-body rotation (due to $\boldsymbol{\omega}$) followed by a strain with stretches along three mutually orthogonal directions (the principal axes of $\mathbf{e}$). It will be seen (§3) that this classical result may be complemented by the result that any infinitesimal deformation may be regarded as the superposition of a rigid-body translation upon a rigid-body rotation followed by a strain with stretches along three oblique directions.

If $\theta$ ($\neq 0, \pi$) is the angle between a pair of material line elements $\mathscr{L}$ and $\mathscr{L}'$ along the unit vectors $\mathbf{s}$ and $\mathbf{t}$ at a point $P$ in an undeformed body, then $\Delta(\mathbf{s}, \mathbf{t})$, the small decrease in the angle between this pair of elements as a result of the infinitesimal deformation, is given by (e.g. [8]; [9])

$$\Delta(\mathbf{s}, \mathbf{t}) \sin\theta = 2e_{ij}s_it_j - (e_{(\mathbf{s})} + e_{(\mathbf{t})})\cos\theta, \tag{2.6}$$

where, for any unit vector $\mathbf{q}$, $e_{(\mathbf{q})}$ is the strain along $\mathbf{q}$ defined by

$$e_{(\mathbf{q})} = e_{ij}q_iq_j. \tag{2.7}$$

In the special case when $\mathbf{s}$ and $\mathbf{t}$ are orthogonal, we retrieve the familiar expression for orthogonal shear, $\Delta(\mathbf{s}, \mathbf{t}) = 2e_{ij}s_it_j$, $\mathbf{s}\cdot\mathbf{t} = 0$.

It follows from (2.6) that the condition for material line elements along $\mathbf{s}$ and $\mathbf{t}$ to be unsheared is

$$2\mathbf{s}\cdot\mathbf{et} = (e_{(\mathbf{s})} + e_{(\mathbf{t})})\mathbf{s}\cdot\mathbf{t}. \tag{2.8}$$

We note in passing that if $L_0$ is the length before deformation of a material line element along the unit vector $\mathbf{q}$ at $P$, then its length after deformation is $L_0(1+e_{(\mathbf{q})})$. In the deformed state the element lies in the direction of the unit vector $\tilde{\mathbf{q}}$, where

$$\tilde{q}_i = (1 - e_{(\mathbf{q})})q_i + \frac{\partial u_i}{\partial X_j} q_j. \tag{2.9}$$

If $\mathbf{c}$ and $\mathbf{d}$ are orthogonal unit vectors in the plane of $\mathbf{s}$ and $\mathbf{t}$, along the internal and external bisectors of the angle $\theta$ between $\mathbf{s}$ and $\mathbf{t}$—the order of the vectors being $\mathbf{s}$, $\mathbf{c}$, $\mathbf{t}$, $\mathbf{d}$—so that

$$\mathbf{s} = \mathbf{c}\cos\frac{\theta}{2} - \mathbf{d}\sin\frac{\theta}{2}, \quad \mathbf{t} = \mathbf{c}\cos\frac{\theta}{2} + \mathbf{d}\sin\frac{\theta}{2}, \tag{2.10}$$

then [1]

$$\Delta(\mathbf{s},\mathbf{t}) = \{e_{(\mathbf{c})} - e_{(\mathbf{d})}\}\sin\theta. \tag{2.11}$$

Let the eigenvalues of $e_{ij}$ be denoted by $e_\alpha$, with $e_3 > e_2 > e_1$. Let the corresponding orthogonal unit eigenvectors be $\mathbf{v}_\alpha$. In general, the eigenvalues are not all positive. However, a positive definite tensor $\mathbf{E}$ may be defined through [7]:

$$E_{ij} = e_{ij} + \lambda\delta_{ij}, \quad \mathbf{E} = \mathbf{e} + \lambda\mathbf{1}, \tag{2.12}$$

where $\lambda$ is chosen such that $e_1 + \lambda > 0$. Similarly to $e_{(\mathbf{h})}$, $E_{(\mathbf{h})}$ is defined by

$$E_{(\mathbf{h})} = E_{ij}h_ih_j = e_{(\mathbf{h})} + \lambda, \tag{2.13}$$

so that, by (2.11),

$$\Delta(\mathbf{s},\mathbf{t}) = \{e_{(\mathbf{c})} - e_{(\mathbf{d})}\}\sin\theta = \{E_{(\mathbf{c})} - E_{(\mathbf{d})}\}\sin\theta. \tag{2.14}$$

The quadric surface $x_iE_{ij}x_j = 1$ associated with the tensor $\mathbf{E}$ is an ellipsoid, which we denote by $\mathscr{E}$. Its principal axes are the principal axes of strain at the point $P$. Any plane $\Pi$ through $P$ will cut $\mathscr{E}$ in an ellipse, which we denote by $\mathscr{S}$. In the special case when the ellipse $\mathscr{S}$ is a circle we denote it by $\mathscr{C}$. If all three principal strains $e_\alpha$ are different from each other, there are two such central circular sections the planes of which are $\mathscr{C}^\pm$ (say). If two and only two of the principal strains are equal, then $\mathscr{E}$ is a spheroid, and there is only one plane of central circular section. Of course, if all three principal strains are equal, then $\mathscr{E}$ is a sphere and every section by $\Pi$ is a circle.

If we assume that $e_3 > e_2 > e_1$, the unit normals $\mathbf{h}^\pm$ (say) to the planes $\mathscr{C}^\pm$ are given by (e.g. [2, p. 90])

$$(e_3 - e_1)^{1/2}\mathbf{h}^\pm = (e_3 - e_2)^{1/2}\mathbf{v}_3 \pm (e_2 - e_1)^{1/2}\mathbf{v}_1, \tag{2.15}$$

and the tensors $\mathbf{e}$ and $\mathbf{E}$ may be written (see, for instance, [2, p. 90])

$$\mathbf{e} = e_2\mathbf{1} + \frac{1}{2}(e_3 - e_1)(\mathbf{h}^+ \otimes \mathbf{h}^- + \mathbf{h}^- \otimes \mathbf{h}^+), \tag{2.16a}$$

$$\mathbf{E} = (\lambda + e_2)\mathbf{1} + \frac{1}{2}(e_3 - e_1)(\mathbf{h}^+ \otimes \mathbf{h}^- + \mathbf{h}^- \otimes \mathbf{h}^+). \tag{2.16b}$$

(When $e_1 = e_2 < e_3$ or $e_1 < e_2 = e_3$, the unit normal to the plane of central circular section is $\mathbf{v}_3$ or $\mathbf{v}_1$, respectively.) We note that

$$(e_3 - e_1)\mathbf{h}^+ \cdot \mathbf{h}^- = e_1 + e_3 - 2e_2, \quad e_{(\mathbf{h}^\pm)} = e_1 + e_3 - e_2. \tag{2.17}$$

So $\mathbf{h}^+ \cdot \mathbf{h}^- = 0$ if

$$2e_2 = e_1 + e_3, \tag{2.18}$$

in which case the normal to one plane $\mathscr{C}^+$ ($\mathscr{C}^-$) lies in the plane of the other $\mathscr{C}^-$ ($\mathscr{C}^+$). Also, if $\mathbf{p}$ is any unit vector lying in $\mathscr{C}^+$ (or $\mathscr{C}^-$), so that $\mathbf{p} \cdot \mathbf{h}^+ = 0$ (or $\mathbf{p} \cdot \mathbf{h}^- = 0$), then, from (2.16a),

$$e_{(\mathbf{p})} = e_2, \tag{2.19}$$

so that the strains of all material infinitesimal line elements in $\mathscr{C}^+$ (or $\mathscr{C}^-$) are equal [6].

To be definite, we consider material elements lying in the plane $\Pi$, with unit normal $\mathbf{n}$, at a point $P$. We suppose also that the ellipse $\mathscr{S}$ in which $\Pi$ cuts the ellipsoid $\mathscr{E}$ has major axis along the unit vector $\mathbf{I}$ and minor axis along the unit vector $\mathbf{J}$:

$$\mathbf{I} \cdot \mathbf{E J} = \mathbf{I} \cdot \mathbf{e J} = 0, \quad e_{(\mathbf{J})} > e_{(\mathbf{I})}. \tag{2.20}$$

If $\Pi$ cuts the ellipsoid $\mathscr{E}$ in a circle $\mathscr{C}$, then all pairs of material line elements in $\mathscr{C}$ are unsheared [6].

If the plane $\Pi$ cuts $\mathscr{E}$ in an ellipse, then the unsheared pairs $(\mathbf{s}, \mathbf{t})$, subtending the angle $\theta$, are given by [4]

$$\sqrt{2}\mathbf{s} = (\mathbf{I} - \mathbf{J}) \sin\frac{\theta}{2} + (\mathbf{I} + \mathbf{J}) \cos\frac{\theta}{2}, \tag{2.21a}$$

$$\sqrt{2}\mathbf{t} = -(\mathbf{I} - \mathbf{J}) \sin\frac{\theta}{2} + (\mathbf{I} + \mathbf{J}) \cos\frac{\theta}{2}, \tag{2.21b}$$

so that

$$2e_{(\mathbf{s})} = e_{(\mathbf{I})} + e_{(\mathbf{J})} + (e_{(\mathbf{I})} - e_{(\mathbf{J})}) \sin\theta, \tag{2.22a}$$

$$2e_{(\mathbf{t})} = e_{(\mathbf{I})} + e_{(\mathbf{J})} - (e_{(\mathbf{I})} - e_{(\mathbf{J})}) \sin\theta, \tag{2.22b}$$

Further,

$$e_{(\mathbf{s})} + e_{(\mathbf{t})} = e_{(\mathbf{I})} + e_{(\mathbf{J})} = I - e_{(\mathbf{n})}, \tag{2.23}$$

where $I = \operatorname{tr} \mathbf{e} = e_{kk}$. Thus, the sum of the stretches along the arms of any unsheared pair in the plane $\Pi$ at $P$ is invariant.

As $\theta$ is varied, an infinite set of unsheared pairs in $\Pi$ is obtained from (2.21). However, as pointed out previously [4], material elements along $(\mathbf{I} + \mathbf{J})$ or $(\mathbf{I} - \mathbf{J})$ have no conjugate elements in $\Pi$. The corresponding directions, which are along the bisectors of the angle between the principal axes of the ellipse $\mathscr{E}$, are called *limiting directions* [4].

Expressions for unsheared pairs may also be given in terms of $\mathbf{h}^+$, $\mathbf{h}^-$ and the unit normal to the plane $\Pi$ [4]. If the normal $\mathbf{n}$ to the plane $\Pi$ is not coplanar with $\mathbf{h}^+$ and $\mathbf{h}^-$, the normals to the planes of central circular section of $\mathscr{E}$ at $P$, i.e. $\mathbf{n} \cdot \mathbf{h}^+ \times \mathbf{h}^- \neq 0$, then the unsheared pairs subtending the angle $\theta$ in $\Pi$ are along $\mathbf{s}$ and $\mathbf{t}$, which are given by

$$\sin\nu\, \mathbf{s} = \widehat{\mathbf{n} \times \mathbf{h}^+} \cos\left(\frac{\pi}{4} + \frac{\theta - \nu}{2}\right) - \widehat{\mathbf{n} \times \mathbf{h}^-} \cos\left(\frac{\pi}{4} + \frac{\theta + \nu}{2}\right), \tag{2.24a}$$

$$\sin\nu\, \mathbf{t} = \widehat{\mathbf{n} \times \mathbf{h}^+} \sin\left(\frac{\pi}{4} + \frac{\theta + \nu}{2}\right) - \widehat{\mathbf{n} \times \mathbf{h}^-} \sin\left(\frac{\pi}{4} + \frac{\theta - \nu}{2}\right), \tag{2.24b}$$

where

$$\cos \nu = (\widehat{\mathbf{n}\times\mathbf{h}^+})\cdot(\widehat{\mathbf{n}\times\mathbf{h}^-}). \tag{2.25}$$

Here $\widehat{\mathbf{n}\times\mathbf{h}^\pm}$ are unit vectors along $\mathbf{n}\times\mathbf{h}^\pm$. As $\theta$ is varied, (2.24) gives all the unsheared pairs in the plane $\Pi$ whose normal $\mathbf{n}$ is not in the plane containing $\mathbf{h}^+$ and $\mathbf{h}^-$ or, equivalently, not in the principal plane containing $\mathbf{v}_1$ and $\mathbf{v}_3$.

However, if $\mathbf{n}$ is in the plane of $\mathbf{h}^+$ and $\mathbf{h}^-$, that is, in the principal $\mathbf{v}_1-\mathbf{v}_3$ plane, then (assuming $\mathbf{n}\times\mathbf{h}^+\neq\mathbf{0}$, $\mathbf{n}\times\mathbf{h}^-\neq\mathbf{0}$) the unsheared pairs subtending the angle $\theta$ in $\Pi$ are [4]

$$\mathbf{s} = \widehat{\mathbf{n}\times\mathbf{h}^+}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \mathbf{n}\times(\widehat{\mathbf{n}\times\mathbf{h}^+})\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right), \tag{2.26a}$$

$$\mathbf{t} = \widehat{\mathbf{n}\times\mathbf{h}^+}\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \mathbf{n}\times(\widehat{\mathbf{n}\times\mathbf{h}^+})\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right). \tag{2.26b}$$

Alternatively, let $\Pi$ contain $\mathbf{t}$ and let $\mathbf{n}$ be normal to $\Pi$. Then $\mathbf{s}$, the direction conjugate to $\mathbf{t}$ in $\Pi$, is given by [4]

$$\begin{aligned}\sigma\mathbf{s} &= [2(\mathbf{n}\times\mathbf{et})\cdot\mathbf{t}]\mathbf{t} + [e_{(\mathbf{t})} - e_{(\mathbf{n}\times\mathbf{t})}]\mathbf{n}\times\mathbf{t}\\ &= [2(\mathbf{n}\times\mathbf{Et})\cdot\mathbf{t}]\mathbf{t} + [E_{(\mathbf{t})} - E_{(\mathbf{n}\times\mathbf{t})}]\mathbf{n}\times\mathbf{t},\end{aligned} \tag{2.27}$$

where $\sigma$ is a scalar such that $\mathbf{s}\cdot\mathbf{s}=1$. This formula always gives a direction conjugate to $\mathbf{t}$ in $\Pi$, unless either (i) $\Pi$ is $\mathscr{C}^+$ or $\mathscr{C}^-$ or (ii) $\mathbf{t}$ is along a limiting direction in $\Pi$ [4].

### *2.1. Remark: unequal strains of arms of unsheared pairs*

Let material elements along $\mathbf{s}$, $\mathbf{t}$ be an unsheared pair, so that equation (2.8) holds. We show that the strains $e_{(\mathbf{s})}$ and $e_{(\mathbf{t})}$ along the arms of the unsheared pair may not be equal, $e_{(\mathbf{s})}\neq e_{(\mathbf{t})}$, unless the plane of $\mathbf{s}$, $\mathbf{t}$ is a plane of central circular section of the ellipsoid $\mathscr{E}$.

