MALGRANGE THEOREMS FOR INFINITE-DIMENSIONAL DIFFERENTIAL OPERATORS

By Henrik Petersson*
School of Mathematics and Systems Engineering,
Växjö University, Sweden

[Received 8 January 2001. Read 24 January 2002. Published 30 June 2003.]

Abstract

We prove Malgrange-type existence and approximation theorems for partial differential operators and spaces in the ring of formal power series $\mathfrak{A} \equiv \prod_{N^{(N)}} C$, $N^{(N)} \equiv \oplus_1^\infty N$, in an infinite number of variables. In particular we study spaces of entire functions within this framework. With the infinite-dimensional Fourier–Borel transform as a tool, we prove existence theorems for the spaces $\mathfrak{A}, \mathfrak{B}, A(X), \mathrm{Exp}(Y)$ and F. Here $\mathfrak{B} \equiv \oplus_{N^{(N)}} C$ is the space of finitely supported polynomials, $A(X)$ and $\mathrm{Exp}(Y)$ are spaces of entire (respectively exponential-type) functions and F is the Fischer–Fock (Hilbert) space. These spaces are related as follows: $\mathfrak{B} \subseteq \mathrm{Exp}(Y) \subseteq F \subseteq A(X) \subseteq \mathfrak{A}$, and can, pairwise, be considered as dual to one another. The key result for the existence theorem on F is a division theorem for the spaces $\mathrm{Exp}(Y), A(X)$ and F. Furthermore, we show that homogeneous solutions can be approximated by homogeneous solutions consisting of exponential finitely supported polynomials.

1. Introduction

We prove Malgrange-type existence and approximation theorems for partial differential operators (PDOs) and for spaces in the ring of formal power series $\mathfrak{A}(\equiv \prod_{N^{(N)}} C$, $N^{(N)} \equiv \oplus_1^\infty N)$ in an infinite number of variables. The theory of infinite-dimensional PDOs, and pseudo-differential operators, in infinite-dimensional domains has been developed by Aron [1], Boland [2], Dineen [3], Dwyer [6; 7; 8], Gupta [9], Lascar [13], Khrennikov [11; 12] and Smolyanov [18]. We prove the surjectivity of PDOs on spaces of entire functions, defined on sequence spaces, in $\mathfrak{A}$. Previous studies have shown that, in order to study infinite-dimensional holomorphy in $\mathfrak{A}$, we should work with nuclear (A-nuclear, fully nuclear) spaces with basis, because then we have monomial expansions (see [4]). However, we show that it is not the nuclearity of the domain space that is most important but rather the nuclearity (or semi-reflexivity) of the corresponding space of entire functions. We obtain a rich theory without requiring any topological structure on the domain space. We introduce a fundamental notion, ℓ_1-closedness, which is a non-topological extension of A-nuclearity and full nuclearity (see Section 2 and [5]). Every sequence space contains a largest ℓ_1-closed sequence space—the ℓ_1-closed core. We prove surjectivity of differential operators on the Fischer–Fock space $F \subseteq \mathfrak{A}$. F is a Hilbert space but not a subring of $\mathfrak{A}$. We show that F is closely related to the rings of entire functions, which we study by the division theorem (4.9). The Fischer–Fock space in n variables has been studied by Shapiro

*E-mail: Henrik.Petersson@msi.vxu.se

[16; 17] and in infinitely many variables by Dwyer [6]. The existence results in Theorem 4.6 reduce, in some cases, to reformulations of known results obtained by Boland [2]. However, we use different techniques, and our non-topological approach puts the results in a different perspective. In [2] it is shown that homogeneous solutions to PDOs can be approximated by homogeneous solutions consisting of exponential polynomials (see also [7]). In Section 5 we show that the (coarser) family of exponential finitely supported polynomials are dense in the kernels of differential operators.

This article is based on parts of the author's doctoral thesis [14].

2. Preliminaries

Given a countable index set A, a sequence space is a subspace $X \subseteq C^A$ such that $C^{(A)} \subseteq X \subseteq C^A$, where $C^{(A)}, C^A$ denote the direct sum and product respectively of the set C of complex numbers. Moreover, we tacitly assume that every sequence space X is completely Reinhardt (CR), i.e. $x_\delta \equiv (\delta_\alpha x_\alpha) \in X$ for all $x = (x_\alpha) \in X$ and $\delta = (\delta_\alpha) \in C^A$ with $|\delta_\alpha| \leq 1$, $\alpha \in A$.

The support of $a \in C^A$ is the set $\operatorname{supp} a \equiv \{\alpha \in A : a_\alpha \neq 0\}$. Hence, $C^{(A)}$ can be described as all sequences with finite support. A polydisc in a sequence space X is a set of the form $C_r = \{x = (x_\alpha) \in X : |x_\alpha| \leq r_\alpha, \ \forall r_\alpha\}$ where $r = (r_\alpha) \in X^+$. Here X^+ denotes the image of X by the map $x \mapsto x^+ \equiv (|x_\alpha|)$.

Let $\Delta(A)$ denote all sequences $\delta = (\delta_\alpha > 1) \in C^A$ such that $\delta^{-1} \equiv (\delta_\alpha^{-1}) \in \ell_1(A)$. We write $\Delta \equiv \Delta(N)$ and $\ell_1 \equiv \ell_1(N)$, where N denotes the set of natural numbers. A set $X \subseteq C^A$ is *ℓ_1-closed* if for every $x \in X$ there is a sequence $\delta \in \Delta(A)$ such that $x_\delta \in X$. We say that X is *uniformly ℓ_1-closed* if we can take the same $\delta \in \Delta(A)$ for all $x \in X$.

Definition 2.1. Let $P \subseteq C^A$. The *sequence polar set* of P is the vector space

$$P^s \equiv \left\{ a = (a_\alpha) \in C^A : \sum_{\alpha \in A} |p_\alpha||a_\alpha| < \infty, \forall p = (p_\alpha) \in P \right\}. \tag{1}$$

The *polydisc topology* on P^s is the topology generated by the seminorms $\|a\|_r \equiv \sum_A r_\alpha |a_\alpha|$, $r \in P^+$.

It is clear that $P^s = \hat{P}^s$, where $\hat{P}$ is the set of all sequences $x \in C^A$ such that $x^+ \leq p_1^+ + \ldots + p_n^+$, for some $p_j \in P$.

A sequence space X and its sequence polar X^s are naturally paired. The bilinear form for this pairing is denoted by $\langle \cdot, \cdot \rangle$.

In the literature, P^s and its topology are also known as Köthe spaces (see, for example, [4]). We tacitly assume that $X = P^s$ is endowed with the polydisc topology. It is easily verified that $X = P^s$ is complete. The notion of ℓ_1-closedness is related to nuclearity of Köthe spaces $X = P^s$ (see also [14]). It is known that $X = P^s$ $(P = \hat{P})$ is nuclear if and only if P is ℓ_1-closed (Grothendieck–Pietsch criterion). This property does not imply that X is ℓ_1-closed. However, $X = P^s$ is A-nuclear (see [5]) if and only if P is uniformly ℓ_1-closed, and in this case X is uniformly ℓ_1-closed. On the other hand, if X is ℓ_1-closed (or uniformly ℓ_1-closed), it is not in general true that $X = P^s$ for some uniformly ℓ_1-closed set P. A similar discussion concerning full nuclearity leads to: every fully nuclear space with basis is of the form P^s where P, P^s, P^{ss} are all ℓ_1-closed. To sum up, ℓ_1-closedness is some sort of non-topological extension of both full and A-nuclearity.

Theorem 2.1. *Let $X = P^s$ where P is an ℓ_1-closed collection. Then X is semi-reflexive.*

The linear hull of any union of CR ℓ_1-closed sequence spaces is again completely Reinhardt and ℓ_1-closed. Consequently, every sequence space contains a largest CR ℓ_1-closed sequence space—the ℓ_1-closed core (see [14] for details). If X is an ℓ_1-closed sequence space, then, by definition, there is a sequence $\delta^j \in \Delta(A)$ such that $x_{\delta^{(j)}} \in X$, $j = 1, \dots$, where $\delta^{(j)} \equiv \delta^1\delta^2 \dots \delta^j \equiv (\delta^1_\alpha \dots \delta^j_\alpha)$. In fact we have the following explicit description of the ℓ_1-closed core.