Suppose that $\mathbf{p}$ is any unit vector in the plane of ($\mathbf{s}$, $\mathbf{t}$). We show that, if (2.8) is valid with $e_{(\mathbf{s})}=e_{(\mathbf{t})}$, then $e_{(\mathbf{p})}=e_{(\mathbf{s})}$ so that the plane of $\mathbf{s}$, $\mathbf{t}$ must be a plane of central circular section of $\mathscr{E}$. Indeed, for any $\mathbf{p}$ in the plane of ($\mathbf{s}$, $\mathbf{t}$), we have $\mathbf{p}=\alpha\mathbf{s}+\beta\mathbf{t}$, where $\alpha$, $\beta$ are related through $\alpha^2+\beta^2+2\alpha\beta\mathbf{s}\cdot\mathbf{t}=1$, to ensure that $\mathbf{p}$ is a unit vector. Now, using (2.8) and $e_{(\mathbf{s})}=e_{(\mathbf{t})}$, we have

$$e_{(\mathbf{p})} = \alpha^2 e_{(\mathbf{s})} + \beta^2 e_{(\mathbf{t})} + 2\alpha\beta\mathbf{s}\cdot\mathbf{et} = e_{(\mathbf{s})}(\alpha^2+\beta^2+2\alpha\beta\mathbf{s}\cdot\mathbf{t}) = e_{(\mathbf{s})}. \tag{2.28}$$

### *2.2. Remark: unsheared pairs with one arm along* $\mathbf{h}^+$ *or* $\mathbf{h}^-$

Let $\mathbf{s}=\mathbf{h}^+$ so that $\mathbf{s}$ is along the normal to the plane $\mathscr{C}^+$. Let an infinitesimal material line element along $\mathbf{t}$ form an unsheared pair with an element along $\mathbf{s}$. Then the unit normal $\mathbf{n}$ to the plane containing $\mathbf{s}$ and $\mathbf{t}$ is along $\mathbf{s}\times\mathbf{t}$ and lies in $\mathscr{C}^+$, and so $e_{(\mathbf{n})}=e_2$, by equation (2.19). It follows from $(2.17)_2$ and $(2.23)_2$ that $e_{(\mathbf{t})}=I-e_{(\mathbf{n})}-e_{(\mathbf{s})}=I-e_2-(e_1+e_3-e_2)=e_2$. Then, if we use (2.16) and (2.17), the condition (2.8) for an unsheared pair ($\mathbf{s}$, $\mathbf{t}$) gives $2\mathbf{h}^+\cdot\mathbf{et}=(e_1+e_3)\mathbf{h}^+\cdot\mathbf{t}$, which reduces to $\mathbf{h}^-\cdot\mathbf{t}=0$, so that $\mathbf{t}$ is any vector in $\mathscr{C}^-$. Similarly, if $\mathbf{s}$ is along the normal to the plane $\mathscr{C}^-$, then all material line elements forming an unsheared pair with an element along $\mathbf{s}$ lie in $\mathscr{C}^+$.

In summary, the situation is that if a material line element $\mathscr{L}$ lies along the normal to one plane of central circular section of $\mathscr{E}$ at a point $P$, then every material line element lying in the other plane of central circular section of $\mathscr{E}$ at the point $P$ forms an unsheared pair with $\mathscr{L}$.

*2.3. Remark: strains and shears of elements along coplanar directions reciprocal to arms of unsheared pairs*

Let material elements along **s**, **t** form an unsheared pair. Then, from the linearly independent set **s**, **t**, $\mathbf{I}\times\mathbf{J}$, we may determine the reciprocal set $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{I}\times\mathbf{J}$. Thus, $\mathbf{s}^*$, $\mathbf{t}^*$ is the coplanar reciprocal set to **s**, **t**. Using (2.21), we have

$$\sqrt{2}\sin\theta\mathbf{s}^* = (\mathbf{I}-\mathbf{J})\sin\frac{\theta}{2} + (\mathbf{I}+\mathbf{J})\cos\frac{\theta}{2}, \tag{2.29a}$$

$$\sqrt{2}\sin\theta\mathbf{t}^* = (\mathbf{I}-\mathbf{J})\sin\frac{\theta}{2} - (\mathbf{I}+\mathbf{J})\cos\frac{\theta}{2}. \tag{2.29b}$$

It is easily checked that pairs of material line elements along $\mathbf{s}^*$, $\mathbf{t}^*$ are also unsheared. It may also be seen directly by noticing that, if $(\mathbf{I}+\mathbf{J})/\sqrt{2}$ bisects the angle $\theta$ between **s** and **t**, it also bisects the angle $\pi-\theta$ between $\mathbf{s}^*$ and $\mathbf{t}^*$.

Further, we may check that

$$e_{(\mathbf{s})} = e_{(\hat{\mathbf{s}}^*)}, \quad e_{(\mathbf{t})} = e_{(\hat{\mathbf{t}}^*)}, \tag{2.30}$$

where $\hat{\mathbf{s}}^*$ is a unit vector in the direction of $\mathbf{s}^*$. This may also be seen from Fig. 1. Along a line $Ox$, let $OI$ represent $e_{(\mathbf{I})}$ and let $OJ$ represent $e_{(\mathbf{J})}$, so that the midpoint $C$, between $I$ and $J$, is such that $OC=(e_{(\mathbf{I})}+e_{(\mathbf{J})})/2$. With $C$ as centre draw a circle through $I$ and $J$. Let $ACB$ be the diameter orthogonal to $IJ$. For pairs of unsheared material elements **s**, **t** subtending an angle $\theta$, the corresponding strains $e_{(\mathbf{s})}$ and $e_{(\mathbf{t})}$ are given by $OS$ and $OT$:

$$OS = OC + CS = (e_{(\mathbf{I})}+e_{(\mathbf{J})})/2 + \{(e_{(\mathbf{I})}-e_{(\mathbf{J})})/2\}\sin\theta = e_{(\mathbf{s})}, \tag{2.31a}$$

$$OT = OC - CT = (e_{(\mathbf{I})}+e_{(\mathbf{J})})/2 - \{(e_{(\mathbf{I})}-e_{(\mathbf{J})})/2\}\sin\theta = e_{(\mathbf{t})}, \tag{2.31b}$$

by (2.22). Also,

$$OS = (e_{(\mathbf{I})}+e_{(\mathbf{J})})/2 + \{(e_{(\mathbf{I})}-e_{(\mathbf{J})})/2\}\sin(\pi-\theta) = e_{(\hat{\mathbf{s}}^*)}, \tag{2.32a}$$

$$OT = (e_{(\mathbf{I})}+e_{(\mathbf{J})})/2 - \{(e_{(\mathbf{I})}-e_{(\mathbf{J})})/2\}\sin(\pi-\theta) = e_{(\hat{\mathbf{t}}^*)}, \tag{2.32b}$$

## 3. A new decomposition of the displacement gradient and its interpretation

Now we depart from the classical decomposition (2.4) of the displacement gradient tensor into the sum of a symmetric tensor and a skew-symmetric tensor. We introduce a new decomposition by writing **H** as $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where $\mathbf{\Omega}$ is any skew-symmetric tensor such that $2\mathbf{\Omega}\neq\mathbf{H}-\mathbf{H}^T$. We show that if $\mathbf{P}=\mathbf{H}-\mathbf{\Omega}$ has three real linearly independent right eigenvectors then the corresponding triad of material elements along these eigenvectors is unsheared. T.J. Laffey (Appendix A) has shown that, provided that **H** is not a scalar multiple of the unit tensor, it is always possible to choose a skew-symmetric tensor $\mathbf{\Omega}$, $2\mathbf{\Omega}\neq\mathbf{H}-\mathbf{H}^T$, such that $\mathbf{P}=\mathbf{H}-\mathbf{\Omega}$ has three real linearly independent right eigenvectors. If, for example, the deformation is a simple shear or a simple extension, it is seen that there is an infinity of such triads. This means that a physical interpretation may be given to the new decomposition $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where **P** has three real linearly independent right

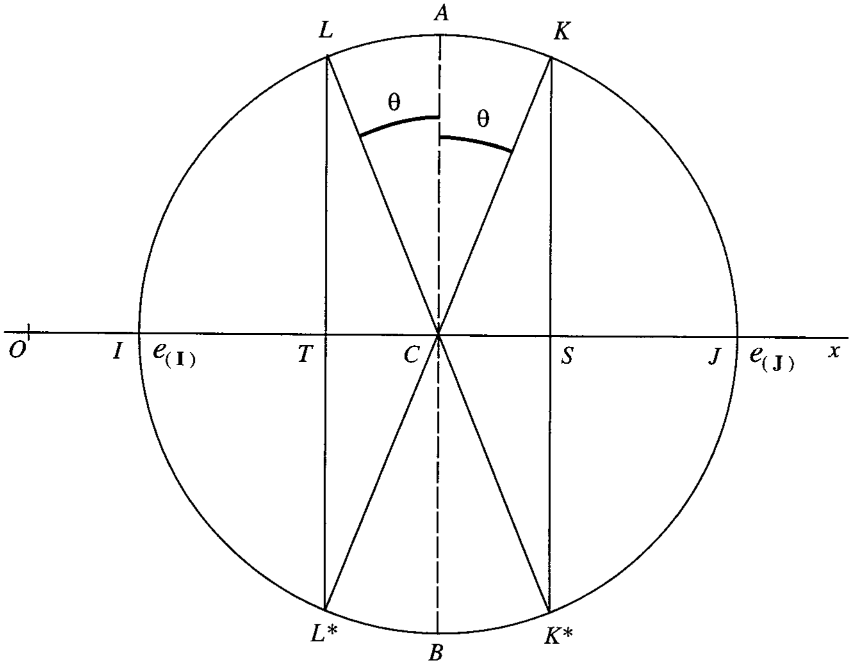


FIG. 1—Geometrical construction of the strains along the arms of the unsheared pairs (**s**,**t**) and (**s***, **t***). Corresponding to the pair (**s**,**t**) subtending the angle $\theta$, draw *CK* and *CL*, making the angle $\theta$ with CA, and, corresponding to the pair (**s***, **t***) subtending the angle $\pi-\theta$, draw *CK** and *CL**, making the angle $\pi-\theta$ with *CA*. The common projection *S* of *K* and *K** and the common projection *T* of *L* and *L** onto the axis *Ox* give the strains along the arms of both unsheared pairs: $e_{(\mathbf{s})}=e_{(\hat{\mathbf{s}}^*)}=OS$, $e_{(\mathbf{t})}=e_{(\hat{\mathbf{t}}^*)}=OT$.

eigenvectors: the infinitesimal deformation at a point *P* may be regarded as a rigid-body translation, followed by a rotation (**Ω**), followed by stretches without shear along the right eigenvectors of **P**. In such a decomposition, we call the tensor **P** a *modified strain tensor* and the tensor **Ω** a *modified rotation tensor*.

First of all we show that in general there is only one unsheared *orthogonal* triad. Then we proceed to introduce the new decomposition.

Clearly if the triad **m**, **s**, **t** is orthonormal, and such that the pairs of material elements along (**m**, **s**), (**s**, **t**) and (**t**, **m**) are unsheared, then, from (2.8), we have

$$\mathbf{m}\cdot\mathbf{es}=\mathbf{s}\cdot\mathbf{et}=\mathbf{t}\cdot\mathbf{em}=0, \quad \mathbf{m}\cdot\mathbf{s}=\mathbf{s}\cdot\mathbf{t}=\mathbf{t}\cdot\mathbf{m}=0, \tag{3.1}$$

or, equivalently,

$$\mathbf{m}\cdot\mathbf{Es}=\mathbf{s}\cdot\mathbf{Et}=\mathbf{t}\cdot\mathbf{Em}=0, \quad \mathbf{m}\cdot\mathbf{s}=\mathbf{s}\cdot\mathbf{t}=\mathbf{t}\cdot\mathbf{m}=0. \tag{3.2}$$

Thus (**m**, **s**) are orthogonal and conjugate with respect to $\mathscr{E}$. So are (**s**, **t**) and (**t**, **m**). It follows that **m**, **s**, **t** are eigenvectors of **E**, that is, along the principal axes of strain. Thus, there is just one unsheared orthogonal triad.

We recall the classical decomposition (2.4) of **H**, the displacement gradient, into the sum of a strain tensor **e** and a rotation tensor $\omega$: $\mathbf{H}=\mathbf{e}+\omega$. Departing from that decomposition, we write instead

$$H_{ij}=\frac{\partial u_i}{\partial X_j}=P_{ij}+\Omega_{ij}, \quad \Omega_{ij}=-\Omega_{ji}, \tag{3.3}$$

where, for the moment, $\mathbf{\Omega}$ is *any* skew-symmetric tensor and $\mathbf{P}=\mathbf{H}-\mathbf{\Omega}$. We note that equation (2.6) may be written

$$\Delta(\mathbf{s},\mathbf{t})\sin\theta=s_i\{H_{ij}t_j-e_{(\mathbf{t})}t_i\}+t_i\{H_{ij}s_j-e_{(\mathbf{s})}s_i\}. \tag{3.4}$$

Now

$$H_{ij}s_j-e_{(\mathbf{s})}s_i=\Omega_{ij}s_j+P_{ij}s_j-e_{(\mathbf{s})}s_i=\Omega_{ij}s_j+(P_{ij}-e_{(\mathbf{s})}\delta_{ij})s_j, \tag{3.5}$$

so that

$$\Delta(\mathbf{s},\mathbf{t})\sin\theta=s_i(P_{ij}-e_{(\mathbf{t})}\delta_{ij})t_j+t_i(P_{ij}-e_{(\mathbf{s})}\delta_{ij})s_j. \tag{3.6}$$

Let $\mathbf{r}$ be a real unit right eigenvector of $\mathbf{P}$. Then the corresponding eigenvalue $\lambda$ is equal to the strain $e_{(\mathbf{r})}$ along the direction of this eigenvector:

$$P_{ij}r_j=\lambda r_i, \quad \text{with} \quad \lambda=r_iP_{ij}r_j=r_iH_{ij}r_j-r_i\Omega_{ij}r_j=e_{(\mathbf{r})}. \tag{3.7}$$

Thus, we note that if $\mathbf{s}$ and $\mathbf{t}$ are right eigenvectors of $\mathbf{P}$ then

$$P_{ij}s_j=e_{(\mathbf{s})}s_i, \quad P_{ij}t_j=e_{(\mathbf{t})}t_i, \tag{3.8}$$

and so

$$\Delta(\mathbf{s},\mathbf{t})=0. \tag{3.9}$$

Thus, the shear of a pair of material elements along the right eigenvectors of $\mathbf{P}$ is zero. Hence, if $\mathbf{\Omega}$ can be chosen such that $\mathbf{P}$ has three real linearly independent right eigenvectors, then material elements along these eigenvectors form an unsheared triad.

(We note that this decomposition (3.3) means that *any* infinitesimal deformation may be regarded locally as the superposition of a rigid-body translation upon a rigid-body rotation (due to $\mathbf{\Omega}$ ) followed by stretches without rotation along three oblique directions—the real right eigenvectors of $\mathbf{P}$.)

Now we present the examples of simple shear and simple extension and show the specific existence of unsheared triads of material elements.