Theorem 2.2. *Let X be any sequence space. A sequence $x \in X$ belongs to the ℓ_1-closed core X_1 of X if and only if there is a sequence $\delta^j \in \Delta(A)$ such that $x_{\delta^{(j)}} \in X$, for all j.*

PROOF. Let X_1^* denote the space of sequences with the property described in the theorem. It is not difficult to prove that X_1^* is a CR ℓ_1-closed vector space contained in X. Hence $X_1^* \subseteq X_1$.

Assume now that $x \in X_1$. As X_1 is ℓ_1-closed, there is a sequence $\delta^1 \in \Delta(A)$ such that $x_{\delta^1} \in X_1 \subseteq X$. But, again, since $x_{\delta^1} \in X_1$, there is a sequence $\delta^2 \in \Delta(A)$ such that $(x_{\delta^1})_{\delta^2} = x_{\delta^{(2)}} \in X_1 \subseteq X$. We conclude that x belongs to X_1^* and, since x was arbitrary, $X_1 \subseteq X_1^*$. ∎

Example 2.1. The space S of complex-valued Schwarz functions on the real line is isomorphic to the following Köthe space:

$$S \simeq P^s \equiv \{((1+k)^m) : m = 1, \dots\}^s \quad (= \{x : ((1+k)^m x_k) \in \ell_1 \ \forall m\})$$

in C^N (see, for example, [15]). The collection P is uniformly ℓ_1-closed, and consequently S is ℓ_1-closed.

Example 2.2. For $r \geq 0$ let $\mathcal{O}(C_r)$ denote the space of all functions that are analytic in some neighbourhood of the closed disc $|z| \leq r$. We identify functions in $\mathcal{O}(C_r)$ with sequences in C^N through the Taylor expansion about the origin. Thus $\mathcal{O}(C_r)$ is the space of all sequences (f_n) such that $\sum |f_n| R^n < \infty$ for some $R > r$. Clearly, $\mathcal{O}(C_r)$ is ℓ_1-closed, and if $r < 1 \leq R$ we have

$$A(C) \subseteq \mathcal{O}(C_R) \subseteq S \subseteq \ell_1 \subseteq \ell_\infty \subseteq \mathcal{O}(C_r).$$

Here $A(C)$ denotes the (ℓ_1-closed) space of entire functions.

3. Formal power series spaces

This section is organised as follows. We introduce the spaces that we study, and show how they are related by using the infinite-dimensional Fourier–Borel transform.

Definition 3.1. The *ring of formal power series* is the sequence space $\mathfrak{A} = C^{N^{(N)}}$, where $N^{(N)} \equiv \oplus_1^\infty N$. The *space of finitely supported polynomials* is the subring $\mathfrak{B} \equiv C^{(N^{(N)})}$. We assume that $\mathfrak{A}$ and $\mathfrak{B}$ are endowed with the product and direct sum topologies respectively.

We identify the elements $e_\alpha = (\delta_\beta^\alpha)$, $\alpha \in N^{(N)}$, with the corresponding (formal) power x^α. Hence, every element $(f_\alpha) \in \mathfrak{A}$ may formally be written as $\sum_\alpha f_\alpha x^\alpha$.

Definition 3.2. The *Fischer–Fock space* is the Hilbert space F consisting of all $\phi \in \mathfrak{A}$ such that $\sum \alpha!|\phi_\alpha|^2 < \infty$, $\alpha! \equiv \alpha_1, \ldots$, endowed with the inner product $(\psi, \phi) \equiv \sum \alpha!\psi_\alpha\overline{\phi_\alpha}$. The norm on F is denoted by $\| \cdot \|$.

Note that F is not a subring of $\mathfrak{A}$, i.e. the product of two Fischer–Fock functions may not belong to F.

In the n-dimensional case, the space of entire functions can, by the map

$$f \mapsto (f^{(\alpha)}(0)/\alpha!), \tag{2}$$

be considered a subspace and subring of the space of formal power series in n variables $\mathfrak{A}_n$. The space of entire functions endowed with the compact open topology is the space $P^s \subseteq C^{N^n}$ where P consists of all powers $(r^\alpha)_{\alpha \in N^n}$, $r = (r_k \geq 0) \in R^n$. We shall study entire functions on sequence spaces $C^{(N)} \subseteq X \subseteq C^N$. Any space of Gâteaux holomorphic functions is embedded in $\mathfrak{A}$ by (2), where $f^{(\alpha)}(0)$ denotes the directional derivative along the 'unit basis' vectors $e_k = (\delta_j^k)$, corresponding to the multi-index $\alpha \in N^{(N)}$, at the origin. We refer to this fundamental observation by (FO). Note that the map defined by (FO) may not be one-to-one.

Let $\Theta = \Theta(X)$ denote the family of all polydiscs in X. The observation above suggests the following definition.

Definition 3.3. Let $X \subseteq C^N$ be a sequence space. The space of entire functions on X in $\mathfrak{A}$ is the space $A_\Theta \equiv P^s \subseteq \mathfrak{A}$, where $P = P_X \equiv \{(r^\alpha)_{\alpha \in N^{(N)}} : r \in X^+\}$.

The topology on A_Θ is thus the topology generated by the seminorms

$$\|f\|_r \equiv \sum_\alpha |f_\alpha| r^\alpha, \quad r \in X^+. \tag{3}$$

Lemma 3.1. *If $\delta \in \Delta$, then $(\delta^{-\alpha}) \in \ell_1(N^{(N)})$. Moreover, if $r \in \ell_1^+$, then $(r^\alpha/\alpha!) \in \ell_1(N^{(N)})$ and $\sum_\alpha r^\alpha/\alpha! = e^{\|r\|_1}$.*

Lemma 3.1 implies that if X is ℓ_1-closed then P, that defines A_Θ, is an ℓ_1-closed collection. Hence, by Theorem 2.1 we obtain the following.

Theorem 3.2. *Let X be an ℓ_1-closed sequence space. Then A_Θ is semi-reflexive.*

Since every metric semi-reflexive space is reflexive, A_Θ is reflexive if X is ℓ_1-closed and can be covered by a countable family of polydiscs.

Let X be a sequence space and $\Theta = \Theta(X)$. We denote by $A_\Theta(X)$ the space of all Gâteaux holomorphic functions f on X (i.e. $f_{x;\xi} \equiv f(x + (\cdot)\xi)$ is entire $\forall x, \xi \in X$) that are bounded on every polydisc $C_r \in \Theta$ (polydisc-bounded entire functions). We denote by $H_G(X)$ the space of Gâteaux holomorphic functions on a (complex) vector space X, and the symbols $D_\xi f(x) = \partial_\xi f(x) \equiv f'_{x;\xi}(0)$, $f \in H_G(X)$, shall be used for the directional derivative. We endow $A_\Theta(X)$ with the topology generated by the seminorms

$$\|f\|_{C_r} \equiv \sup_{x \in C_r} |f(x)|, \quad C_r \in \Theta. \tag{4}$$

It is easily verified that $A_\Theta(X)$ is complete.

Now, every element (f_α) in A_Θ defines a function in $A_\Theta(X)$ by $f(x) \equiv \sum f_\alpha x^\alpha$ where the series converges absolutely in $A_\Theta(X)$. On the other hand, every function $f \in A_\Theta(X)$ defines a sequence in $\mathfrak{A}$ by virtue of (FO), i.e. by $f \mapsto (f^{(\alpha)}(0)/\alpha!) \in \mathfrak{A}$.

Let X be a sequence space and let C_r be a polydisc in X. Let X_{C_r} denote the space $\bigcup_{n=1}^\infty nC_r$ endowed with the topology generated by the Minkowski functional for C_r. We denote by X_Θ the space X endowed with the finest topology (not necessarily locally convex) such that the embeddings $X_{C_r} \to X$, $C_r \in \Theta(X)$, are continuous. Every function $f \in A_\Theta(X)$ is continuous on X_Θ, i.e. continuous on every X_{C_r} (see [5, prop. 3.7] or [14, lemma 3.3]).

Theorem 3.3. *Let X be an ℓ_1-closed sequence space. Then the spaces A_Θ and $A_\Theta(X)$ are (topologically) isomorphic by (FO);*

$$A_\Theta(X) \overset{FO}{\simeq} A_\Theta.$$

Moreover, the power series $\sum_\alpha f_\alpha(\cdot)^\alpha$ converges pointwise to $f \in A_\Theta(X)$ for every x in the ℓ_1-closed core of X and absolutely in $A_\Theta(X)$ if X is ℓ_1-closed.