*Example 1: simple shear*
Let

$$\mathbf{H}=\begin{pmatrix}0 & k & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.10}$$

which corresponds to a shear of amount $k$. Then we may write $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where

$$\mathbf{\Omega}=\begin{pmatrix}0 & l & 0\\ -l & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix}0 & k-l & 0\\ l & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.11}$$

and $l(\neq 0)$ is arbitrary. The eigenvalues of $\mathbf{P}$ are $\pm p$ and 0, where

$$p=\sqrt{l(k-l)}, \tag{3.12}$$

with corresponding right eigenvectors

$$\mathbf{p}_1=(p, l, 0), \quad \mathbf{p}_2=(p, -l, 0), \quad \mathbf{p}_3=(0, 0, 1), \tag{3.13}$$

which are linearly independent. Thus, the triad of material elements along $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ is unsheared.

In writing (3.12), we assume that $l$ has been chosen so that $l(k-l)>0$. If we assume without loss of generality that $k>0$, then $l$ has to be chosen such that $k>l>0$. For any given $k>0$, there is an infinity of choices of $l$ satisfying $k>l>0$ and hence there is an infinity of unsheared triads.

*Example 2: simple extension*
As another example, we consider simple extension. Let

$$\mathbf{H}=\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix}, \quad e_1 \neq e. \tag{3.14}$$

Then we may write $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where

$$\mathbf{\Omega}=\begin{pmatrix} 0 & -k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix} e_1 & k & 0 \\ -k & e & 0 \\ 0 & 0 & e \end{pmatrix}, \tag{3.15}$$

and $k$ is arbitrary.

Now $\mathbf{P}$ has eigenvalues $e$, $\mu^{\pm}=\frac{1}{2}\{e_1+e\pm\{(e_1-e)^2-4k^2\}^{1/2}\}$ and corresponding right eigenvectors

$$\mathbf{p}_1=(0, 0, 1), \quad \mathbf{p}_2=(e-\mu^{+}, k, 0), \quad \mathbf{p}_3=(e-\mu^{-}, k, 0). \tag{3.16}$$

Again, material elements along $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ form an unsheared triad. Of course, $k$ must be chosen sufficiently small so that $(e_1-e_2)^2>4k^2$. Thus, for $e_1\neq e$, there is an infinity of possible choices of $k$ and therefore also an infinity of unsheared triads.

## 4. Construction of the decomposition $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$ from a given unsheared triad

Here we assume that the displacement gradient $\mathbf{H}$ is given at a point $P$. We also assume that a known triad of linearly independent unit vectors $\mathbf{s}, \mathbf{t}, \mathbf{m}$ forms the edges of an unsheared triad. We show how the *modified strain tensor* $\mathbf{P}$ may be constructed and prove that $\mathbf{H}-\mathbf{P}$ is skew-symmetric.

Firstly, using the given $\mathbf{H}$, we construct the infinitesimal strain tensor $\mathbf{e}=(\mathbf{H}+\mathbf{H}^T)/2$ and then determine $e_{(\mathbf{s})}$, $e_{(\mathbf{t})}$, $e_{(\mathbf{m})}$. Let $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ be the set of vectors reciprocal to $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$. Then $\mathbf{P}$, with right eigenvectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ and corresponding eigenvalues $e_{(\mathbf{s})}$, $e_{(\mathbf{t})}$, $e_{(\mathbf{m})}$, is given by

$$\mathbf{P}=e_{(\mathbf{s})}\mathbf{s}\otimes\mathbf{s}^*+e_{(\mathbf{t})}\mathbf{t}\otimes\mathbf{t}^*+e_{(\mathbf{m})}\mathbf{m}\otimes\mathbf{m}^*. \tag{4.1}$$

Write $\mathbf{\Omega}=\mathbf{H}-\mathbf{P}$. We show that $\mathbf{\Omega}+\mathbf{\Omega}^T=0$, and thus $\mathbf{\Omega}$ is skew-symmetric. Recalling the condition (2.8) for unsheared pairs, we have here

$$2\mathbf{s}\cdot\mathbf{e}\mathbf{t}=(e_{(\mathbf{s})}+e_{(\mathbf{t})})\mathbf{s}\cdot\mathbf{t}, \quad 2\mathbf{t}\cdot\mathbf{e}\mathbf{m}=(e_{(\mathbf{t})}+e_{(\mathbf{m})})\mathbf{t}\cdot\mathbf{m}, \quad 2\mathbf{m}\cdot\mathbf{e}\mathbf{s}=(e_{(\mathbf{m})}+e_{(\mathbf{s})})\mathbf{m}\cdot\mathbf{s}. \tag{4.2}$$

Now,

$$2\mathbf{s}\cdot\mathbf{et}=\mathbf{s}\cdot(\mathbf{H}+\mathbf{H}^T)\mathbf{t}=\mathbf{s}\cdot(\mathbf{P}+\mathbf{P}^T+\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{t}. \tag{4.3}$$

But, by (4.1),

$$\mathbf{s}\cdot(\mathbf{P}+\mathbf{P}^T)\mathbf{t}=(e_{(\mathbf{s})}+e_{(\mathbf{t})})\mathbf{s}\cdot\mathbf{t}, \tag{4.4}$$

and so $\mathbf{s}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{t}=0$. Thus, we have

$$\mathbf{s}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{t}=0, \quad \mathbf{t}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{m}=0, \quad \mathbf{m}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{s}=0. \tag{4.5}$$

Also,

$$e_{(\mathbf{s})}=\mathbf{s}\cdot\mathbf{es}=\mathbf{s}\cdot(\mathbf{P}+\mathbf{P}^T+\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{s}=e_{(\mathbf{s})}+\mathbf{s}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{s}, \tag{4.6}$$

by (4.1). Thus, we have

$$\mathbf{s}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{s}=0, \quad \mathbf{t}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{t}=0, \quad \mathbf{m}\cdot(\mathbf{\Omega}+\mathbf{\Omega}^T)\mathbf{m}=0. \tag{4.7}$$

Because **s**, **t**, **m** is a linearly independent set, it follows that $\mathbf{\Omega}+\mathbf{\Omega}^T=\mathbf{0}$, and so $\mathbf{\Omega}=\mathbf{H}-\mathbf{P}$ is skew-symmetric. Hence, with **P** defined by (4.1), we may write

$$\mathbf{H}=\mathbf{P}+\mathbf{\Omega}, \tag{4.8}$$

where the tensor $\mathbf{\Omega}$, the *modified rotation tensor*, is skew-symmetric. We also note that, because $\mathbf{\Omega}$ is skew-symmetric, we have

$$2\mathbf{e}=\mathbf{P}+\mathbf{P}^T=e_{(\mathbf{s})}(\mathbf{s}\otimes\mathbf{s}^*+\mathbf{s}^*\otimes\mathbf{s})+e_{(\mathbf{t})}(\mathbf{t}\otimes\mathbf{t}^*+\mathbf{t}^*\otimes\mathbf{t})+e_{(\mathbf{m})}(\mathbf{m}\otimes\mathbf{m}^*+\mathbf{m}^*\otimes\mathbf{m}). \tag{4.9}$$

*4.1. Remark: connection between unsheared triads corresponding to* **P** *and* $\mathbf{P}^T$

Suppose that a definite choice of $\mathbf{\Omega}$ and **P** has been made such that

$$\mathbf{H}=\mathbf{\Omega}+\mathbf{P}, \quad \mathbf{\Omega}=-\mathbf{\Omega}^T. \tag{4.10}$$

Equally well, we may write

$$\mathbf{H}=\tilde{\mathbf{\Omega}}+\mathbf{P}^T, \quad \tilde{\mathbf{\Omega}}=-\tilde{\mathbf{\Omega}}^T, \tag{4.11}$$

where $\tilde{\mathbf{\Omega}}=\mathbf{\Omega}+\mathbf{P}-\mathbf{P}^T$. We know that if **P** has three linearly independent right eigenvectors **s**, **t**, **m** then these form the edges of an unsheared triad. If we assume this to be so, the reciprocal set $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ consists of three linearly independent right eigenvectors of $\mathbf{P}^T$ (or, equivalently, three linearly independent left eigenvectors of **P**), and thus, from (4.11) and §3, the vectors $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ are also along the edges of an unsheared triad of material line elements. Moreover, recalling (3.7), we note that, because **P** and $\mathbf{P}^T$ have the same eigenvalues with corresponding right eigenvectors forming reciprocal sets, we have

$$e_{(\mathbf{s})}=e_{(\hat{\mathbf{s}}^*)}, \quad e_{(\mathbf{t})}=e_{(\hat{\mathbf{t}}^*)}, \quad e_{(\mathbf{m})}=e_{(\hat{\mathbf{m}}^*)}, \tag{4.12}$$

where $\hat{\mathbf{s}}^*$, $\hat{\mathbf{t}}^*$, $\hat{\mathbf{m}}^*$ are unit vectors along $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$. Thus, for any unsheared triad of material line elements, the corresponding triad of material line elements along its reciprocal set is also unsheared, and the stretches along the corresponding edges are the same. This may also be proved by direct computation, as shown in the next section. Hence we have the result: corresponding to one oblique unsheared triad with edges along **s**, **t**, **m** (say), there is always another oblique unsheared triad with edges along $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$, the reciprocal set.

Also, recalling (2.4), the definition of the rotation tensor $\boldsymbol{\omega}$, and using (4.10) and (4.11), we have $2\boldsymbol{\omega} = \mathbf{H} - \mathbf{H}^T = (\mathbf{P} + \boldsymbol{\Omega}) - (\mathbf{P}^T + \tilde{\boldsymbol{\Omega}})^T$, so that

$$2\boldsymbol{\omega} = \boldsymbol{\Omega} + \tilde{\boldsymbol{\Omega}}. \tag{4.13}$$

Thus we have the result that the rotation tensor $\boldsymbol{\omega}$ is the mean of the modified rotation tensors $\boldsymbol{\Omega}$ and $\tilde{\boldsymbol{\Omega}}$ corresponding to an unsheared triad $\mathbf{s}, \mathbf{t}, \mathbf{m}$ and to its reciprocal set $\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$.

## 5. Unsheared triad and reciprocal triad

Here we consider, for given strain tensor $\mathbf{e}$, an unsheared triad of material line elements along the three linearly independent unit vectors $\mathbf{s}, \mathbf{t}, \mathbf{m}$. First we show that the dilatation $I = \operatorname{tr} \mathbf{e}$ is given by

$$I = e_{(\mathbf{s})} + e_{(\mathbf{t})} + e_{(\mathbf{m})}, \tag{5.1}$$

or, equivalently, if $\mathbf{s}\cdot\mathbf{t} \neq 0$, $\mathbf{t}\cdot\mathbf{m} \neq 0$, $\mathbf{m}\cdot\mathbf{s} \neq 0$, by

$$I = \frac{\mathbf{m}\cdot\mathbf{es}}{\mathbf{m}\cdot\mathbf{s}} + \frac{\mathbf{s}\cdot\mathbf{et}}{\mathbf{s}\cdot\mathbf{t}} + \frac{\mathbf{t}\cdot\mathbf{em}}{\mathbf{t}\cdot\mathbf{m}}. \tag{5.2}$$

Next it is shown by direct computation that if $\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$ is the set of vectors reciprocal to $\mathbf{s}, \mathbf{t}, \mathbf{m}$ then the stretches along $\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$ are equal to the stretches along $\mathbf{s}, \mathbf{t}, \mathbf{m}$, respectively, and the triad of material elements along $\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$ is unsheared.

The set of vectors $\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$ reciprocal to $\mathbf{s}, \mathbf{t}, \mathbf{m}$ is given by

$$\mathscr{V}\mathbf{s}^* = \mathbf{t}\times\mathbf{m}, \quad \mathscr{V}\mathbf{t}^* = \mathbf{m}\times\mathbf{s}, \quad \mathscr{V}\mathbf{m}^* = \mathbf{s}\times\mathbf{t}, \quad \text{with} \quad \mathscr{V} = \mathbf{s}\times\mathbf{t}\cdot\mathbf{m}. \tag{5.3}$$

Because $\mathbf{s}, \mathbf{t}, \mathbf{m}$ are along the edges of an unsheared triad, we have, recalling (2.8),

$$\begin{aligned} 2\mathbf{s}\cdot\mathbf{et} &= (e_{(\mathbf{s})} + e_{(\mathbf{t})})\mathbf{s}\cdot\mathbf{t}, \\ 2\mathbf{t}\cdot\mathbf{em} &= (e_{(\mathbf{t})} + e_{(\mathbf{m})})\mathbf{t}\cdot\mathbf{m}, \\ 2\mathbf{m}\cdot\mathbf{es} &= (e_{(\mathbf{m})} + e_{(\mathbf{s})})\mathbf{m}\cdot\mathbf{s}. \end{aligned} \tag{5.4}$$

Now, using these, we have

$$\begin{aligned} I = \operatorname{tr}\mathbf{e} &= \mathbf{s}\cdot\mathbf{es}^* + \mathbf{t}\cdot\mathbf{et}^* + \mathbf{m}\cdot\mathbf{em}^* \\ &= e_{(\mathbf{s})}\mathbf{s}^*\cdot\mathbf{s}^* + e_{(\mathbf{t})}\mathbf{t}^*\cdot\mathbf{t}^* + e_{(\mathbf{m})}\mathbf{m}^*\cdot\mathbf{m}^* \\ &\quad + 2(\mathbf{s}\cdot\mathbf{et})\mathbf{s}^*\cdot\mathbf{t}^* + 2(\mathbf{t}\cdot\mathbf{em})\mathbf{t}^*\cdot\mathbf{m}^* + 2(\mathbf{m}\cdot\mathbf{es})\mathbf{m}^*\cdot\mathbf{s}^* \\ &= e_{(\mathbf{s})}\mathbf{s}^*\cdot\mathbf{s}^* + e_{(\mathbf{t})}\mathbf{t}^*\cdot\mathbf{t}^* + e_{(\mathbf{m})}\mathbf{m}^*\cdot\mathbf{m}^* + (e_{(\mathbf{s})} + e_{(\mathbf{t})})(\mathbf{s}\cdot\mathbf{t})\mathbf{s}^*\cdot\mathbf{t}^* \\ &\quad + (e_{(\mathbf{t})} + e_{(\mathbf{m})})(\mathbf{t}\cdot\mathbf{m})\mathbf{t}^*\cdot\mathbf{m}^* + (e_{(\mathbf{m})} + e_{(\mathbf{s})})(\mathbf{m}\cdot\mathbf{s})\mathbf{m}^*\cdot\mathbf{s}^* \\ &= e_{(\mathbf{s})}\mathbf{s}^*\cdot\mathbf{s} + e_{(\mathbf{t})}\mathbf{t}^*\cdot\mathbf{t} + e_{(\mathbf{m})}\mathbf{m}^*\cdot\mathbf{m} = e_{(\mathbf{s})} + e_{(\mathbf{t})} + e_{(\mathbf{m})}, \end{aligned} \tag{5.5}$$

on noting the fact that

$$\mathbf{s}^* = (\mathbf{s}^*\cdot\mathbf{s}^*)\mathbf{s} + (\mathbf{s}^*\cdot\mathbf{t}^*)\mathbf{t} + (\mathbf{s}^*\cdot\mathbf{m}^*)\mathbf{m} \quad \text{etc.} \tag{5.6}$$

Thus, we have shown that (5.1) and then (5.2) follow directly from (5.4). The sum of the stretches along the edges of an unsheared triad is invariant. It is equal to the dilatation.