PROOF. Let $r \in X^+$ be arbitrary. For any given $(f_\alpha) \in A_\Theta$, we have $f(x) \equiv \sum f_\alpha x^\alpha \in A_\Theta(X)$ and

$$\|f\|_{C_r} = \sup_{x \in C_r} |\sum f_\alpha x^\alpha| \le \sum |f_\alpha r^\alpha| = \|f\|_r.$$

This proves continuity of the embedding $A_\Theta \to A_\Theta(X)$.

Next, assume that X is ℓ_1-closed. Let $f \in A_\Theta(X)$ be arbitrary and put $f_\alpha \equiv f^{(\alpha)}(0)/\alpha!$. Let $r \in X^+$ be arbitrary and choose $\delta \in \Delta$ such that $r_\delta \in X$. By the Cauchy formula we obtain the estimate $|f_\alpha r^\alpha| \le \delta^{-\alpha}\|f\|_{C_{r_\delta}}$ and hence $(f_\alpha) \in A_\Theta$ by

Lemma 3.1. In fact we have

$$\|f\|_r \leq \sum \delta^{-\alpha}\|f\|_{C_{r_\delta}} = \|\delta^{-1}\|_1\|f\|_{C_{r_\delta}}.$$

This proves that the map defined by (FO) maps $A_\Theta(X)$ into A_Θ continuously. It remains to prove that the map $A_\Theta(X) \to A_\Theta$ is one-to-one. This means that a function $f \in A_\Theta(X)$ is uniquely determined by its restriction to the direct sum $C^{(N)}$. We must prove that, if $f \in A_\Theta(X)$ and $f^{(\alpha)}(0) = 0$ for all α, then $f = 0$. Hence, it suffices to prove the last part of the theorem.

Let $x \in X$ be a sequence in the ℓ_1-closed core. Then there is a sequence $\delta \in \Delta$ such that $x_\delta \in X$. Let $r_k \equiv |x_k|$ and $r = (r_k)$. As X is completely Reinhardt, $r_\delta \in X$. From the first part of the proof, the power series for every $f \in A_\Theta(X)$ defines a function in $A_{\Theta_1}(X_1)$. Here X_1 and Θ_1 denote the ℓ_1-closed core of X and the collection of polydiscs in X_1 respectively. Thus, f and its power series are both continuous on X_B where $B = C_{r_\delta}$. By the assumption on δ it follows that $C^{(N)} \ni x^{(n)} \equiv \sum_{k=1}^n x_k e_k \to x$ in X_B. This proves pointwise convergence as f and its power series coincide on the direct sum.

Finally, assume that X is ℓ_1-closed and $f \in A_\Theta(X)$. By the first part of the proof, the power series for f converges absolutely to some function in $A_\Theta(X)$, and, from the above, it converges to f. ∎

As a common notation for the spaces A_Θ, $A_\Theta(X)$ we use $A(X)$. If X is an A-nuclear space (or fully nuclear), and thus ℓ_1-closed, then $A(X)$ is the 'usual' space of entire functions (compact open topology).

Definition 3.4. Let X be a sequence space. The space of exponential-type functions with respect to X in $\mathfrak{A}$ is the space $\mathrm{Exp}_\Theta \equiv \bigcup_{r\in X^+}\mathrm{Exp}_r$ where Exp_r is the space of all formal power series $(\varphi_\alpha) \in \mathfrak{A}$ such that, for some $M > 0$, $|\varphi_\alpha| \leq Mr^\alpha/\alpha!$, $\forall\alpha$. Each Exp_r is endowed with the (Banach) norm topology generated by the norm

$$\|\varphi\|_r \equiv \inf\{M : |\varphi_\alpha| \leq Mr^\alpha/\alpha!,\ \forall\alpha \in N^{(N)}\}, \tag{5}$$

and Exp_Θ with the corresponding inductive locally convex topology.

Let X, Y be given sequence spaces such that $Y \subseteq X^s$. We denote by $\mathrm{Exp}_\Theta(Y)$ the space of all $\varphi \in H_G(Y)$ such that

$$\|\varphi\|_{C_r} \equiv \sup_{y\in Y}|\varphi(y)|e^{-\|y\|_r} < \infty, \tag{6}$$

for some $r \in X^+$. We have $\mathrm{Exp}_\Theta(Y) = \bigcup_{r\in X^+}\mathrm{Exp}_r(Y)$, where $\mathrm{Exp}_r(Y)$ is the space of all functions $\varphi \in H_G(Y)$ that satisfies (6). We endow each $\mathrm{Exp}_r(Y)$ with the (Banach) norm topology generated by the norm (6) and $\mathrm{Exp}_\Theta(Y)$ with the corresponding inductive locally convex topology.

The space Y ($\subseteq X^s$) endowed with the polydisc topology is denoted by Y_Θ. Previous arguments show that every $\varphi \in \mathrm{Exp}_\Theta(Y)$ is continuous on Y_Θ.

Theorem 3.4. *Let X, Y be sequence spaces such that $Y \subseteq X^s$. Then Exp_Θ, $Exp_\Theta(Y)$ are (topologically) isomorphic by (FO);*

$$Exp_\Theta(Y) \overset{FO}{\simeq} Exp_\Theta.$$

Moreover, the power series $\sum \varphi_\alpha(\cdot)^\alpha$ converges pointwise to φ for every $\varphi \in Exp_\Theta(Y)$. If X is ℓ_1-closed, the series converges absolutely in $Exp_\Theta(Y)$.

PROOF. We prove that the embedding $(\varphi_\alpha) \mapsto \sum \varphi_\alpha y^\alpha \equiv \varphi(y)$ between Exp_Θ and $\mathrm{Exp}_\Theta(Y)$ is continuous. We must prove that the restriction to every Exp_r is continuous. Let $r \in X^+$ be arbitrary. By Lemma 3.1 we have

$$|\varphi(y)| \leq \sum |\varphi_\alpha||y^\alpha| \leq \|\varphi\|_r \sum |r^\alpha||y^\alpha|/\alpha! = \|\varphi\|_r e^{\|y\|_r}. \tag{7}$$

Hence, Exp_r is mapped into $\mathrm{Exp}_r(Y)$ continuously with $\|\varphi\|_{C_r} \leq \|(\varphi_\alpha)\|_r$. This proves the assertion.

Let $\varphi \in \mathrm{Exp}_r(Y)$ be arbitrary. Then $|\varphi(y)| \leq \|\varphi\|_{C_r} e^{\|y\|_{C_r}}$. The Cauchy formula gives the estimate $|\varphi^{(\alpha)}(0)| \leq \|\varphi\|_{C_r}(er)^\alpha$. Indeed, we can assume that $r_k \neq 0$ for all $k \in \operatorname{supp}\alpha$ because otherwise $\varphi^{(\alpha)}(0) = 0$. In view of this the Cauchy formula gives the desired estimate by integrating on the polycircle $|z_k| = \alpha_k/r_k$, $k \in \operatorname{supp}\alpha$. Hence, if $\varphi_\alpha \equiv \varphi^{(\alpha)}(0)/\alpha!$, then $(\varphi_\alpha) \in \mathrm{Exp}_{er}$ and $\|(\varphi_\alpha)\|_{er} \leq \|\varphi\|_{C_r}$. Hence, the map induced by (FO) maps $\mathrm{Exp}_r(Y)$ into Exp_{er} continuously, and $\mathrm{Exp}_\Theta(Y) \to \mathrm{Exp}_\Theta$ is continuous.

From the first part of the proof, the power series for every function in $\mathrm{Exp}_\Theta(Y)$ defines a function in $\mathrm{Exp}_\Theta(Y)$. Moreover, every function $\varphi \in \mathrm{Exp}_\Theta(Y)$ coincides with its power series on the direct sum $C^{(N)}$. Hence, pointwise convergence follows by the continuity of φ on Y_Θ, together with the observation that $C^{(N)} \ni y^{(n)} \equiv \sum_k^n y_k e_k \to y$ in Y_Θ. Assume that X is ℓ_1-closed and $\varphi \in \mathrm{Exp}_r(Y)$. If $\delta \in \Delta$ is chosen so that $r_\delta \in X$, then

$$|\varphi_\alpha y^\alpha| \leq \|\varphi\|_{C_r}(er)^\alpha|y^\alpha|/\alpha! \leq \|\varphi\|_{C_r}\delta^{-\alpha}(er_\delta)^\alpha|y^\alpha|/\alpha! \leq \|\varphi\|_{C_r}\delta^{-\alpha}e^{\|y\|_{er_\delta}}.$$

Hence, $\varphi_\alpha(\cdot)^\alpha$ belongs to $\mathrm{Exp}_{er_\delta}(Y)$ for all α with norm not greater than $\|\varphi\|_{C_r}\delta^{-\alpha}$. This shows absolute convergence and proves the theorem. ∎

We use the symbol $\mathrm{Exp}(Y)$ for the spaces $\mathrm{Exp}_\Theta(Y)$ and Exp_Θ.