*5.1. Remark: unequal strains of arms of unsheared triads*

It has been shown in Remark 2.1 that the strains $e_{(\mathbf{s})}$ and $e_{(\mathbf{t})}$ are unequal for any unsheared pair along **s** and **t**, both of which do not lie in the plane $\mathscr{C}^+$ or the plane $\mathscr{C}^-$. Thus, for an unsheared triad along **s**, **t**, **m** we have $e_{(\mathbf{s})} \neq e_{(\mathbf{t})} \neq e_{(\mathbf{m})} \neq e_{(\mathbf{s})}$, unless two arms of the triad both lie in the plane $\mathscr{C}^+$ or in the plane $\mathscr{C}^-$ (triads with two edges in the same plane of central circular section of $\mathscr{E}$ are considered in §7).

Now we consider the strains along the unsheared triad **s**, **t**, **m** and the reciprocal triad **s***, **t***, **m***. If $\hat{\mathbf{s}}^*$ is a unit vector along **s***, then, from the respective definitions,

$$\mathscr{V}^2\mathbf{s}^* \cdot \mathbf{s}^* e_{(\hat{\mathbf{s}}^*)} = (\mathbf{t} \times \mathbf{m}) \cdot \mathbf{e}(\mathbf{t} \times \mathbf{m}). \tag{5.7}$$

Using now the identity

$$(\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathrm{tr}\,\mathbf{e} = \mathbf{a} \cdot \mathbf{e}(\mathbf{b} \times \mathbf{c}) + \mathbf{b} \cdot \mathbf{e}(\mathbf{c} \times \mathbf{a}) + \mathbf{c} \cdot \mathbf{e}(\mathbf{a} \times \mathbf{b}), \tag{5.8}$$

which is valid for any three vectors, we take $\mathbf{a} = \mathbf{t}$, $\mathbf{b} = \mathbf{m}$, $\mathbf{c} = \mathbf{t} \times \mathbf{m}$ and obtain

$$\begin{aligned}\mathscr{V}^2\mathbf{s}^* \cdot \mathbf{s}^* e_{(\hat{\mathbf{s}}^*)} &= (\mathbf{t} \times \mathbf{m}) \cdot (\mathbf{t} \times \mathbf{m})\mathrm{tr}\,\mathbf{e} - e_{(\mathbf{t})}\mathbf{m} \cdot \mathbf{m} - e_{(\mathbf{m})}\mathbf{t} \cdot \mathbf{t} + 2(\mathbf{t} \cdot \mathbf{em})\mathbf{t} \cdot \mathbf{m} \\ &= \mathscr{V}^2\mathbf{s}^* \cdot \mathbf{s}^*\mathrm{tr}\,\mathbf{e} - \{1 - (\mathbf{t} \cdot \mathbf{m})^2\}(e_{(\mathbf{t})} + e_{(\mathbf{m})}) \\ &= \mathscr{V}^2\mathbf{s}^* \cdot \mathbf{s}^*(\mathrm{tr}\,\mathbf{e} - e_{(\mathbf{t})} - e_{(\mathbf{m})}) \\ &= \mathscr{V}^2\mathbf{s}^* \cdot \mathbf{s}^* e_{(\mathbf{s})},\end{aligned} \tag{5.9}$$

where (5.3), (5.4) and (5.1) have been used. Thus, $e_{(\hat{\mathbf{s}}^*)} = e_{(\mathbf{s})}$, and, proceeding similarly for $\hat{\mathbf{t}}^*$ and $\hat{\mathbf{m}}^*$ (unit vectors along **t*** and **m***), we have

$$e_{(\hat{\mathbf{s}}^*)} = e_{(\mathbf{s})}, \quad e_{(\hat{\mathbf{t}}^*)} = e_{(\mathbf{t})}, \quad e_{(\hat{\mathbf{m}}^*)} = e_{(\mathbf{m})}. \tag{5.10}$$

We have thus shown by explicit calculation that the stretches along the three edges **s**, **t**, **m** of an unsheared triad are equal, in turn, to the stretches along their reciprocal directions **s***, **t***, **m***.

Next we consider the shear of a pair of material line elements along **s***, **t***. Again using (5.4) and the identity (5.8), we take $\mathbf{a} = \mathbf{m}$, $\mathbf{b} = \mathbf{s}$, $\mathbf{c} = \mathbf{t} \times \mathbf{m}$ to obtain

$$\begin{aligned}&2(\mathbf{m} \times \mathbf{s}) \cdot (\mathbf{t} \times \mathbf{m})\mathrm{tr}\,\mathbf{e} - 2(\mathbf{m} \times \mathbf{s}) \cdot \mathbf{e}(\mathbf{t} \times \mathbf{m}) \\ &\quad = 2\mathbf{m}[\mathbf{e} \cdot \{\mathbf{s} \times (\mathbf{t} \times \mathbf{m})\}] + 2\mathbf{s}[\mathbf{e} \cdot \{(\mathbf{t} \times \mathbf{m}) \times \mathbf{m}\}] \\ &\quad = 2(\mathbf{m} \cdot \mathbf{et})\mathbf{s} \cdot \mathbf{m} - 2e_{(\mathbf{m})}\mathbf{s} \cdot \mathbf{t} + 2(\mathbf{s} \cdot \mathbf{em})\mathbf{t} \cdot \mathbf{m} - 2\mathbf{s} \cdot \mathbf{et} \\ &\quad = \{e_{(\mathbf{s})} + e_{(\mathbf{t})}\}(\mathbf{m} \cdot \mathbf{t})(\mathbf{s} \cdot \mathbf{m}) + \{e_{(\mathbf{s})} + e_{(\mathbf{m})}\}(\mathbf{s} \cdot \mathbf{m})(\mathbf{t} \cdot \mathbf{m}) - \{e_{(\mathbf{s})} + e_{(\mathbf{t})} + 2e_{(\mathbf{m})}\}\mathbf{s} \cdot \mathbf{t} \\ &\quad = \{e_{(\mathbf{s})} + e_{(\mathbf{t})} + 2e_{(\mathbf{m})}\}(\mathbf{m} \times \mathbf{s}) \cdot (\mathbf{t} \times \mathbf{m}).\end{aligned} \tag{5.11}$$

Hence,

$$(\mathbf{m} \times \mathbf{s}) \cdot (\mathbf{t} \times \mathbf{m})\{e_{(\mathbf{s})} + e_{(\mathbf{t})}\} = 2(\mathbf{m} \times \mathbf{s}) \cdot \mathbf{e}(\mathbf{t} \times \mathbf{m}), \tag{5.12}$$

or

$$\mathbf{s}^* \cdot \mathbf{t}^*\{e_{(\mathbf{s})} + e_{(\mathbf{t})}\} = 2\mathbf{s}^* \cdot \mathbf{et}^*, \tag{5.13}$$

or, using (5.10),

$$2\mathbf{s}^* \cdot \mathbf{et}^* = \{e_{(\hat{\mathbf{s}}^*)} + e_{(\hat{\mathbf{t}}^*)}\}\mathbf{s}^* \cdot \mathbf{t}^*. \tag{5.14}$$

Thus, the pair of material elements along **s***, **t*** is unsheared. Similarly, the pairs of material elements along **t***, **m*** and **m***, **s*** are also unsheared. Thus, we have shown by explicit calculation that if we have one triad of unsheared material elements, along **s**, **t**, **m**, another triad of unsheared material line elements is along **s***, **t***, **m***.

We conclude this section with some numerical examples.

*Example 3*: *decompositions of* **H**

At the point $P$ in the body let **H** be given by

$$\mathbf{H} = \varepsilon \begin{pmatrix} -16 & 16 & 8 \\ -28 & 52 & 4 \\ 28 & -22 & 12 \end{pmatrix}, \tag{5.15}$$

where $\varepsilon$ is a small quantity, $\varepsilon^2 \ll |\varepsilon|$, so that the linearised strain theory is appropriate. Then

$$\mathbf{e} = \varepsilon \begin{pmatrix} -16 & -6 & 18 \\ -6 & 52 & -9 \\ 18 & -9 & 12 \end{pmatrix}. \tag{5.16}$$

First we write $\mathbf{H} = \mathbf{P} + \mathbf{\Omega}$, where

$$\mathbf{\Omega} = \varepsilon \begin{pmatrix} 0 & -12 & 0 \\ 12 & 0 & -4 \\ 0 & 4 & 0 \end{pmatrix}, \quad \mathbf{P} = \varepsilon \begin{pmatrix} -16 & 28 & 8 \\ -40 & 52 & 8 \\ 28 & -26 & 12 \end{pmatrix}. \tag{5.17}$$

Then it is easily checked that the right unit eigenvectors of **P** are **s**, **t**, **m** given by

$$\mathbf{s} = (2, 2, -1)/3, \quad \mathbf{t} = (2, 2, 1)/3, \quad \mathbf{m} = (-1, -2, 2)/3, \tag{5.18}$$

with corresponding eigenvalues $8\varepsilon$, $16\varepsilon$, $24\varepsilon$, respectively. Also, the set **s***, **t***, **m***, reciprocal to **s**, **t**, **m**, is given by

$$\mathbf{s}^* = (18, -15, -6)/4, \quad \mathbf{t}^* = (-6, 9, 6)/4, \quad \mathbf{m}^* = (3, -3, 0). \tag{5.19}$$

We have $e_{(\hat{\mathbf{s}}^*)} = e_{(\mathbf{s})} = 8\varepsilon$, $e_{(\hat{\mathbf{t}}^*)} = e_{(\mathbf{t})} = 16\varepsilon$, $e_{(\hat{\mathbf{m}}^*)} = e_{(\mathbf{m})} = 24\varepsilon$. It may be checked that the pairs of material line elements along (**s**, **t**), (**t**, **m**), (**m**, **s**) and along (**s***, **t***), (**t***, **m***), (**m***, **s***) are all unsheared.

A second choice is to take

$$\mathbf{\Omega} = \varepsilon \begin{pmatrix} 0 & 16 & 8 \\ -16 & 0 & 4 \\ -8 & -4 & 0 \end{pmatrix}, \quad \mathbf{P} = \varepsilon \begin{pmatrix} -16 & 0 & 0 \\ -12 & 52 & 0 \\ 36 & -18 & 12 \end{pmatrix}. \tag{5.20}$$

Then, for the right eigenvectors **s**, **t**, **m** of **P**, we have

$$\mathbf{s} \sim (1, 3/17, -279/238), \quad \mathbf{t} \sim (0, 1, -9/20), \quad \mathbf{m} \sim (0, 0, 1), \tag{5.21}$$

where we use the symbol $\sim$ to indicate that two vectors have the same direction. The corresponding eigenvalues are $-16\varepsilon$, $52\varepsilon$, $12\varepsilon$, respectively. For the reciprocal set **s***, **t***, **m*** we have

$$\mathbf{s}^* \sim (1, 0, 0), \quad \mathbf{t}^* \sim (-3/17, 1, 0), \quad \mathbf{m}^* \sim (153/140, 9/20, 1). \tag{5.22}$$

Material line elements along **s**, **t**, **m** form an unsheared triad, as do material line elements along **s***, **t***, **m***.

Yet again, we may take $\mathbf{P}^T$ instead of $\mathbf{P}$ given by (5.20) and write $\mathbf{H}=\tilde{\mathbf{\Omega}}+\mathbf{P}^T$, with

$$\tilde{\mathbf{\Omega}}=\varepsilon\begin{pmatrix} 0 & 28 & -28 \\ -28 & 0 & 22 \\ 28 & -22 & 0 \end{pmatrix}, \quad \mathbf{P}^T=\varepsilon\begin{pmatrix} -16 & -12 & 36 \\ 0 & 52 & -18 \\ 0 & 0 & 12 \end{pmatrix}. \tag{5.23}$$

The right eigenvectors of $\mathbf{P}^T$ are $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ given by (5.22), and the reciprocal set is $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ given by (5.21).

## 6. Explicit determination of unsheared triads

Suppose that the pair of material elements along $\mathbf{s}$ and $\mathbf{t}$ at $P$ is unsheared. Here it is shown how to determine $\mathbf{m}$, explicitly, so that the material elements along $\mathbf{m}$, $\mathbf{s}$ and $\mathbf{t}$ form an unsheared triad.

First we note that, if $\alpha$, $\beta$, $\gamma$ are the lengths of the infinitesimal material elements at $P$ along $\mathbf{m}$, $\mathbf{s}$, $\mathbf{t}$, respectively, before the deformation, then the volume of the tetrahedral material element with edges $\alpha\mathbf{m}$, $\beta\mathbf{s}$, $\gamma\mathbf{t}$ is $V_0$ (say), given by

$$V_0=\frac{1}{6}\alpha\beta\gamma\mathbf{m}\times\mathbf{s}\cdot\mathbf{t}. \tag{6.1}$$

After the deformation the edges are approximately of lengths $\alpha(1+e_{(\mathbf{m})})$, $\beta(1+e_{(\mathbf{s})})$, $\gamma(1+e_{(\mathbf{t})})$, so that the volume after deformation is $V_1$ (say), given by

$$V_1=\frac{1}{6}\alpha(1+e_{(\mathbf{m})})\beta(1+e_{(\mathbf{s})})\gamma(1+e_{(\mathbf{t})})\mathbf{m}\times\mathbf{s}\cdot\mathbf{t} \tag{6.2}$$

because the angles between the edges in the tetrahedron do not change. Thus, within the approximation of the infinitesimal strain theory,

$$V_1=V_0(1+e_{(\mathbf{m})}+e_{(\mathbf{s})}+e_{(\mathbf{t})}). \tag{6.3}$$

But, from the general theory of infinitesimal strain, corresponding material volume elements $V_0$ and $V_1$ before and after deformation are related through [8]

$$V_1=V_0(1+I), \quad I=\operatorname{tr}\mathbf{e}. \tag{6.4}$$

So, for an unsheared triad along material elements along $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, assumed to be linearly independent, we recover (5.1)

$$e_{(\mathbf{m})}+e_{(\mathbf{s})}+e_{(\mathbf{t})}=I. \tag{6.5}$$

The condition (6.5) on the sum of the elongations along the edges of any unsheared triad is valid only for triads consisting of three linearly independent vectors. Such triads will be said to be *genuine*, whilst triads consisting of three linearly dependent vectors will be said to be *coplanar*.

From (2.6), for the unsheared pair ($\mathbf{m}$, $\mathbf{s}$) we have

$$\mathbf{m}\cdot\{[e_{(\mathbf{m})}+e_{(\mathbf{s})}]\mathbf{s}-2\mathbf{e}\mathbf{s}\}=0 \tag{6.6}$$

or, equivalently,

$$\mathbf{m}\cdot\{(I-e_{(\mathbf{t})})\mathbf{s}-2\mathbf{e}\mathbf{s}\}=0. \tag{6.7}$$

Similarly, for the unsheared pair (**m**,**t**),

$$\mathbf{m}\cdot\{(I-e_{(\mathbf{s})})\mathbf{t}-2\mathbf{et}\}=0. \tag{6.8}$$

Hence,

$$\mu\mathbf{m}=\{[I-e_{(\mathbf{t})}]\mathbf{s}-2\mathbf{es}\}\times\{[I-e_{(\mathbf{s})}]\mathbf{t}-2\mathbf{et}\}, \tag{6.9}$$

where $\mu$ is a scalar factor such that $\mathbf{m}\cdot\mathbf{m}=1$.