In studying PDOs on spaces in $\mathfrak{A}$, it is interesting to characterise the image of the Fourier (Borel) transform $\mathscr{F}$ defined by $\mathscr{F} : \lambda \mapsto ((\mathscr{F}\lambda)_\alpha) \in \mathfrak{A}$, $\lambda \in \mathfrak{A}^*$,

$$(\mathscr{F}\lambda)_\alpha \equiv \langle \lambda, x^\alpha/\alpha! \rangle = \langle \lambda, e_\alpha/\alpha! \rangle. \tag{8}$$

Suppose that $A \subseteq \mathfrak{A}$ and that $B \subseteq \mathfrak{A}$ is the Fourier image of A', where A is endowed with a topology for which the elements e_α form a (weak) basis. Then we may put A and B into duality by the formula

$$\langle a, b \rangle = \sum \alpha! a_\alpha b_\alpha \quad (= \langle \lambda, a \rangle,\ \mathscr{F}\lambda = b). \tag{9}$$

This is useful, as the transpose of differential operators becomes multiplication (in $\mathfrak{A}$) by the corresponding symbol (see Section 3.1).

We now have the following Paley–Wiener–Schwartz-type theorem. In view of Theorems 3.3 and 3.4 the proof is quite elementary and we omit it.

Theorem 3.5. *Let X be an ℓ_1-closed sequence space. Then the Fourier transform $\mathscr{F}$ is an isomorphism between A'_Θ (strong topology) and Exp_Θ ($\simeq Exp_\Theta(Y)$, $Y \subseteq X^s$). The formula*

$$\langle f, \varphi \rangle = \sum \alpha! f_\alpha \varphi_\alpha, \quad f \in A(X), \ \varphi \in Exp(Y), \quad (= \langle \lambda, f \rangle, \quad \mathscr{F}\lambda = \varphi), \tag{10}$$

puts $A(X)$ and $Exp(Y)$ into duality such that $A'(X) = Exp(Y)$ and $Exp'(Y) = A(X)$. Moreover, if $\varphi \in Exp_r(Y)$, then

$$|\langle f, \varphi \rangle| \leq \|f\|_{er} \|\varphi\|_{C_r} \leq D_\delta \|f\|_{eC_{r_\delta}} \|\varphi\|_{C_r}, \quad D_\delta \equiv \sum \delta^{-\alpha}, \tag{11}$$

where $\delta \in \Delta$ and $r_\delta \in X$.

Remark 3.1. When we identify Exp_Θ with $\mathrm{Exp}_\Theta(Y)$, the Fourier transform is given by $\lambda \mapsto \langle \lambda, e_y \rangle$. Here $e_y \in A_\Theta(X)$ is the kernel defined by $e_y(x) \equiv e^{\langle x,y \rangle}$. We refer to this map as the Fourier transform between $A'(X)$ and $\mathrm{Exp}_\Theta(Y)$ and denote it by $\mathscr{F}$. The dual operator $\mathscr{F}' : \mathrm{Exp}'(Y) \to A''(X) = A(X)$ is given by $\mathscr{F}'\mu(x) = \langle \mu, e_x \rangle$ where $e_x = e^{\langle x,\cdot \rangle} \in \mathrm{Exp}_\Theta(Y)$.

Consider the spaces $\mathfrak{A} \equiv C^{N^{(N)}}$ and $\mathfrak{B} \equiv C^{(N^{(N)})}$. As $N^{(N)}$ is countable, $\mathfrak{A}$ is a reflexive Fréchet space with respect to the product topology. Moreover, $\mathfrak{B}$ endowed with the direct sum topology is the strong dual of $\mathfrak{A}$ with duality (9). Hence we may put $\mathfrak{A}$ and $\mathfrak{B}$ into duality by the bracket (9), and we have

$$\mathfrak{A}' = \mathfrak{B}, \quad \mathfrak{B}' = \mathfrak{B}^* = \mathfrak{A}. \tag{12}$$

Remark 3.2. (12) is the analogue of Theorem 3.5 for the spaces $\mathfrak{A}$ and $\mathfrak{B}$. Further, $\mathscr{F}F = \mathscr{F}F' = F$ because this is precisely $F = F'$ in the Hilbert space sense.

If $X \subseteq \ell_2$, we have a natural continuous embedding $i : \mathrm{Exp}(Y) \to F$, $\sum \varphi_\alpha y^\alpha \mapsto (\varphi_\alpha)$ with dense image (in some literature such a continuously and densely embedded space is called a rigged Hilbert space). Thus $\mathrm{Exp}(Y)$ is a subring of $\mathfrak{A}$ densely embedded in F. The transpose $j = {}^t i : F \to A(X)$ is the injection $j\phi = \sum \phi_\alpha x^\alpha$. In the present situation, we are dealing with the following sequence of injections

$$\mathfrak{B} \to \mathrm{Exp}(Y) \to F \to A(X) \to \mathfrak{A}, \quad X \subseteq \ell_2. \tag{13}$$

3.1. Differential operators

We introduce differential operators on the spaces $\mathfrak{A}, \mathfrak{B}, A(X), \mathrm{Exp}(Y)$ and F. A more general discussion, concerning differential operators on spaces in the ring $\mathfrak{A}$, can be found in [14].

Assume that $A \subseteq \mathfrak{A}$ is a subring endowed with a topology for which the elements $e_\alpha = x^\alpha$, $\alpha \in N^{(N)}$, form a weak basis ($\sigma(A', A)$) and for which the multiplication operator $c \mapsto ac$, $c \in A$ is continuous for every $a \in A$. As $(e_\alpha)_\alpha$ is a basis, the Fourier transform on A' is one-to-one. Assume that $B \equiv \mathscr{F}A'$ is the Fourier image and put A and B into duality by (9). Let $a \in A$ and define $a(D)$ by $\mathscr{F}a'\mathscr{F}^{-1}$, i.e. $a(D) = {}^t a$, where a' and ${}^t a$ denote the adjoint and the transpose (duality between A and B) respectively of the multiplication operator a. Hence $a(D)$ is, by definition, continuous for the topology $\sigma(B, A)$ with ${}^t a(D) = a$. It is also easy to prove that $a(D)$ is given by $a(D)b = \sum_\alpha a_\alpha D^\alpha b$ where the series converges with respect to $\sigma(B, A)$. Here $D^\alpha = \partial^\alpha$, $\alpha \in N^{(N)}$, denotes the operator on $\mathfrak{A}$, corresponding to the directional derivative along the vectors e_k. In the other direction, we can define differential operators on A with symbols in B. The set $(e_\alpha)_\alpha$ is a basis for $(B, \sigma(B, A))$. Moreover, the Fourier transform image of $B' = A$ for this topology is A. Hence, if B is endowed with any dual topology (i.e. a topology for which the dual is A) and B is continuously closed under multiplication, we can define $b(D) : A \to A$, $b \in B$, in the same way.

Multiplication $b \mapsto ab$, $a, b \in \mathfrak{A}$, is easily seen to be continuous in the ring $\mathfrak{A}$, and thus the operator $a(D) : \mathfrak{B} \to \mathfrak{B}$, defined by $a(D) \equiv {}^t a = \mathscr{F}a'\mathscr{F}^{-1}$, is given by

$$a(D)b = \sum a_\alpha D^\alpha b \tag{14}$$

with weak convergence. Moreover, it is weakly continuous for the pairing (9) with transpose ${}^t a(D) = a$. In the same way, $\mathfrak{B} = \mathscr{F}\mathfrak{A}'$ is a subring of $\mathfrak{A}$ and continuously closed under multiplication. Thus, if $b \in \mathfrak{B}$, we may define $b(D) \equiv {}^t b : \mathfrak{A} \to \mathfrak{A}$ so that ${}^t b(D) = b$ and $b(D)a$ is given by (14) with the roles of a and b interchanged. To sum up, we have defined

$$a(D) \equiv {}^t a : \mathfrak{B} \to \mathfrak{B}, \quad b(D) \equiv {}^t b : \mathfrak{A} \to \mathfrak{A}, \quad a \in \mathfrak{A},\ b \in \mathfrak{B}.$$

Next, assume that X is ℓ_1-closed and that $Y \subseteq X^s$. It is easily checked that both $A(X)$ and $\mathrm{Exp}(Y)$ are continuously closed under multiplication by functions in the respective space. If $f \in A(X)$ and $\varphi \in \mathrm{Exp}(Y)$, we define

$$f(D) \equiv {}^t f : \mathrm{Exp}(Y) \to \mathrm{Exp}(Y), \quad \varphi(D) \equiv {}^t \varphi : A(X) \to A(X).$$

From the discussion above, $f(D)\varphi = \sum f_\alpha D^\alpha \varphi$ and $\varphi(D)f = \sum \varphi_\alpha D^\alpha f$ with weak convergence. In particular this means pointwise convergence for

$$f(D)\varphi(y) = \langle e_y, f(D)\varphi\rangle = \sum f_\alpha \langle e_y, D^\alpha \varphi\rangle = \sum f_\alpha D^\alpha \varphi(y)$$

and similarly for $\varphi(D)f$. This can be strengthened as follows.

Theorem 3.6. *Let X, Y be sequence spaces where X is ℓ_1-closed and $Y \subseteq X^s$. If $f = \sum f_\alpha (\cdot)^\alpha \in A(X)$ and $\varphi = \sum \varphi_\alpha (\cdot)^\alpha \in Exp(Y)$ then*

$$f(D)\varphi = \sum f_\alpha D^\alpha \varphi, \quad \varphi(D)f = \sum \varphi_\alpha D^\alpha f,$$

with absolute convergence in Exp(Y) and A(X) respectively. Moreover, the operators f(D) and φ(D) are both continuous.

Proof. That the series converges pointwise is already established. Next assume that $\varphi \in \mathrm{Exp}_r(Y)$, $|\varphi(y)| \leq \|\varphi\|_{C_r} e^{\|y\|_r}$. Then

$$|D^\alpha \varphi(y)| \leq \|\varphi\|_{C_r}(er)^\alpha e^{\|y\|_r}$$

(see the proof of Theorem 3.4). Hence $f_\alpha D^\alpha \varphi \in \mathrm{Exp}_r(Y)$ for all α with norm not greater than $\|\varphi\|_{C_r}|f_\alpha|(er)^\alpha$. This gives absolute convergence and continuity of the map $f(D)$ on $\mathrm{Exp}(Y)$.

Next assume that $\varphi \in \mathrm{Exp}_r(Y)$, $r \in X^+$ and hence that $|\varphi_\alpha| \leq \|\varphi\|_{C_r}(er)^\alpha/\alpha!$ (Theorem 3.4). Let $R \in X^+$ and choose $\delta \in \Delta$ so that $r_\delta \in X$. If $R' = er_\delta + R$, the Cauchy estimates imply that

$$(er)^\alpha |D^\alpha f(x)| \leq \frac{\alpha!}{\delta^\alpha}\|f\|_{C_{R'}}$$

for all $x \in C_R$. Hence, $\|\varphi_\alpha D^\alpha f\|_{C_R} \leq \|\varphi\|_{C_r}\delta^{-\alpha}\|f\|_{C_{R'}}$ for all α. By Lemma 3.1 this shows that the series for $\varphi(D)f$ converges absolutely and the map $\varphi(D)$ is continuous on $A(X)$. ■

Next we consider operators on F. Let $\varphi \in \mathrm{Exp}(Y)$ be any symbol and assume that $X \subseteq \ell_2$. Then $f \mapsto \varphi f$, $f \in D$, where $D \equiv \{f : \varphi f \in F\}$ is a densely defined operator on F. We define

$$\varphi(D) \equiv {}^t\varphi : F \supseteq \mathscr{D} \to F, \quad \varphi \in \mathrm{Exp}(Y),$$

where ${}^t\varphi$ is the transpose of φ for the duality between F and itself with respect to the bilinear form (9).

Theorem 3.7. *Let X, Y be sequence spaces where $X \subseteq \ell_2$ is ℓ_1-closed and $Y \subseteq X^s$. Let $\varphi \in Exp(Y)$ be any symbol. Then $\varphi : F \supseteq D \to F$ is a densely defined closed operator on F. Thus $\varphi(D)$ is a densely defined closed operator on F with adjoint $\varphi(D)^* = \varphi^*$ where φ^* is obtained from φ by conjugating its coefficients. Further, $\varphi(D) = \bar{\varphi}(D) \subseteq \hat{\varphi}(D)$ where $\bar{\varphi}(D)$ denotes the closure in F of $\varphi(D) : \mathfrak{B} \to \mathfrak{B}$, and $\hat{\varphi}(D) : F \supseteq \hat{D} \to F$, $\hat{D} \equiv \{f \in F : \varphi(D)f \in F\}$, thus defined, is closed.*

Proof. We prove that φ is weakly closed. Assume that $D \ni f^\gamma \to f$ and that $\varphi f^\gamma \equiv g^\gamma \to g$ in $(F, \sigma(F,F))$. Since $g^\gamma \to g$ weakly, $g^\gamma_\alpha \to g_\alpha$ for every α. By the $(\mathfrak{A}, \mathfrak{B})$ duality we obtain that

$$\begin{aligned}\alpha!(\varphi f)_\alpha &= \langle \varphi f, (\cdot)^\alpha \rangle = \langle f, \varphi(D)(\cdot)^\alpha \rangle \\ &= \lim \langle f^\gamma, \varphi(D)(\cdot)^\alpha \rangle = \lim \langle \varphi f^\gamma, (\cdot)^\alpha \rangle = \lim \alpha! g^\gamma_\alpha = \alpha! g_\alpha.\end{aligned}$$

Thus $\varphi f = g$, which proves the assertion.

As φ is closed, we conclude that ${}^t\varphi(D) = \varphi$, which gives the statement concerning the adjoint.

Next we prove that $\varphi(D) = {}^t\varphi$ is an extension of $\bar{\varphi}(D)$ (it is easily checked that $\varphi(D) : \mathfrak{B} \to \mathfrak{B}$ is closable). Assume that $f \in \bar{D}$ where $\bar{D}$ is the domain of definition of $\bar{\varphi}(D)$. This means that there is a net $f^\gamma \in \mathfrak{B}$ such that $f^\gamma \to f$ and $\varphi(D)f^\gamma \to \bar{\varphi}(D)f = g$ in F for the weak topology $\sigma(F, F)$. Hence, the duality between $\mathfrak{A}$ and $\mathfrak{B}$ gives

$$\langle \varphi h, f \rangle = \lim \langle \varphi h, f^\gamma \rangle = \lim \langle h, \varphi(D) f^\gamma \rangle = \langle h, g \rangle$$

for all $h \in D$. This proves the statement.

We prove that ${}^t\varphi \subseteq \bar{\varphi}(D)$ or, equivalently, that φ is an extension of ${}^t\bar{\varphi}(D)$ (since $\bar{\varphi}(D)$ is weakly closed). Assume that f belongs to the domain of definition of ${}^t\bar{\varphi}(D)$. This means that there is a $g = {}^t\bar{\varphi}(D)f$ such that $\langle f, \bar{\varphi}(D)h \rangle = \langle g, h \rangle$ for all $h \in \bar{D}$. Since the monomials belong to $\mathfrak{B} \subseteq \bar{D}$, the duality between $\mathfrak{A}$ and $\mathfrak{B}$ implies that

$$\alpha! g_\alpha = \langle g, (\cdot)^\alpha \rangle = \langle f, \varphi(D)(\cdot)^\alpha \rangle = \langle \varphi f, (\cdot)^\alpha \rangle = \alpha! (\varphi f)_\alpha.$$

Thus $\varphi f = g$. Hence $f \in D$ and φ is an extension of ${}^t\bar{\varphi}(D)$.

From the proof that φ is closed we deduce that $\hat{\varphi}(D)$ is weakly closed.

Finally we prove that $\hat{\varphi}(D)$ is an extension of $\varphi(D) = {}^t\varphi$. Assume that f belongs to the domain of ${}^t\varphi$. This means that there is a $g = {}^t\varphi f$ such that $\langle f, \varphi h \rangle = \langle g, h \rangle$ for all $h \in D$. Since the monomials belong to D, we obtain, by the duality between $A(X)$ and $\mathrm{Exp}(Y)$, that

$$a! g_\alpha = \langle g, (\cdot)^\alpha \rangle = \langle f, \varphi(\cdot)^\alpha \rangle = \langle \varphi(D) f, (\cdot)^\alpha \rangle = \alpha! (\varphi(D) f)_\alpha.$$

Hence $f \in \hat{D}$ and $g = \hat{\varphi}(D)f$. ■

4. Existence theorems

The 'finite-dimensional' version of the following theorem is proved by Treves in [20, p. 29] (see also [19, p. 96]).