Thus, in general, for a given strain tensor **e**, if **s**, **t** are known to be an unsheared pair, then there is a unique third direction **m**, given explicitly by (6.9), such that (**s**, **t**, **m**) is an unsheared triad. However, special cases may occur. These are discussed in the next sections (§§7 and 8).

Also, to make the connection with the decomposition $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, we note that, if **s** and **t** are right eigenvectors of **P** forming an unsheared pair, then **m**, given by equation (6.9), is also a right eigenvector of **P**, provided, of course, that **P** has three linearly independent right eigenvectors. This is proved in Appendix B.

Finally, examples of constructions of unsheared triads are now presented.

*Example 4: construction of unsheared triads*

We take **e**, given by (5.16), from Example 3, pick an unsheared pair, and construct the corresponding unsheared triad and its reciprocal unsheared triad.

(i) Let

$$\mathbf{s}=(1,\ 0,\ 0),\quad \mathbf{t}=(3,\ -17,\ 0)/\sqrt{298}. \tag{6.10}$$

Using (5.16), we have

$$e_{(\mathbf{s})}=-16\varepsilon,\quad e_{(\mathbf{t})}=52\varepsilon,\quad I=48\varepsilon,\quad \mathbf{s}\cdot\mathbf{t}=3/\sqrt{298},\quad 2\mathbf{s}\cdot\mathbf{et}=108\varepsilon/\sqrt{298}, \tag{6.11}$$

and (2.8) holds, so that **s**, **t** form an unsheared pair. Now, we construct **m**, using equation (6.9). We obtain (the symbol $\sim$ indicates that two vectors have the same direction)

$$\{(I-e_{(\mathbf{t})})\mathbf{s}-2\mathbf{es}\}\sim(7,\ 3,\ -9), \tag{6.12a}$$

$$\{(I-e_{(\mathbf{s})})\mathbf{t}-2\mathbf{et}\}\sim(54,\ -902,\ 207), \tag{6.12b}$$

so that

$$\mathbf{m}\sim(153,\ 63,\ 140), \tag{6.13}$$

and $e_{(\mathbf{m})}=12\varepsilon$. The linearly independent vectors **s**, **t**, **m** are along the edges of an unsheared triad for the strain field given by (5.16).

The directions of the vectors **s***, **t***, **m*** of the reciprocal set are given by

$$\mathbf{s}^*\sim(238,\ 42,\ 279),\quad \mathbf{t}^*\sim(0,\ 20,\ -9),\quad \mathbf{m}^*\sim(0,\ 0,\ 1). \tag{6.14}$$

It may be checked that $e_{(\hat{\mathbf{s}}^*)}=-16\varepsilon$, $e_{(\hat{\mathbf{t}}^*)}=52\varepsilon$, $e_{(\hat{\mathbf{m}}^*)}=12\varepsilon$. It may also be checked that the vectors **s***, **t***, **m*** are along the edges of an unsheared triad for the strain field given by (5.16).

We now proceed to determine another pair of unsheared triads for the same strain field.
(ii) Let

$$\mathbf{s} = (0,\ 1,\ 0), \quad \mathbf{t} = (0,\ 9,\ 20)/\sqrt{481}. \tag{6.15}$$

Using (5.16), we have

$$e_{(\mathbf{s})} = 52\varepsilon, \quad e_{(\mathbf{t})} = 12\varepsilon, \quad I = 48\varepsilon, \quad \mathbf{s}\cdot\mathbf{t} = 9/\sqrt{481}, \quad 2\mathbf{s}\cdot\mathbf{et} = 576\varepsilon/\sqrt{481}, \tag{6.16}$$

so that (2.8) holds. Thus **s**, **t** form an unsheared pair. Also,

$$\{(I - e_{(\mathbf{t})})\mathbf{s} - 2\mathbf{es}\} \sim (6,\ -34,\ 9), \tag{6.17a}$$

$$\{(I - e_{(\mathbf{s})})\mathbf{t} - 2\mathbf{et}\} \sim (306,\ 306,\ 199). \tag{6.17b}$$

Hence, using (6.9), we obtain **m**. We get

$$\mathbf{m} \sim (-238,\ 39,\ 306) \tag{6.18}$$

and obtain $e_{(\mathbf{m})} = -16\varepsilon$. The linearly independent vectors **s**, **t**, **m** are along the edges of an unsheared triad for the strain field given by (5.16).

The directions of the vectors **s***, **t***, **m*** of the reciprocal set are given by

$$\mathbf{s}^* \sim (-141,\ 340,\ -153), \quad \mathbf{t}^* \sim (9,\ 0,\ 7), \quad \mathbf{m}^* \sim (-1,\ 0,\ 0). \tag{6.19}$$

It may be checked that the vectors **s***, **t***, **m*** are along the edges of an unsheared triad for the strain field given by (5.16).

## 7. Special cases: coplanar triads

Here we consider the possibility of coplanar unsheared triads. We first show that all coplanar unsheared triads consist of three material line elements all of which lie either in the plane $\mathscr{C}^+$ or in the plane $\mathscr{C}^-$. We then determine all unsheared triads (coplanar or genuine) with two edges in either the plane $\mathscr{C}^+$ or the plane $\mathscr{C}^-$.

Assume that a coplanar triad (**s**, **t**, **m**) is unsheared. Then, in the plane of this triad, the unit vector **s** has at least two companions **t** and **m** such that (**s**, **t**) and (**s**, **m**) are unsheared pairs. It follows that the plane of the triad must be either the plane $\mathscr{C}^+$ or the plane $\mathscr{C}^-$, because in any other plane each direction has just one companion to form an unsheared pair (except for the limiting directions, which have no companions). Hence, all coplanar unsheared triads consist of three arbitrary line elements lying either in the plane $\mathscr{C}^+$ or in the plane $\mathscr{C}^-$.

Consider now any pair (**s**, **t**) lying in the plane $\mathscr{C}^-$, for instance. It is an unsheared pair [6]. Then all unit vectors **m** in this plane complete with **s** and **t** coplanar unsheared triads. However, in general, (6.9) also yields a unique **m** completing with **s** and **t** a genuine unsheared triad (**s**, **t**, **m**). Indeed, because **s** and **t** lie in the plane $\mathscr{C}^-$, we now have, recalling (2.16a),

$$[I - e_{(\mathbf{t})}]\mathbf{s} - 2\mathbf{es} = (e_1 + e_3 - 2e_2)\mathbf{s} - (e_3 - e_1)(\mathbf{h}^+\cdot\mathbf{s})\mathbf{h}^-, \tag{7.1a}$$

$$[I - e_{(\mathbf{s})}]\mathbf{t} - 2\mathbf{et} = (e_1 + e_3 - 2e_2)\mathbf{t} - (e_3 - e_1)(\mathbf{h}^+\cdot\mathbf{t})\mathbf{h}^-. \tag{7.1b}$$

Hence, in this case, equation (6.9) yields

$$\mu\mathbf{m} = (e_1 + e_3 - 2e_2)^2 \mathbf{s} \times \mathbf{t} - (e_1 + e_3 - 2e_2)(e_3 - e_1)\mathbf{h}^- \times \{(\mathbf{s} \times \mathbf{t}) \times \mathbf{h}^+\}. \tag{7.2}$$

But $\mathbf{s} \times \mathbf{t} = \sin\theta\, \mathbf{h}^-$, where $\theta$ is the angle between $\mathbf{s}$ and $\mathbf{t}$. Hence, if we recall (2.17), equation (7.2) may be written as

$$\mu\mathbf{m} = (e_1 + e_3 - 2e_2)(e_3 - e_1)\mathbf{h}^+. \tag{7.3}$$

For general deformations, that is, for deformations such that $e_1 + e_3 \neq 2e_2$, $\mathbf{h}^+$ does not lie in the plane $\mathscr{C}^-$. Hence, provided that $e_1 + e_3 \neq 2e_2$, the unit vector $\mathbf{m} = \mathbf{h}^+$, normal to the plane $\mathscr{C}^+$, completes with $(\mathbf{s}, \mathbf{t})$ in the plane $\mathscr{C}^-$ a genuine unsheared triad.

For special deformations such that $e_1 + e_3 = 2e_2$, as is the case, for instance, in simple shear (for simple shear we have $e_2 = 0$ and $e_3 = -e_1$), $\mathbf{h}^+$ lies in the plane $\mathscr{C}^-$, and (7.3) reduces to $\mu\mathbf{m} = \mathbf{0}$. Then, the only possible unsheared triads with $\mathbf{s}$ and $\mathbf{t}$ in the plane $\mathscr{C}^-$ are the coplanar triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$ with $\mathbf{m}$ arbitrary in this plane.

Here we summarise the results for the unsheared triads with two arms $\mathbf{s}$, $\mathbf{t}$ in a plane of central circular section of the ellipsoid $\mathscr{E}$ (plane $\mathscr{C}^+$ or $\mathscr{C}^-$).

*Summary*

(a) General deformations: $e_1 + e_3 \neq 2e_2$

When $\mathbf{s}$ and $\mathbf{t}$ both lie in the same plane of central circular section of the ellipsoid $\mathscr{E}$, the corresponding unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$ are: a unique genuine triad with $\mathbf{m}$ along the normal to the other plane of central circular section of the ellipsoid $\mathscr{E}$, and an infinity of coplanar triads with $\mathbf{m}$ arbitrary in the plane of $\mathbf{s}$ and $\mathbf{t}$.

(b) Special deformations: $e_1 + e_3 = 2e_2$

When $\mathbf{s}$ and $\mathbf{t}$ both lie in the same plane of central circular section of the ellipsoid $\mathscr{E}$, the corresponding unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$ are: an infinity of coplanar triads with $\mathbf{m}$ arbitrary in the plane of $\mathbf{s}$ and $\mathbf{t}$.

## 8. Special cases: no triad, undetermined triads

Here we consider the special choices of the unsheared pairs $(\mathbf{s}, \mathbf{t})$ for which $\mu\mathbf{m}$ given by (6.9) is such that $\mathbf{s} \times \mathbf{t} \cdot \mu\mathbf{m} = 0$. For these unsheared pairs, (6.9) does not provide a genuine triad, and it is seen that, in general, there is no $\mathbf{m}$ completing with $\mathbf{s}$ and $\mathbf{t}$ an unsheared triad. These unsheared pairs are said to be *singular*. However, for some particular choices of $\mathbf{s}$ and $\mathbf{t}$ there is an infinity of genuine unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$.

Using the condition (2.8) for an unsheared pair, and taking the dot product of (6.9) with $\mathbf{s} \times \mathbf{t}$, we note that $\mathbf{s} \times \mathbf{t} \cdot \mu\mathbf{m} = 0$ occurs when

$$\{1 - (\mathbf{s} \cdot \mathbf{t})^2\}(I - 2e_{(\mathbf{s})} - e_{(\mathbf{t})})(I - 2e_{(\mathbf{t})} - e_{(\mathbf{s})}) = 0. \tag{8.1}$$

Because $\mathbf{s} \cdot \mathbf{t} \neq \pm 1$, $\mathbf{s} \times \mathbf{t} \cdot \mu\mathbf{m} = 0$ occurs when the pair $(\mathbf{s}, \mathbf{t})$ is such that $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I$, or $2e_{(\mathbf{t})} + e_{(\mathbf{s})} = I$. The second possibility may be obtained from the first by permuting the roles of $\mathbf{s}$ and $\mathbf{t}$, so that, without loss of generality, we may discuss only the possibility

$$2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I. \tag{8.2}$$

As was done in §7, it will prove useful to distinguish between general deformations, for which $e_1 + e_3 \neq 2e_2$, and special deformations, for which $e_1 + e_3 = 2e_2$.

(a) General deformations: $e_1+e_3\neq 2e_2$

The assumption that $e_1+e_3\neq 2e_2$, together with (8.2), excludes the possibility that $e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_2$ and thus the possibility that **s** and **t** lie in the plane $\mathscr{C}^+$, or the plane $\mathscr{C}^-$, and hence also excludes coplanar unsheared triads (see §7). We now consider the possibility of genuine unsheared triads (**s**, **t**, **m**) for which **s** and **t** satisfy (8.2). For such triads the condition (6.5) on the sum of the extensions $e_{(\mathbf{s})}$, $e_{(\mathbf{t})}$, $e_{(\mathbf{m})}$ holds, so that (8.2) yields $e_{(\mathbf{s})}=e_{(\mathbf{m})}$. It follows that **s**, **m** must both lie in a plane of central circular section of the ellipsoid $\mathscr{E}$ and hence

$$e_{(\mathbf{s})}=e_{(\mathbf{m})}=e_2, \quad e_{(\mathbf{t})}=e_1+e_3-e_2. \tag{8.3}$$

Hence, recalling that for a genuine unsheared triad with two edges in the plane $\mathscr{C}^-$ the third edge is necessarily along $\mathbf{h}^+$ (§7), we conclude that genuine unsheared triads with **s** and **t** satisfying (8.2) are possible only when $\mathbf{t}=\mathbf{h}^+$ and **s**, **m** lie in the plane $\mathscr{C}^-$, or when $\mathbf{t}=\mathbf{h}^-$ and **s**, **m** lie in the plane $\mathscr{C}^+$.

Thus we conclude that, if $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$ with $\mathbf{t}\neq\mathbf{h}^+$ and $\mathbf{t}\neq\mathbf{h}^-$, then there is no unsheared triad (**s**, **t**, **m**) corresponding to the unsheared pair (**s**, **t**). Such unsheared pairs will be called *singular pairs*.

If $\mathbf{t}=\mathbf{h}^+$, then **s** forming an unsheared pair with $\mathbf{h}^+$ necessarily lies in the plane $\mathscr{C}^-$ (see Remark 2.2), and we have $e_{(\mathbf{t})}=e_1+e_3-e_2$, $e_{(\mathbf{s})}=e_2$, and $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$. Then, all unit vectors **m** in the plane $\mathscr{C}^-$ complete with **s** and **t** genuine unsheared triads. Similarly, if $\mathbf{t}=\mathbf{h}^-$, then **s** forming an unsheared pair with $\mathbf{h}^-$ necessarily lies in the plane $\mathscr{C}^+$, and all unit vectors **m** in the plane $\mathscr{C}^+$ complete with **s** and **t** genuine unsheared triads.

Thus, all triads consisting of $\mathbf{t}=\mathbf{h}^+$ (or $\mathbf{h}^-$), the unit normal to a plane of central circular section of the ellipsoid $\mathscr{E}$, and of any two vectors in the plane $\mathscr{C}^-$ (or $\mathscr{C}^+$), the other plane of central circular section of this ellipsoid, are unsheared. They form an infinite family of genuine unsheared triads with a common edge along the normal to a plane of central circular section of the ellipsoid $\mathscr{E}$. There are two such families: one with the common edge along $\mathbf{t}=\mathbf{h}^+$, and the other with the common edge along $\mathbf{t}=\mathbf{h}^-$.

In Appendix C, it is shown that, when $\mathbf{t}=\mathbf{h}^+$ (or $\mathbf{h}^-$), (6.9) yields $\mu\mathbf{m}=\mathbf{0}$, whilst, when $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$ with $\mathbf{t}\neq\mathbf{h}^+$ and $\mathbf{t}\neq\mathbf{h}^-$, (6.9) yields $\mu\mathbf{m}$ along **s**. Thus $\mu\mathbf{m}=\mathbf{0}$ corresponds to the case of an infinity of triads, whilst $\mu\mathbf{m}$ along **s** corresponds to singular pairs, for which there is no unsheared triad.