Theorem 4.1. *$a(D)\mathfrak{B} = \mathfrak{B}$ and $b(D)\mathfrak{A} = \mathfrak{A}$ for every non-zero $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$.*

PROOF. A linear operator is surjective if and only if its algebraic transpose is injective. Hence, to prove that $a(D)\mathfrak{B} = \mathfrak{B}$ we show that $a(D)^*$ is one-to-one. Since ${}^t a(D) = a$ and $\mathfrak{B}^* = \mathfrak{A}$, we have $a(D)^* = a$. The multiplication operator a is obviously injective. This establishes the first part of the theorem.

Next we prove that $b(D)$ is surjective. $\mathfrak{A}$ is a Fréchet space with strong dual $\mathfrak{B}$. Hence, it suffices to prove that the transpose b is one-to-one and has closed image. The one-to-one property is clear. By (12), $\mathfrak{B}^* = \mathfrak{A} = \mathfrak{B}'$ ($\mathfrak{B} \simeq C^{(N)}$), and hence every linear form on $\mathfrak{B}$ is continuous. This implies that every subspace of $\mathfrak{B}$ is closed. In particular, the image of b is closed. ■

Surjectivity on $A(X)$ and $\mathrm{Exp}(Y)$

For $S \subseteq N$ and $\alpha \in N^{(N)}$ let $|\alpha|_S \equiv \sum_{k \in S} \alpha_k$ and $|\alpha|_N = |\alpha|$. A polynomial in $\mathfrak{A}$ is an element $a \in \mathfrak{A}$ such that $\sup_{a_\alpha \neq 0} |\alpha| < \infty$ (the degree of a). Similarly, a is a polynomial with respect to $S \subseteq N$ when $\sup_{a_\alpha \neq 0} |\alpha|_S < \infty$ (the degree of a with respect to S).

For a sequence space X and $S \subseteq N$, let $X_S \equiv \{x \in X : \operatorname{supp} x \subseteq S\}$ and $N_S \equiv \{\alpha \in N^{(N)} : \operatorname{supp} \alpha \subseteq S\}$. We have a natural decomposition $X = X_S \oplus X_{S'}$ and (projection) map $X \ni x \mapsto x_S \in X_S$, where $S' \subseteq N$ is the complement of S. We write $(x_S, x_{S'}) \equiv x_S + x_{S'}$.

If X is ℓ_1-closed, then $P \in A_\Theta(X)$ (and similarly for functions in $\mathrm{Exp}_\Theta(Y)$) can be written uniquely in the form

$$P(x) = \sum_{\beta \in N_S} u_\beta(x) x_S^\beta, \tag{15}$$

where each $u_\beta \in A_\Theta(X)$ does not depend on the variables x_k, $k \in S$. It is clear that P is a polynomial with respect to S if and only if $\sup_{u_\beta \neq 0} |\beta| = \sup_{u_\beta \neq 0} |\beta|_S < \infty$, and this finite integer gives the degree. The principal part (with respect to S) is defined by

$$P_S(x) \equiv \sum_{\substack{|\beta|=m \\ \beta \in N_S}} u_\beta(x) x_S^\beta,$$

where m is the degree of P with respect to S. We say that P is a proper polynomial with respect to S when the functions u_β, $|\beta| = m$ are constants, that is, the principal part depends only on the variables x_k, $k \in S$.

In the finite-dimensional case we have the following (compare with [20, lemma 8.1]).

Lemma 4.2. *Let* $P \in A(C^n)$, $P(z) = \sum_{\alpha \in N^n} a_\alpha z^\alpha$, *be a polynomial of degree* m *with respect to* $S \subseteq \{1, \ldots, n\}$. *For arbitrary* $\varepsilon > 0$ *and* $c \in C_S^n$ *we have*

$$|P_S(c, z_{S'}) h(z)| \leq \varepsilon^{-m} \sup_{\zeta \in C,\ |\zeta| \leq \varepsilon} |P(\zeta c + z) h(\zeta c + z)|, \tag{16}$$

for all $z \in C^n$ *and every entire function* $h \in A(C^n)$.

Proof. Assume first that P is a polynomial in one complex variable, $P(z) = Az^m + \tilde{P}(z)$, where $\tilde{P}$ is of degree less than m and $A \neq 0$. Consider the polynomial $Q(z) \equiv z^m P^*(1/z)$, where P^* is obtained from P by conjugating its coefficients. If $|z| = r$, we have that $Q(z) = z^m P^*(\bar{z}/|z|^2) = z^m \overline{P_r(z)}$ where $P_r(z) \equiv P(z/r^2)$. By the maximum modulus theorem, we obtain for any entire function h that

$$\begin{aligned} |Ah(0)| &= |Q(0) h_r(0)| \\ &\leq \sup_{|z|=r} |Q(z) h_r(z)| = r^m \sup_{|z|=r} |P_r(z) h_r(z)| = r^m \sup_{|z|=1/r} |P(z) h(z)|. \end{aligned} \tag{17}$$

Here $h_r(z) \equiv h(z/r^2)$.

Now assume that $P \in A(C^n)$ is an entire function of the form $P(z_1, \ldots, z_n) = A(z_2, \ldots, z_n) z_1^m + \tilde{P}(z_1, \ldots, z_n)$ where $\tilde{P}$ is of degree less than m in the variable z_1 and $A \neq 0$ is an entire function. Let $z_1, \ldots, z_n$ be fixed complex numbers. Consider the polynomial in one variable defined by $P_{z_1,\dots,z_n}(z) \equiv P(z+z_1, z_2, \dots, z_n)$. It follows that $P_{z_1,\dots,z_n}$ is of the form $Az^m + \tilde{P}_1$ where $\tilde{P}_1$ is of degree less than m in z. By (17)

$$\begin{aligned}|Ah(z_1,\dots,z_n)| &= |Ah_{z_1,\dots,z_n}(0)| \\ &\le r^m \sup_{|z'|=1/r} |P_{z_1,\dots,z_n}(z')h_{z_1,\dots,z_n}(z')| \\ &\le r^m \sup_{|z_1'|=1/r} |P(z_1+z_1',\dots,z_n)h(z_1+z_1',\dots,z_n)|. \qquad (18)\end{aligned}$$

Here $h_{z_1,\dots,z_n}$ is defined in the same way as $P_{z_1,\dots,z_n}$.

Finally, let P be an arbitrary polynomial with respect to S. We may suppose that $S = \{1,\dots,k\}$ for some $k \le n$. Moreover, we can assume that $(C^n_S \ni)\ c = (c_1,\dots,c_k,0,\dots,0) \ne 0$ and, for simplicity, that $c_1 \ne 0$. Consider the change of variables $\zeta = Lz$, described by $\zeta_1 = c_1 z_1$, $\zeta_j = c_j z_1 - z_j, j = 2,\dots,k$, $\zeta_j = z_j, j = k+1,\dots,n$. Since $c_1 \ne 0$, the transformation L is bijective and the composition $P_L \equiv PL$ has the form $P_L(z) = P_S(c, z_{S'})z_1^m + \tilde{P}(z)$, where $\tilde{P}$ is of degree less than m in the variable z_1. Moreover, $(L^{-1}z)_{S'} = z_{S'}$. Hence by (18) we obtain that

$$\begin{aligned}|P_S(c,z_{S'})h(z)| &= |P_S(c,(L^{-1}z)_{S'})h_L(L^{-1}z)| \\ &\le \varepsilon^{-m} \sup_{\substack{|\zeta'|\le\varepsilon \\ \zeta=(\zeta',0,\dots,0)}} |P_L(\zeta+L^{-1}z)h_L(\zeta+L^{-1}z)| \\ &\le \varepsilon^{-m} \sup_{\substack{|\zeta'|\le\varepsilon \\ \zeta=(\zeta',0,\dots,0)}} |P(L\zeta+z)h(L\zeta+z)| \\ &\le \varepsilon^{-m} \sup_{\substack{|\zeta|\le\varepsilon \\ \zeta\in C}} |P(\zeta c+z)h(\zeta c+z)|. \qquad \blacksquare\end{aligned}$$

Our next objective is to generalise Lemma 4.2 to functions in $A_\Theta(X)$. If $Y \subseteq X^s$, a (Y-) cylinder function is a function $h \in A_\Theta(X)$ of the form $h = u \circ \pi_F$, $F = (y_1,\dots,y_n) \in Y^n$, where $u \in A(C^n)$ and π_F is the projector $\pi_F : X \to C^n$ defined by $\pi_F(x) \equiv (\langle x, y_k\rangle)$. If X is ℓ_1-closed, Theorem 3.3 shows that the space of (Y-) cylinder functions $A_Y(X)$ is a dense subspace of $A_\Theta(X)$. If $f \in H_G(X)$, $f^{[n]}$ is the entire function in n complex variables defined by $f(z) \equiv f(\sum_1^n z_k e_k)$. Moreover, we write $x^{[n]} = (x_1,\dots,x_n) \in C^n$ and $x^{(n)} = \sum_1^n x_k e_k \in C^{(N)}$ for $x = (x_k) \in X$. Lemma 4.2 can be generalised in the following way.