(b) Special deformations: $e_1+e_3=2e_2$

When $e_1+e_3=2e_2$, we note that $I=3e_2$. Then (8.2) becomes

$$2e_{(\mathbf{s})}+e_{(\mathbf{t})}=3e_2. \tag{8.4}$$

When (8.4) is satisfied with $e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_2$, then **s** and **t** both lie in a plane of central circular section of the ellipsoid $\mathscr{E}$, and we retrieve a case discussed in §7. The corresponding unsheared triads are all coplanar. They consist of any three unit vectors in a plane of central circular section of the ellipsoid $\mathscr{E}$. In this case (6.9) yields $\mu\mathbf{m}=\mathbf{0}$ (see §7 or Appendix C).

When (8.4) is satisfied with $e_{(\mathbf{s})}\neq e_{(\mathbf{t})}$, then there is no coplanar unsheared triad (**s**, **t**, **m**). Moreover, it may be seen that there is also no genuine unsheared triad. Indeed, assume that (**s**, **t**, **m**) is an unsheared triad. Then the condition (6.5) on the sum of the extensions $e_{(\mathbf{s})}$, $e_{(\mathbf{t})}$, $e_{(\mathbf{m})}$, where now $e_{(\mathbf{s})}+e_{(\mathbf{t})}+e_{(\mathbf{m})}=3e_2$, yields $e_{(\mathbf{s})}=e_{(\mathbf{m})}$ and hence (see Remark 2.1) $e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_{(\mathbf{m})}=e_2$, which contradicts the assumption.

TABLE 1—General deformations: $e_1+e_3\neq 2e_2$. For each type of unsheared pair (**s**, **t**), the result of (6.9) for $\mu\mathbf{m}$, and all vectors **m** completing corresponding genuine or coplanar unsheared triads (**s**, **t**, **m**), are presented.

| Unsheared pair (**s**, **t**) | $\mu\mathbf{m}$ given by (6.9) | Unsheared triads (**s**, **t**, **m**) | Type |
|---|---|---|---|
| $2e_{(\mathbf{s})}+e_{(\mathbf{t})}\neq I$<br>$2e_{(\mathbf{t})}+e_{(\mathbf{s})}\neq I$<br>**s**, **t** not both in $\mathscr{C}^{\pm}$ | $\mu\mathbf{m}\neq\mathbf{0}$ | unique **m** | genuine |
| **s**, **t** both in $\mathscr{C}^{\pm}$:<br>$e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_2$ | $\mu\mathbf{m}$ along $\mathbf{h}^{\mp}$ | $\mathbf{m}=\mathbf{h}^{\mp}$ and<br>**m** arbitrary in $\mathscr{C}^{\pm}$ | genuine<br>coplanar |
| $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$ with<br>$\mathbf{t}\neq\mathbf{h}^{+}$ and $\mathbf{t}\neq\mathbf{h}^{-}$ | $\mu\mathbf{m}$ along **s** | no triad | — |
| $2e_{(\mathbf{t})}+e_{(\mathbf{s})}=I$ with<br>$\mathbf{s}\neq\mathbf{h}^{+}$ and $\mathbf{s}\neq\mathbf{h}^{-}$ | $\mu\mathbf{m}$ along **t** | no triad | — |
| $\mathbf{t}=\mathbf{h}^{\pm}$ ($\Rightarrow$ **s** in $\mathscr{C}^{\mp}$)<br>$\mathbf{t}=\mathbf{h}^{\pm}$ ($\Rightarrow$ **t** in $\mathscr{C}^{\mp}$) | $\mu\mathbf{m}=\mathbf{0}$<br>$\mu\mathbf{m}=\mathbf{0}$ | **m** arbitrary in $\mathscr{C}^{\mp}$<br>**m** arbitrary in $\mathscr{C}^{\mp}$ | genuine<br>genuine |

Thus, if $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I=3e_2$ with $e_{(\mathbf{s})}\neq e_{(\mathbf{t})}$, then there is no unsheared triad (**s**, **t**, **m**) corresponding to the unsheared pair (**s**, **t**). This unsheared pair is singular. In Appendix C, it is shown that in this case (6.9) yields $\mu\mathbf{m}$ along **t**.

The results presented in this section are summarised below. For the sake of completeness, the results corresponding to $2e_{(\mathbf{t})}+e_{(\mathbf{s})}=I$, which may be read off from the above results by permuting the roles of **s** and **t**, are also stated. Moreover, all the results of §§6–8 are summarised in Tables 1 and 2.

*Summary*

(a) General deformations: $e_1+e_3\neq 2e_2$

When **s** and **t** forming an unsheared pair satisfy $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$, and **t** is not normal to a plane of central circular section of the ellipsoid $\mathscr{E}$ ($\mathbf{t}\neq\mathbf{h}^{+}$ and $\mathbf{t}\neq\mathbf{h}^{-}$), then the pair (**s**, **t**) is singular and there is no unsheared triad (**s**, **t**, **m**).

When **s** and **t** forming an unsheared pair satisfy $2e_{(\mathbf{t})}+e_{(\mathbf{s})}=I$, and **s** is not normal to a plane of central circular section of the ellipsoid $\mathscr{E}$ ($\mathbf{s}\neq\mathbf{h}^{+}$ and $\mathbf{s}\neq\mathbf{h}^{-}$), then the pair (**s**, **t**) is singular and there is no unsheared triad (**s**, **t**, **m**).

When **t** in the unsheared pair (**s**, **t**) is normal to a plane of central circular section of the ellipsoid $\mathscr{E}$, $\mathbf{t}=\mathbf{h}^{+}$ (or $\mathbf{h}^{-}$), so that **s** lies in the plane $\mathscr{C}^{-}$ (or $\mathscr{C}^{+}$) and $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=I$, there is an infinity of genuine unsheared triads (**s**, **t**, **m**) with **m** arbitrary in the plane $\mathscr{C}^{-}$ (or $\mathscr{C}^{+}$).

When **s** in the unsheared pair (**s**, **t**) is normal to a plane of central circular section of the ellipsoid $\mathscr{E}$, $\mathbf{s}=\mathbf{h}^{+}$ (or $\mathbf{h}^{-}$), so that **t** lies in the plane $\mathscr{C}^{-}$ (or $\mathscr{C}^{+}$) and $2e_{(\mathbf{t})}+e_{(\mathbf{s})}=I$, there is an infinity of genuine unsheared triads (**s**, **t**, **m**) with **m** arbitrary in the plane $\mathscr{C}^{-}$ (or $\mathscr{C}^{+}$).

(b) Special deformations: $e_1+e_3=2e_2$

When **s** and **t** forming an unsheared pair satisfy either $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=3e_2$ or $2e_{(\mathbf{t})}+e_{(\mathbf{s})}=3e_2$, with $e_{(\mathbf{s})}\neq e_{(\mathbf{t})}$, then the pair (**s**, **t**) is singular and there is no unsheared triad (**s**, **t**, **m**).

TABLE 2—Special deformations: $e_1 + e_3 = 2e_2$. For each type of unsheared pair $(\mathbf{s}, \mathbf{t})$, the result of (6.9) for $\mu\mathbf{m}$, and all vectors $\mathbf{m}$ completing corresponding genuine or coplanar unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$, are presented. For special deformations, $\mathbf{h}^+ \cdot \mathbf{h}^- = 0$, and $I = 3e_2$.

| Unsheared pair $(\mathbf{s}, \mathbf{t})$ | $\mu\mathbf{m}$ given by (6.9) | Unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$ | Type |
|---|---|---|---|
| $2e_{(\mathbf{s})} + e_{(\mathbf{t})} \neq 3e_2$ and $2e_{(\mathbf{t})} + e_{(\mathbf{s})} \neq 3e_2$ | $\mu\mathbf{m} \neq \mathbf{0}$ | unique $\mathbf{m}$ | genuine |
| $\mathbf{s}$, $\mathbf{t}$ both in $\mathscr{C}^\pm$: $e_{(\mathbf{s})} = e_{(\mathbf{t})} = e_2$ | $\mu\mathbf{m} = \mathbf{0}$ | $\mathbf{m}$ arbitrary in $\mathscr{C}^\pm$ | coplanar |
| $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = 3e_2$ with $e_{(\mathbf{s})} \neq e_{(\mathbf{t})}$ | $\mu\mathbf{m}$ along $\mathbf{s}$ | no triad | — |
| $2e_{(\mathbf{t})} + e_{(\mathbf{s})} = 3e_2$ with $e_{(\mathbf{s})} \neq e_{(\mathbf{t})}$ | $\mu\mathbf{m}$ along $\mathbf{t}$ | no triad | — |

When $\mathbf{s}$ and $\mathbf{t}$ both lie in a plane of central circular section of the ellipsoid $\mathscr{E}$, so that $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = 2e_{(\mathbf{t})} + e_{(\mathbf{s})} = 3e_2$, we retrieve a conclusion of §7: there is an infinity of coplanar unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$ with $\mathbf{m}$ arbitrary in the plane of $\mathbf{s}$ and $\mathbf{t}$.

*8.1. Remark: determination of unsheared pairs* $(\mathbf{s}, \mathbf{t})$ *such that* $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I$

Unsheared pairs $(\mathbf{s}, \mathbf{t})$ for which (8.2) holds play a crucial role in the discussion of the special cases presented here. Now, we show how to determine, in any given plane $\Pi$ with unit normal $\mathbf{n}$, all unsheared pairs $(\mathbf{s}, \mathbf{t})$ satisfying (8.2).

Recalling (2.23), we note that (8.2) also reads

$$e_{(\mathbf{s})} = e_{(\mathbf{n})}. \tag{8.5}$$

The pair $(\mathbf{s}, \mathbf{t})$, being unsheared, is given by (2.21), and thus it follows from (2.22) that (8.5) also reads

$$e_{(\mathbf{I})} + e_{(\mathbf{J})} - (e_{(\mathbf{J})} - e_{(\mathbf{I})}) \sin\theta = 2e_{(\mathbf{n})}. \tag{8.6}$$

If the unit normal $\mathbf{n}$ is such that $e_{(\mathbf{I})} < e_{(\mathbf{n})} < e_{(\mathbf{J})}$, then (8.6) yields two values of $\theta$: an angle $\theta_{(\mathbf{n})}$ and its supplement $\pi - \theta_{(\mathbf{n})}$. Provided that $\theta_{(\mathbf{n})} \neq 0$ or $\pi$, equation (2.21) gives two corresponding unsheared pairs in the plane $\Pi$: one pair $(\mathbf{s}, \mathbf{t})$ corresponding to $\theta_{(\mathbf{n})}$ and the other $(\hat{\mathbf{s}}^*, \hat{\mathbf{t}}^*)$, along its coplanar reciprocal pair, corresponding to $\pi - \theta_{(\mathbf{n})}$. For these two unsheared pairs, (8.2) is satisfied.

If the unit normal $\mathbf{n}$ is such that $e_{(\mathbf{n})} < e_{(\mathbf{I})}$ or $e_{(\mathbf{n})} > e_{(\mathbf{J})}$, then (8.6) may not be satisfied and there is no unsheared pair $(\mathbf{s}, \mathbf{t})$ satisfying (8.2) in the plane $\Pi$.

Finally, if $e_{(\mathbf{n})} = e_{(\mathbf{I})}$ (or $e_{(\mathbf{n})} = e_{(\mathbf{J})}$), then (8.6) yields $\theta = \pi/2$ (or $\theta = -\pi/2$) and (2.21) gives only one pair satisfying (8.2) in the plane $\Pi$: the orthogonal pair $(\mathbf{I}, \mathbf{J})$.

Thus, in a given plane $\Pi$ with unit normal $\mathbf{n}$, (8.6) with (2.21) yields the unsheared pairs $(\mathbf{s}, \mathbf{t})$, which are such that $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I$. For general deformations $(e_1 + e_3 \neq 2e_2)$, these are singular pairs except if $\mathbf{t}$ is normal to a plane of central circular section of the ellipsoid $\mathscr{E}$. For special deformations $(e_1 + e_3 = 2e_2)$, these are singular pairs except if $\Pi$ is a plane of central circular section of the ellipsoid $\mathscr{E}$, in which case $\mathbf{s}$ and $\mathbf{t}$ are arbitrary in this plane.

*Example 5: no triad, infinity of triads*
Here we consider a particular strain. We present two examples of unsheared pairs for which (8.2) holds. It is seen that one pair ($\mathbf{s}$, $\mathbf{t}$) is singular so that there is no $\mathbf{m}$ forming an unsheared triad with $\mathbf{s}$ and $\mathbf{t}$. However, the other pair ($\mathbf{s}^*$, $\mathbf{t}^*$) are the common edges of an infinity of triads.

At the point $P$ in the body let the infinitesimal strain tensor $\mathbf{e}$ be given by

$$\mathbf{e}=\varepsilon\begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 3 & 3/2 \\ 0 & 3/2 & 1 \end{pmatrix}, \tag{8.7}$$

where $\varepsilon$ is a small quantity, $\varepsilon^2 \ll |\varepsilon|$, so that the linearised strain theory is appropriate. Then, $I=5\varepsilon$,

$$e_1=\varepsilon(2-\sqrt{14}/2), \quad e_2=\varepsilon, \quad e_3=\varepsilon(2+\sqrt{14}/2), \tag{8.8}$$

and hence $e_1+e_3\neq 2e_2$. The normals $\mathbf{h}^{\pm}$ to the planes of central circular section of the ellipsoid $\mathscr{E}$ are

$$\mathbf{h}^+=(0,\ 1,\ 0), \quad \mathbf{h}^-=(1,\ 2,\ 3)/\sqrt{14}. \tag{8.9}$$

Consider first the pair

$$\mathbf{s}=(1,\ 0,\ 0), \quad \mathbf{t}=(1,\ 2,\ 0)/\sqrt{5}. \tag{8.10}$$

Because $\mathbf{s}\cdot\mathbf{t}=1/\sqrt{5}$, $\mathbf{s}\cdot\mathbf{et}=2\varepsilon/\sqrt{5}$, $e_{(\mathbf{s})}=\varepsilon$, $e_{(\mathbf{t})}=3\varepsilon$, the condition (2.8) is satisfied and ($\mathbf{s}, \mathbf{t}$) is an unsheared pair. But, because $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=5\varepsilon=I$ with $\mathbf{t}\neq\mathbf{h}^{\pm}$, this pair is singular: there is no $\mathbf{m}$ forming with $\mathbf{s}$ and $\mathbf{t}$ an unsheared triad.

Consider now the pair

$$\hat{\mathbf{s}}^*=(2,\ -1,\ 0)/\sqrt{5}, \quad \hat{\mathbf{t}}^*=(0,\ 1,\ 0), \tag{8.11}$$

which is along the coplanar reciprocal of the pair ($\mathbf{s}, \mathbf{t}$) given by (8.10). It is also an unsheared pair (see Remark 2.3), and, because $e_{(\hat{\mathbf{s}}^*)}=e_{(\mathbf{s})}$, $e_{(\hat{\mathbf{t}}^*)}=e_{(\mathbf{t})}$, we have $2e_{(\hat{\mathbf{s}}^*)}+e_{(\hat{\mathbf{t}}^*)}=5\varepsilon=I$. However, $\hat{\mathbf{t}}^*=\mathbf{h}^+$. Hence, there is an infinity of $\hat{\mathbf{m}}^*$ forming with $\hat{\mathbf{s}}^*$ and $\hat{\mathbf{t}}^*$ unsheared triads. Indeed, all triads ($\hat{\mathbf{s}}^*$, $\hat{\mathbf{t}}^*$, $\hat{\mathbf{m}}^*$) with $\hat{\mathbf{m}}^*$ arbitrary in the central plane with normal $\mathbf{h}^-$ are unsheared.