Lemma 4.3. *Let X be an ℓ_1-closed sequence space. Let $P \in A_\Theta(X)$, $P(x) = \sum_{\beta\in N_S} u_\beta(x)x_S^\beta$, be a polynomial of degree m with respect to S. For arbitrary $\varepsilon > 0$ and $c \in X_S$ we have that*

$$|P_S(c,x_{S'})f(x)| \le \varepsilon^{-m} \sup_{|\xi|\le\varepsilon,\ \xi\in C} |P(x+\xi c)f(x+\xi c)|, \qquad (19)$$

for all $x \in X$ and every entire function $f \in A_\Theta(X)$.

PROOF. By our assumption, $P^{[n]} \in A(C^n)$ is a polynomial with respect to $S^{[n]} \equiv S \cap \{1,\dots,n\}$ for every n. We first prove that the degree of P with respect to S is equal to the degree of $P^{[n]}$ with respect to $S^{[n]}$ for n sufficiently large. It follows that $u_\beta^{[n]}$, $l(\beta) \leq n$, are the 'coefficient functions' for $P^{[n]}$ in the expansion (15). Hence the degree of $P^{[n]}$ is less than or equal to the degree of P. If $u_{\beta_0}(x_0) \neq 0$, then, by continuity of the functions in $A_\Theta(X)$ on X_Θ, $u_{\beta_0}^{[n]}(x_0^{[n]}) \neq 0$ for all n sufficiently large. It follows that the degree of P is less than or equal to the degree of $P^{[n]}$ for all large n. We also have $(P_S)^{[n]} = (P^{[n]})_{S^{[n]}}$ for the principal parts. For arbitrary $\varepsilon > 0$ and $c \in X_S$, Lemma 4.2 shows that for all n sufficiently large and $x \in C_r$

$$\begin{aligned}|(P_S)^{[n]}(c^{[n]}, x_{S'}^{[n]})f^{[n]}(x^{[n]})| &\leq |(P^{[n]})_{S^{[n]}}(c^{[n]}, (x^{[n]})_{S^{[n]'}})f^{[n]}(x^{[n]})| \\ &\leq \varepsilon^{-m} \sup_{|\xi|\leq\varepsilon, \xi\in C} |P^{[n]}(\xi c^{[n]} + x^{[n]})f^{[n]}(\xi c^{[n]} + x^{[n]})| \\ &\leq \varepsilon^{-m} \sup_{|\xi|\leq\varepsilon, \xi\in C} |P(\xi c^{(n)} + x^{(n)})f(\xi c^{(n)} + x^{(n)})|. \qquad (20)\end{aligned}$$

Since $P_S, f \in A_\Theta(X)$, and functions in $A_\Theta(X)$ are continuous on X_Θ, the left-hand side of (20) tends to $P_S(c, x_{S'})f(x)$ as n tends to infinity. Thus we are finished if we can prove that for every $g \in A_\Theta(X)$ and $\delta > 0$ there exists $n_\delta \in N$ such that

$$\sup_{|\xi|\leq\varepsilon} |g(x^{(n)} + \xi c^{(n)})| \leq \delta + \sup_{|\xi|\leq\varepsilon} |g(x + \xi c)|, \qquad (21)$$

for all $n \geq n_\delta$ (put $g = Pf$). Note that for any $h \in A_Y(X)$, $Y = C^{(N)}$, there is an integer $n_\delta \in N$ such that $\sup_{|\xi|\leq\varepsilon} |h(x + \xi c) - h(x^{(n)} + \xi c^{(n)})| \leq \delta$ for $n \geq n_\delta$. Indeed, h is of the form $h = u \circ \pi_F$ where u is entire and $\pi_F : X \to C^m$ is a projector. Put $r_k \equiv |x_k| + \varepsilon|c_k|$ and $r = (r_k) \in X^+$. Then the set $\pi_F(C_r)$ is contained in some compact set in C^m. As u is uniformly continuous on compact sets, the assertion now follows from the fact that π_F is weakly continuous and $x^{(n)} \to x, c^{(n)} \to c$ weakly $(\sigma(X, Y))$. Next, as $A_Y(X)$ is dense in $A_\Theta(X)$, we can find $g_\delta \in A_Y(X)$ such that $\|g_\delta - g\|_{C_r} \leq \delta/3$. Choose $n_\delta \in N$ such that $\sup_{|\xi|\leq\varepsilon} |g_\delta(x+\xi c) - g_\delta(x^{(n)} + \xi c^{(n)})| \leq \delta/3$ for all $n \geq n_\delta$. If $n \geq n_\delta$ and $|\xi| \leq \varepsilon$, we obtain that

$$\begin{aligned}|g(x^{(n)} + \xi c^{(n)})| &\leq |g(x^{(n)} + \xi c^{(n)}) - g_\delta(x^{(n)} + \xi c^{(n)})| + |g_\delta(x^{(n)} + \xi c^{(n)}) \\ &\quad - g_\delta(x + \xi c)| + |g_\delta(x + \xi c) - g(x + \xi c)| + |g(x + \xi c)| \\ &\leq \delta + |g(x + \xi c)|.\end{aligned}$$

Taking the supremum over $\xi \leq \varepsilon$ yields (21). ∎

Theorem 4.4. *Let X, Y be sequence spaces where X is ℓ_1-closed and $Y \subseteq X^s$. Let $P(D) : Exp(Y) \to Exp(Y)$ be a differential operator with symbol $P \in A(X)$ where P is a proper polynomial with respect to some $S \subseteq N$. Then $P(D) : Exp(Y) \to Exp(Y)$ is a surjection.*

Proof. Let $\varphi_0 = \mathscr{F}\lambda_0 \in \mathrm{Exp}_\Theta(Y)$, $\lambda_0 \in A'_\Theta(X)$ be arbitrary. By (19) and the assumption that P is a proper polynomial with respect to S, it follows that $P : A_\Theta(X) \to A_\Theta(X)$ is one-to-one with continuous inverse $P^{-1} : \mathrm{Im}P \to A_\Theta(X)$. Indeed, by an appropriate choice of c, $P_S(c) \neq 0$, and (19) gives, for any $r \in X^+$,

$|P_S(c)|\|f\|_{C_r} \le \varepsilon^{-m}\|Pf\|_{C_{r+\varepsilon c^+}}$ for all $f \in A_\Theta(X)$ where $\varepsilon > 0$ is arbitrary. Consequently, $\lambda_P \equiv \lambda_0 \circ P^{-1} \in (\mathrm{Im} P)'$. By the Hahn–Banach theorem, λ_P has a continuous extension $\lambda \in A'_\Theta(X)$. With $\varphi \equiv \mathscr{F}\lambda \in \mathrm{Exp}_\Theta(Y)$, we obtain that $P(D)\varphi(y) = \langle \lambda, Pe_y\rangle = \varphi_0(y)$. This proves that $P(D)$ is a surjection. ■

The following fundamental theorem is proved in [20, theorem B, p. 22].

Theorem 4.5. *Let E, F be locally convex Hausdorff spaces, with topological duals E', F', and let (E_0, E_0'), (F_0, F_0') be any (non-degenerate) paired spaces. Assume that $u : E \to F$, $u_0 : E_0 \to F_0$, $a : E_0 \to E$, $b : F_0 \to F$ are weakly continuous, that a, b have dense images and that $b \circ u_0 = u \circ a$. If the following hold true:*

1. *${}^t u_0(y_0) \in {}^t a(E')$, $y_0 \in F_0' \Rightarrow y_0 \in {}^t b(F')$, that is, ${}^t u_0{}^{-1}({}^t a(E')) \subseteq {}^t b(F')$,*
2. *$\ker {}^t u_0 = \{0\}$ and ${}^t u_0(F_0')$ is weakly closed,*

then $a(\ker u_0)$ is dense in $\ker u$, and if, in addition, E and F are Fréchet spaces, u is a surjection.