## 9. Special decompositions of the displacement gradient

Here, we introduce special decompositions $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$ of the displacement gradient $\mathbf{H}$ into the sum of a skew-symmetric tensor $\mathbf{\Omega}$ and a tensor $\mathbf{P}$ that has three linearly independent real right eigenvectors.

In §9.1 we assume that $e_1+e_3\neq 2e_2$ and consider the two infinite families of genuine unsheared triads with a common edge along $\mathbf{h}^+$, or $\mathbf{h}^-$, a normal to a plane of central circular section of the ellipsoid $\mathscr{E}$. We determine the decompositions corresponding to these unsheared triads. It is shown that a decomposition is obtained for each of these families. We call these two decompositions the *CCS-decompositions* of $\mathbf{H}$ on account of the essential role played by the central circular sections (*CCS*) of the ellipsoid $\mathscr{E}$.

In §9.2 we consider special decompositions $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$ for which, in a given rectangular coordinate system, $\mathbf{P}$ has three zero off-diagonal components. There are six such decompositions, and, for two of them, the components of **P** form a triangular matrix (lower or upper triangular matrix). For this reason we call these *triangular decompositions* of **H**.

*9.1. CCS-decompositions*

Here, for general deformations ($e_1+e_3\neq 2e_2$), we consider the infinite family of unsheared triads (**s**, **t**, **m**) with a common edge $\mathbf{t}=\mathbf{h}^+$ and the two other edges along **s**, **m** arbitrary in the plane $\mathscr{C}^-$. Thus, $e_{(\mathbf{s})}=e_{(\mathbf{m})}=e_2$ and $e_{(\mathbf{t})}=e_3+e_1-e_2$. For the corresponding reciprocal triads ($\mathbf{s}^*, \mathbf{t}^*, \mathbf{m}^*$), which are also unsheared, we have $(e_3+e_1-2e_2)\mathbf{t}^*=(e_3-e_1)\mathbf{h}^-$, and $\mathbf{s}^*, \mathbf{m}^*$ lie in the plane $\mathscr{C}^+$. Hence, because $\mathbf{m}\otimes\mathbf{m}^*+\mathbf{s}\otimes\mathbf{s}^*=\mathbf{1}-\mathbf{t}\otimes\mathbf{t}^*$, the corresponding expression (4.1) for **P** becomes $\mathbf{P}=\mathbf{P}_\odot$ (say), with

$$\mathrm{P}_\odot=e_2\mathbf{1}+(e_3-e_1)\mathbf{h}^+\otimes\mathbf{h}^-. \tag{9.1}$$

Similarly, considering the infinite family of unsheared triads with a common edge $\mathbf{t}=\mathbf{h}^-$ and the two other edges along **s**, **m** arbitrary in the plane $\mathscr{C}^+$, we obtain from (4.1) $\mathbf{P}=\mathbf{P}_\odot^T$. Hence, because the triads of the two families are unsheared, the displacement gradient may be decomposed as

$$\mathbf{H}=\mathbf{P}_\odot+\boldsymbol{\Omega}_\odot=\mathbf{P}_\odot^T+\tilde{\boldsymbol{\Omega}}_\odot, \tag{9.2}$$

where both $\boldsymbol{\Omega}_\odot$ and $\tilde{\boldsymbol{\Omega}}_\odot=\boldsymbol{\Omega}_\odot+\mathbf{P}_\odot-\mathbf{P}_\odot^T$ are skew-symmetric. This may be checked explicitly, because, recalling (2.16a), we note that $\mathbf{P}_\odot+\mathbf{P}_\odot^T=2\mathbf{e}$, so that $\mathbf{H}+\mathbf{H}^T=2\mathbf{e}+\boldsymbol{\Omega}_\odot+\boldsymbol{\Omega}_\odot^T$, and hence $\boldsymbol{\Omega}_\odot+\boldsymbol{\Omega}_\odot^T=\mathbf{0}$. The decompositions (9.2) of **H** will be called *CCS-decompositions* of the displacement gradient.

The tensor $\mathbf{P}_\odot$, defined by (9.1), has the simple eigenvalue $e_3+e_1-e_2$ with corresponding right eigenvector $\mathbf{h}^+$, and the double eigenvalue $e_2$ with corresponding right eigenvectors arbitrary in the plane $\mathscr{C}^-$. In components, in the orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ along the principal axes of strain, we have

$$\mathbf{P}_\odot=\begin{pmatrix} e_1 & 0 & \varepsilon \\ 0 & e_2 & 0 \\ -\varepsilon & 0 & e_3 \end{pmatrix}, \quad \boldsymbol{\Omega}_\odot=\boldsymbol{\omega}+\begin{pmatrix} 0 & 0 & -\varepsilon \\ 0 & 0 & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \tag{9.3}$$

where

$$\varepsilon=\{(e_3-e_2)(e_2-e_1)\}^{1/2}. \tag{9.4}$$

The decomposition (9.2) means that the deformation may be regarded locally as the superposition of a rigid-body translation upon a rigid-body rotation (due to $\boldsymbol{\Omega}_\odot$ or $\tilde{\boldsymbol{\Omega}}_\odot$) followed by stretches of the same amount $e_2$ of all elements in a plane of central circular section of $\mathscr{E}$ and a stretch of amount $e_3+e_1-e_2$ along the normal to the other plane of central circular section of $\mathscr{E}$.

Finally, we note that $\mathbf{P}_\odot$ given by (9.1) remains defined even for special deformations with $e_3+e_1=2e_2$, so that the two *CCS*-decompositions (9.2) remain valid in this case. Then, $\varepsilon=(e_3-e_1)/2$, and $\mathbf{P}_\odot$ has the triple eigenvalue $e_2$. Corresponding to this triple eigenvalue there is a double infinity of right eigenvectors consisting of all vectors in the plane $\mathscr{C}^-$.

*9.2. Triangular decompositions*

Let $u_{i,j}=\partial u_i/\partial X_j$ denote the components of the displacement gradient $\mathbf{H}$ in a fixed rectangular Cartesian coordinate system $Ox_1x_2x_3$. Special decompositions $\mathbf{H}=\mathbf{P}+\boldsymbol{\Omega}$ for which $\mathbf{P}$ has, in this coordinate system, three zero off-diagonal components are introduced here. We require, for example, $P_{12}=P_{13}=P_{23}=0$. Because $\boldsymbol{\Omega}$ must be skew-symmetric, these conditions determine uniquely $\boldsymbol{\Omega}$ and $\mathbf{P}$. Indeed, we have $u_{1,2}=\Omega_{12}$, $u_{1,3}=\Omega_{13}$, $u_{2,3}=\Omega_{23}$, so that $\boldsymbol{\Omega}$ is uniquely determined, and the decomposition is

$$\mathbf{H}=\mathbf{P}+\boldsymbol{\Omega};\quad \mathbf{P}=\begin{pmatrix} e_{11} & 0 & 0 \\ 2e_{12} & e_{22} & 0 \\ 2e_{13} & 2e_{23} & e_{33} \end{pmatrix},\quad \boldsymbol{\Omega}=\begin{pmatrix} 0 & u_{1,2} & u_{1,3} \\ -u_{1,2} & 0 & u_{2,3} \\ -u_{1,3} & -u_{2,3} & 0 \end{pmatrix}. \tag{9.5}$$

Because the components of $\mathbf{P}$ form a lower triangular matrix, this decomposition will be called a *triangular decomposition* of the displacement gradient $\mathbf{H}$. The eigenvalues of $\mathbf{P}$ are $e_{11}$, $e_{22}$, $e_{33}$, and we assume that $e_{11}\neq e_{22}\neq e_{33}\neq e_{11}$ so that the corresponding right eigenvectors are necessarily linearly independent. These are eigenvectors along

$$\begin{aligned} &\mathbf{p}_1=((e_{33}-e_{11})(e_{11}-e_{22}),\ 2e_{12}(e_{33}-e_{11}),\ -2e_{13}(e_{11}-e_{22})-4e_{12}e_{23}),\\ &\mathbf{p}_2=(0,\ e_{22}-e_{33},\ 2e_{23}),\\ &\mathbf{p}_3=(0,\ 0,\ 1). \end{aligned} \tag{9.6}$$

These three linearly independent vectors are along the edges of an unsheared triad.

Clearly, the decomposition

$$\mathbf{H}=\mathbf{P}^T+\tilde{\boldsymbol{\Omega}};\quad \mathbf{P}^T=\begin{pmatrix} e_{11} & 2e_{12} & 2e_{13} \\ 0 & e_{22} & 2e_{23} \\ 0 & 0 & 0 \end{pmatrix},\quad \tilde{\boldsymbol{\Omega}}=\begin{pmatrix} 0 & -u_{2,1} & -u_{3,1} \\ u_{2,1} & 0 & -u_{3,2} \\ u_{3,1} & u_{3,2} & 0 \end{pmatrix}, \tag{9.7}$$

is also possible. The right eigenvectors of $\mathbf{P}^T$ (or, equivalently, the left eigenvectors of $\mathbf{P}$) are $\mathbf{p}_1^*$, $\mathbf{p}_2^*$, $\mathbf{p}_3^*$ given, up to scalar factors, by

$$\begin{aligned} &\mathbf{p}_1^*=(1,\ 0,\ 0),\\ &\mathbf{p}_2^*=(2e_{12},\ e_{11}-e_{22},\ 0),\\ &\mathbf{p}_3^*=(2e_{13}(e_{33}-e_{22})-4e_{12}e_{23},\ 2e_{23}(e_{33}-e_{11}),\ (e_{33}-e_{11})(e_{33}-e_{22})). \end{aligned} \tag{9.8}$$

These three linearly independent eigenvectors are also along the edges of an unsheared triad.

Finally, we note that, instead of requiring $P_{12}=P_{13}=P_{23}=0$, we might require $P_{23}=P_{21}=P_{31}=0$, which again leads to two decompositions $\mathbf{H}=\mathbf{P}^T+\boldsymbol{\Omega}=\mathbf{P}^T+\tilde{\boldsymbol{\Omega}}$ that we also call triangular, or require $P_{31}=P_{32}=P_{12}=0$, which again leads to two triangular decompositions. There are thus six possible triangular decompositions of the displacement gradient $\mathbf{H}$, in which $\mathbf{P}$ has three zero off-diagonal components.

## 10. Unsheared areal triads

As a body is deformed, material planar elements at $P$ are stretched and rotated. In the context of the classical infinitesimal strain theory, a material planar areal element $d\mathbf{A}$ at $\mathbf{X}$

is infinitesimally deformed into $d\mathbf{a}$ at $\mathbf{x}$, where

$$d\mathbf{a} = (1 + I)d\mathbf{A} - \mathbf{H}^T d\mathbf{A}, \quad da_i = (1 + I)dA_i - H_{ji}dA_j. \tag{10.1}$$

It follows that, if $A_0$ is the area before deformation of an infinitesimal material planar areal element with unit normal $\mathbf{S}$ at $P$, then its area after deformation is $A_0(1 + I - e_{(\mathbf{S})})$. In the deformed state the planar element has unit normal $\tilde{\mathbf{S}}$, where

$$\tilde{S}_i = (1 + e_{(\mathbf{S})})S_i - \frac{\partial u_j}{\partial X_i} S_j. \tag{10.2}$$

Consider now a pair of material areal elements at $P$ with unit normal vectors $\mathbf{S}$ and $\mathbf{T}$ in the undeformed state, and corresponding unit normal vectors $\tilde{\mathbf{S}}$, $\tilde{\mathbf{T}}$ in the deformed state. Let $\psi(\neq 0, \pi)$ be the angle between $\mathbf{S}$ and $\mathbf{T}$, and let $\psi - \delta(\mathbf{s}, \mathbf{t})$ be the angle between $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{T}}$, so that $\delta(\mathbf{s}, \mathbf{t})$ is the *areal shear* [3], the decrease in the angle between the pair of normals as a result of the deformation. Then,

$$\cos(\psi - \delta(\mathbf{s}, \mathbf{t})) = \tilde{\mathbf{S}} \cdot \tilde{\mathbf{T}} = (1 + e_{(\mathbf{S})} + e_{(\mathbf{T})})\mathbf{S} \cdot \mathbf{T} - 2\mathbf{S} \cdot \mathbf{e}\mathbf{T}, \tag{10.3}$$

and hence

$$\delta(\mathbf{s}, \mathbf{t}) \ \sin\psi = -2e_{ij}S_iT_j + (e_{(\mathbf{S})} + e_{(\mathbf{T})})\cos\psi, \tag{10.4}$$

within the context of the infinitesimal theory.

We note that, apart from the sign, this equation for the areal shear is precisely of the same form as equation (2.6) for the shear of a pair of material line elements. The theory of the shear of planar material elements runs parallel to the theory of the shear of material line elements.

The areal shear $\delta(\mathbf{s}, \mathbf{t})$ is zero if

$$2\mathbf{S} \cdot \mathbf{e}\mathbf{T} = (e_{(\mathbf{S})} + e_{(\mathbf{T})})\mathbf{S} \cdot \mathbf{T}, \tag{10.5}$$

which is precisely of the same form as (2.8).

For an unsheared triad of material line elements along $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, we may consider the pair ($\mathbf{t}$, $\mathbf{m}$) as lying in a material planar element with normal along $\mathbf{s}^*$. Similarly the normal to the material planar element containing ($\mathbf{m}$, $\mathbf{s}$) is along $\mathbf{t}^*$. The condition (10.5), that a pair of material planar elements is unsheared, is satisfied with $\mathbf{S} = \mathbf{s}^*$ and $\mathbf{T} = \mathbf{t}^*$ because of (5.14). Thus, it follows that, if $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ form the edges of an unsheared triad of material line elements, then $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ (along the reciprocal set) are normals to an unsheared triad of material areal elements. Also, it has been seen (§5) that, corresponding to the unsheared triad of material line elements along $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, there is also an unsheared triad of material line elements along the reciprocal set $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$. It follows, therefore, that the triad of planar elements with normals $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ is also unsheared (provided that the triad of material line elements along $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ is unsheared). Conversely, if the triad of material planar elements with unit normals $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ is unsheared, then so is the triad of planar elements with normals $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ and also the two triads of material line elements along $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ and $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$.

## 11. Concluding remarks

Here we summarise the main developments presented in the paper.

The idea of unsheared triads of infinitesimal material line elements has been introduced. For a given deformation, it has been seen that, at a point $P$, there is only one orthogonal triad of unsheared line elements. However, there is an infinity of oblique unsheared triads. If an oblique triad with line elements along **s**, **t**, **m** is known to be unsheared, then the triad with line elements along the reciprocal set **s***, **t***, **m*** is also unsheared. Moreover, the extensions along the directions **s**, **t**, **m** are the same as the extensions along the directions **s***, **t***, **m***, respectively.

New decompositions $\mathbf{H} = \mathbf{P} + \boldsymbol{\Omega}$ of the displacement gradient **H** have been introduced, where $\boldsymbol{\Omega}$ is skew-symmetric and **P** is non-symmetric but has three linearly independent real right eigenvectors. There is an infinity of such decompositions. A link between these decompositions and unsheared triads has been exhibited. Indeed, the three linearly independent real right eigenvectors **s**, **t**, **m** of **P** are along the edges of an unsheared triad. Then, the basic deformation may be regarded as being made up of a rigid-body translation, followed by a rigid-body rotation associated with $\boldsymbol{\Omega}$, followed by three (generally unequal) extensions without rotation along **s**, **t**, **m**. Conversely, to a given unsheared triad with line elements along **s**, **t**, **m** corresponds a decomposition $\mathbf{H} = \mathbf{P} + \boldsymbol{\Omega}$, where **s**, **t**, **m** are the right eigenvectors of **P** and $\boldsymbol{\Omega}$ is skew-symmetric.

Given an unsheared pair of material line elements, a constructive way has been presented of determining the direction of a third line element to form an unsheared triad. This direction is in general unique. The circumstances under which there is no unsheared triad, or an infinity of unsheared triads corresponding to a given unsheared pair, have been examined in detail.

Finally, after recalling the concept of areal shear, we also considered unsheared triads of material areal elements. It has been seen that line elements along three directions **s**, **t**, **m** form an unsheared triad of material line elements (and thus also line elements along **s***, **t***, **m***) if and only if areal elements normal to **s**, **t**, **m** form an unsheared triad of material areal elements (and also areal elements normal to **s***, **t***, **m***).

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## APPENDIX A

### Possibility of the decomposition $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$

It is shown here that, provided that $\mathbf{e}$ is not a scalar multiple of the unit tensor, there is an infinity of decompositions $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where $\mathbf{\Omega}$ is skew-symmetric and $\mathbf{P}$ is not symmetric but has three linearly independent real right eigenvectors. The proof is due to T.J. Laffey (pers. comm.).

Any second-order tensor $\mathbf{H}$ may be written as $\mathbf{H}=\mathbf{e}+\boldsymbol{\omega}$, where $\mathbf{e}$ and $\boldsymbol{\omega}$ are its symmetric and skew-symmetric parts, respectively.

If $\mathbf{e}=e\mathbf{1}$, then writing $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$ yields $\mathbf{P}=e\mathbf{1}+\boldsymbol{\omega}-\mathbf{\Omega}$, and this tensor has the eigenvalue $e$ and two complex conjugate eigenvalues, unless $\boldsymbol{\omega}=\mathbf{\Omega}$, which is excluded because we are looking for non-symmetric tensors $\mathbf{P}$.

If $\mathbf{e}$ is not spherical, then, without loss of generality, we may assume that its eigenvalues $e_\alpha$, $(\alpha=1, 2, 3)$, are such that $e_1 \neq e_2$. We then write $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, thus $\mathbf{P}=\mathbf{e}+\boldsymbol{\omega}-\mathbf{\Omega}$, and we may choose $\mathbf{\Omega}$ so that, in components in the principal axes of $\mathbf{e}$,

$$\boldsymbol{\omega}-\mathbf{\Omega}=\begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix} e_1 & \alpha & 0 \\ -\alpha & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}. \tag{A1}$$

Clearly, one of the eigenvalues of $\mathbf{P}$ is $e_3$, and the corresponding right eigenvector is (0, 0, 1). The other two eigenvalues are the solutions for $\lambda$ of the quadratic

$$\lambda^2-(e_1+e_2)\lambda+(e_1e_2+\alpha^2)=0. \tag{A2}$$

For

$$|\alpha|<|e_1-e_2|/2 \tag{A3}$$

the discriminant of the quadratic (A2) is positive, and hence, besides the eigenvalue $e_3$, the tensor $\mathbf{P}$ has two other real eigenvalues with corresponding linearly independent real right eigenvectors. There is an infinity of choices of $\alpha$ satisfying (A3), and thus an infinity of decompositions $\mathbf{H}=\mathbf{P}+\mathbf{\Omega}$, where $\mathbf{\Omega}$ is skew-symmetric.

## APPENDIX B

### Determination of the third right eigenvector of $\mathbf{P}$

Let $\mathbf{P}$ be a non-symmetric tensor with three linearly independent right eigenvectors $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$. Let $\mathbf{e}=(\mathbf{P}+\mathbf{P}^T)/2$ be the symmetric part of $\mathbf{P}$, so that the real eigenvalues of $\mathbf{P}$ corresponding to $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$ are $e_{(\mathbf{s})}$, $e_{(\mathbf{t})}$, $e_{(\mathbf{m})}$, respectively. Here, it is shown that, if we know $\mathbf{P}$ and the eigenvectors $\mathbf{s}$ and $\mathbf{t}$, the third right eigenvector $\mathbf{m}$ may, in general, be obtained from (6.9), viz. is along the vector product $\mathbf{a}\times\mathbf{b}$, where

$$\mathbf{a}=(I-e_{(\mathbf{t})})\mathbf{s}-2\mathbf{e}\mathbf{s}, \quad \mathbf{b}=(I-e_{(\mathbf{s})})\mathbf{t}-2\mathbf{e}\mathbf{t}. \tag{B1}$$

Let $\mathbf{s}^*$, $\mathbf{t}^*$, $\mathbf{m}^*$ be the set of vectors reciprocal to $\mathbf{s}$, $\mathbf{t}$, $\mathbf{m}$, so that $\mathbf{P}$ is given by (4.1). Then

$$\mathbf{a}=e_{(\mathbf{m})}\mathbf{s}-\mathbf{P}^T\mathbf{s}=(e_{(\mathbf{m})}-e_{(\mathbf{s})})(\mathbf{s}\cdot\mathbf{s})\mathbf{s}^*+(e_{(\mathbf{m})}-e_{(\mathbf{t})})(\mathbf{s}\cdot\mathbf{t})\mathbf{t}^*, \tag{B2}$$

$$\mathbf{b} = e_{(\mathbf{m})}\mathbf{t} - \mathbf{P}^T\mathbf{t} = (e_{(\mathbf{m})} - e_{(\mathbf{s})})(\mathbf{t}\cdot\mathbf{s})\mathbf{s}^* + (e_{(\mathbf{m})} - e_{(\mathbf{t})})(\mathbf{t}\cdot\mathbf{t})\mathbf{t}^*, \tag{B3}$$

where $e_{(\mathbf{m})}$ is given by $e_{(\mathbf{m})} = I - e_{(\mathbf{s})} - e_{(\mathbf{t})}$. Hence, recalling that $\mathbf{s}\times\mathbf{t} = \mathscr{V}\mathbf{m}^*$ and $\mathscr{V}\mathbf{s}^*\times\mathbf{t}^* = \mathbf{m}$, with $\mathscr{V} = \mathbf{s}\times\mathbf{t}\cdot\mathbf{m}$, we have that

$$\mathbf{a}\times\mathbf{b} = \mathscr{V}(e_{(\mathbf{m})} - e_{(\mathbf{s})})(e_{(\mathbf{m})} - e_{(\mathbf{t})})(\mathbf{m}^*\cdot\mathbf{m}^*)\mathbf{m}. \tag{B4}$$

Hence, provided that $e_{(\mathbf{m})}$ is different from $e_{(\mathbf{s})}$ and $e_{(\mathbf{t})}$, the vector product $\mathbf{a}\times\mathbf{b}$ yields the direction of $\mathbf{m}$, the third right eigenvector of the tensor $\mathbf{P}$.

## Appendix C

### Result for $\mu\mathbf{m}$ when $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I$

When the unsheared pair $(\mathbf{s}, \mathbf{t})$ is such that $2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I$, we have shown in §8 that $\mu\mathbf{m}$ given by the vector product (6.9) satisfies $\mathbf{s}\times\mathbf{t}\cdot\mu\mathbf{m} = 0$. Here we show that either $\mu\mathbf{m}$ is along $\mathbf{s}$ or $\mu\mathbf{m} = \mathbf{0}$. The result that $\mu\mathbf{m}$ is along $\mathbf{s}$ occurs when $(\mathbf{s}, \mathbf{t})$ is a singular pair, in which case there is no unsheared triad $(\mathbf{s}, \mathbf{t}, \mathbf{m})$. For general deformations ($e_1 + e_3 \neq 2e_2$), the result $\mu\mathbf{m} = \mathbf{0}$ occurs when $\mathbf{t} = \mathbf{h}^+$ (or $\mathbf{h}^-$), in which case there is an infinity of genuine unsheared triads $(\mathbf{s}, \mathbf{t}, \mathbf{m})$. For special deformations ($e_1 + e_3 = 2e_2$), the result $\mu\mathbf{m} = \mathbf{0}$ occurs when $e_{(\mathbf{s})} = e_{(\mathbf{t})}$, in which case there is an infinity of unsheared triads, all coplanar.

Let $(\mathbf{s}, \mathbf{t})$ be an unsheared pair such that

$$2e_{(\mathbf{s})} + e_{(\mathbf{t})} = I. \tag{C1}$$

Then (6.9) reduces to

$$\mu\mathbf{m} = 2\{e_{(\mathbf{s})}\mathbf{s} - \mathbf{es}\}\times\{[e_{(\mathbf{s})} + e_{(\mathbf{t})}]\mathbf{t} - 2\mathbf{et}\}. \tag{C2}$$

But, recalling the condition (2.8) for unsheared pairs, we can easily see that both factors $e_{(\mathbf{s})}\mathbf{s} - \mathbf{es}$ and $[e_{(\mathbf{s})} + e_{(\mathbf{t})}]\mathbf{t} - 2\mathbf{et}$ of the vector product are orthogonal to $\mathbf{s}$. Hence, $\mu\mathbf{m}$ given by (C2) is along $\mathbf{s}$, unless $\mu\mathbf{m} = \mathbf{0}$.

We now seek to determine when $\mu\mathbf{m} = \mathbf{0}$. Let $\mathbf{n}$ be the unit normal to the plane $\Pi$ of $\mathbf{s}$ and $\mathbf{t}$, so that $\mathbf{s}\times\mathbf{t} = \sin\theta\,\mathbf{n}$. Because both factors of the vector product are in the plane orthogonal to $\mathbf{s}$, they may be written in the orthonormal basis $\mathbf{n}$, $\mathbf{n}\times\mathbf{s}$ of this plane. We so obtain

$$e_{(\mathbf{s})}\mathbf{s} - \mathbf{es} = \frac{1}{2\sin\theta}[e_{(\mathbf{s})} - e_{(\mathbf{t})}](\mathbf{s}\cdot\mathbf{t})\mathbf{n}\times\mathbf{s} - (\mathbf{n}\cdot\mathbf{es})\mathbf{n}, \tag{C3}$$

$$[e_{(\mathbf{s})} + e_{(\mathbf{t})}]\mathbf{t} - 2\mathbf{et} = \frac{1}{\sin\theta}[e_{(\mathbf{s})} - e_{(\mathbf{t})}]\mathbf{n}\times\mathbf{s} - 2(\mathbf{n}\cdot\mathbf{et})\mathbf{n}, \tag{C4}$$

on using $\sin\theta\,\mathbf{n}\times\mathbf{s} = \mathbf{t} - (\mathbf{s}\cdot\mathbf{t})\mathbf{s}$ and recalling (2.8). Introducing (C3) and (C4) into (C2), we obtain

$$\sin\theta\mu\mathbf{m} = 2[e_{(\mathbf{s})} - e_{(\mathbf{t})}]\{(\mathbf{s}\cdot\mathbf{t})(\mathbf{n}\cdot\mathbf{et}) - \mathbf{n}\cdot\mathbf{es}\}\mathbf{s} \tag{C5}$$

or, equivalently,

$$\mu\mathbf{m} = 2[e_{(\mathbf{s})} - e_{(\mathbf{t})}]\{(\mathbf{n}\times\mathbf{t})\cdot\mathbf{en}\}\mathbf{s}. \tag{C6}$$

We now consider separately the case of general deformations ($e_1+e_3\neq 2e_2$) and of special deformations ($e_1+e_3=2e_2$).

(a) General deformations: $e_1+e_3\neq 2e_2$
For general deformations, the assumption that $e_1+e_3\neq 2e_2$, together with (8.2), excludes the possibility that $e_{(\mathbf{s})}=e_{(\mathbf{t})}$. Thus, $e_{(\mathbf{s})}\neq e_{(\mathbf{t})}$, and the condition for $\mu\mathbf{m}$ to be zero is

$$(\mathbf{n}\times\mathbf{t})\cdot\mathbf{en}=0. \tag{C7}$$

Also, recalling (2.23), and noting that $e_{(\mathbf{n})}+\mathrm{e}_{(\mathbf{t})}+e_{(\mathbf{t}\times\mathbf{n})}=I$, we have

$$\mathbf{n}\cdot\mathbf{en}=(\mathbf{t}\times\mathbf{n})\cdot\mathbf{e}(\mathbf{t}\times\mathbf{n})=e_{(\mathbf{s})}. \tag{C8}$$

Because (C7) means that the orthogonal unit vectors $\mathbf{n}$ and $\mathbf{t}\times\mathbf{n}$ are conjugate with respect to the ellipsoid $\mathscr{E}$, these must be along the principal axes of an elliptical section of $\mathscr{E}$ by a central plane. Moreover, (C8) means that these principal axes have the same length. Hence, the condition for $\mu\mathbf{m}$ to be zero is that $\mathbf{n}$ and $\mathbf{t}\times\mathbf{n}$ lie in a plane of central circular section of the ellipsoid $\mathscr{E}$, thus that $\mathbf{t}=\mathbf{h}^+$ or $\mathbf{h}^-$.

Thus, for general deformations, $\mu\mathbf{m}=\mathbf{0}$ occurs when $\mathbf{t}=\mathbf{h}^+$ or $\mathbf{h}^-$. Otherwise $\mu\mathbf{m}$ is along $\mathbf{s}$.

(b) Special deformations: $e_1+e_3=2e_2$
For these deformations (C1) becomes $2e_{(\mathbf{s})}+e_{(\mathbf{t})}=3e_2$. When $e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_2$, thus when $\mathbf{s}$ and $\mathbf{t}$ are in the same plane of central circular section of the ellipsoid $\mathscr{E}$, (C6) yields $\mu\mathbf{m}=\mathbf{0}$. Also, when $(\mathbf{n}\times\mathbf{t})\cdot\mathbf{en}=0$, we conclude, as in the case of general deformations, that $\mathbf{t}=\mathbf{h}^+$ or $\mathbf{h}^-$. But now, $\mathbf{h}^+$, or $\mathbf{h}^-$, lies in the plane $\mathscr{C}^-$, or $\mathscr{C}^+$, respectively, so that $e_{(\mathbf{t})}=e_2$ and $e_{(\mathbf{n})}=e_{(\mathbf{t}\times\mathbf{n})}=e_{(\mathbf{s})}=e_2$. Hence, we retrieve $e_{(\mathbf{s})}=e_{(\mathbf{t})}=e_2$.

Thus, for special deformations, we conclude that $\mu\mathbf{m}=\mathbf{0}$ occurs when $e_{(\mathbf{s})}=e_{(\mathbf{t})}$. Otherwise $\mu\mathbf{m}$ is along $\mathbf{s}$.