If the sequence space X can be covered by a countable family of polydiscs, $A(X)$ is Fréchet. For such spaces we have the following.

Theorem 4.6. *Let X, Y be sequence spaces where X is ℓ_1-closed and $Y \subseteq X^s$. Assume that X can be covered by a countable family of polydiscs. Let $P(D) : A(X) \to A(X)$ be a differential operator with symbol $P \in Exp(Y)$, where P is a proper polynomial with respect to some $S \subseteq N$. Then $P(D)$ is a surjection.*

PROOF. We can assume that $Y = C^{(N)}$. By hypothesis, $A_\Theta(X)$ is a reflexive Fréchet space. We apply Theorem 4.5, with

$$u = P(D) : E \equiv A_\Theta(X) \to F \equiv A_\Theta(X),$$
$$u_0 = P_a(D) : E_0 \equiv \mathrm{Exp}_\Sigma(X) \to F_0 \equiv \mathrm{Exp}_\Sigma(X),$$

and pairings $(A_\Theta(X), \mathrm{Exp}_\Theta(Y))$ and $(A_\Sigma(Y), \mathrm{Exp}_\Sigma(X))$ where $\Sigma = \Theta(Y)$ denotes the set of polydiscs in Y, $P_a \in A_\Sigma(Y)$ the image of P under the natural embedding $\mathrm{Exp}_\Theta(Y) \to A_\Sigma(Y)$, and $a = b : \mathrm{Exp}_\Sigma(X) \to A_\Theta(X)$ the continuous embeddings with dense images. It is clear that the embedding $\mathrm{Exp}_\Theta(Y) \to A_\Sigma(Y)$ is the transpose of a and that $P(D) \circ a = a \circ P_a(D)$.

Property 1 of Theorem 4.5 follows from Lemma 4.3 and estimate (19). Indeed, ${}^t a$ is the embedding $\mathrm{Exp}_\Theta(Y) \to A_\Sigma(Y)$ and ${}^t P_a(D) = P_a : A_\Sigma(Y) \to A_\Sigma(Y)$. Thus we must prove that, if $f \in A_\Sigma(Y)$ and $P_a f = \varphi$ for some $\varphi \in \mathrm{Exp}_\Theta(Y)$, then $f \in \mathrm{Exp}_\Theta(Y)$. It is clear that P_a is a proper polynomial with respect to S with principal part $a(P_S)$. Choose c such that $P_S(c) \neq 0$ and let $A_\varepsilon(c) \equiv \varepsilon^{-m}|P_S(c)|^{-1}$, where m is the degree of P (P_a) with respect to S. By Lemma 4.3, if $\varphi \in \mathrm{Exp}_r(Y)$ and $r \in X^+$,

$$|f(y)| \le A_\varepsilon(c) \sup_{|\xi|\le\varepsilon} |P_a f(y + \xi c)| \le A_\varepsilon(c)\|\varphi\|_{C_r} \exp \sup_{|\xi|\le\varepsilon} \|y + \xi c\|_r$$
$$\le A_\varepsilon(c)\|\varphi\|_{C_r} e^{\|y\|_r} e^{\varepsilon\|c\|_r} = A_\varepsilon(c) B_\varepsilon(c, r)\|\varphi\|_{C_r} e^{\|y\|_r},$$

where $B_\varepsilon(c, r) \equiv e^{\varepsilon\|c\|_r}$. Hence $f \in \mathrm{Exp}_\Theta(Y)$ and property 1 holds.

Since Y is ℓ_1-closed and $X \subseteq X^{ss} \subseteq Y^s$, u_0 is a surjection by Theorem 4.4. Hence the transpose ${}^t u_0 = P_a$ is a weak injective strict morphism, that is, $P_a : A_\Sigma(Y) \to \mathrm{Im}P_a \subseteq A_\Sigma(Y)$ is an isomorphism for the weak topologies $\sigma(A_\Sigma(Y), \mathrm{Exp}_\Sigma(X))$ and $\sigma(\mathrm{Im}P_a, \mathrm{Exp}_\Sigma(X))$. Since $\mathrm{Exp}_\Sigma(X)$ is the dual of the Fréchet space $A_\Sigma(Y)$, $P_a : A_\Sigma(Y) \to \mathrm{Im}P_a$ is an isomorphism for the initial metric topologies [10, p. 265]. Hence $\mathrm{Im}P_a$ is complete and hence closed in $A_\Sigma(Y)$. Thus $\mathrm{Im}P_a$ is closed for the weak topology $\sigma(A_\Sigma(Y), \mathrm{Exp}_\Sigma(X)) = \sigma(A_\Sigma(Y), A'_\Sigma(Y))$. Hence property 2 of Theorem 4.5 holds, and thus the theorem. ■

With $X = \mathrm{Exp}(C), S', C^{(N)}, \mathcal{O}(C_r)$, $r \geq 0$, and the notation used in Section 2 and Example 2.2, Theorem 4.6 can be applied. Theorem 4.4 can be applied to these spaces and to $X = A(C), S, C^N$.

Surjectivity on F

Our next objective is to prove that $P(D)$ is a surjection on F for a large class of symbols $P \in \mathrm{Exp}(Y)$.

Let X be an ℓ_1-closed sequence space and let $Y \subseteq X^s$. Denote by B the bilinear form $B : A(X) \times \mathrm{Exp}(Y) \to H_G(X \times Y)$ defined by $B(f, \varphi)(x, y) \equiv e^{-\langle x,y \rangle} \langle f e_y, \varphi e_x \rangle$. We note that B maps $A(X) \times \mathrm{Exp}(Y)$ into the space $S(X \times Y)$ of all Gâteaux holomorphic functions $\psi \in H_G(X \times Y)$ with the following property: for every $r \in X^+$ there are $R_r \in X^+$ and $M_r > 0$ such that

$$\sup_{x \in C_r} |\psi(x, y)| = \|\psi(\cdot, y)\|_{C_r} \leq M_r e^{\|y\|_{R_r}}, \quad y \in Y.$$

By Theorems 3.3 and 3.4 and fixing one variable at a time, we have the following.

Theorem 4.7. *Let X be an ℓ_1-closed sequence space and let $Y \subseteq X^s$. Let $f \in A(X)$ and $\varphi \in Exp(Y)$. If $f_\alpha = \partial^\alpha f$ and $\varphi_\beta = \partial^\beta \varphi$, then*

$$B(f, \varphi)(x, y) = \sum_\alpha f_\alpha(D)\varphi(y) x^\alpha / \alpha! \tag{22}$$

$$= \sum_\beta \varphi_\beta(D) f(x) y^\beta / \beta! \tag{23}$$

$$= \sum_{\alpha\beta} \frac{\langle f_\alpha, \varphi_\beta \rangle}{\alpha! \beta!} x^\alpha y^\beta, \tag{24}$$

where all series converge pointwise to $B(f, \varphi)$ and (22) *and* (24) *converge absolutely in $A(X)$ for fixed $y \in Y$ while* (23) *and* (24) *converge absolutely in $Exp(Y)$ for fixed $x \in X$.*

Shapiro and Newman [17] show that, in n dimensions, if $f\varphi \in F_n$ ($\varphi \in \mathrm{Exp}$, $f \in A$), then $B_n(f, \varphi) \in F_n^2$ and $\|f\varphi\|_n = \|B_n(f, \varphi)\|_n$. Here F_n and F_n^2 denote the Fock spaces for C^n resp. $C^n \times C^n$, $\|\cdot\|_n$ the corresponding norms and B_n the n-dimensional analogue of B. The next theorem generalises this result to F and F^2. Here F^2

is the subset of $\mathfrak{A}^2 \equiv C^{N^{(N)}\times N^{(N)}}$ consisting of all sequences $\phi = (\phi_{\alpha\beta})$ such that $\|\phi\|^2 \equiv \sum_{\alpha\beta} \alpha!\beta!|\phi_{\alpha\beta}|^2 < \infty$.

Theorem 4.8. *Let X be an ℓ_1-closed sequence space and let $Y \subseteq X^s$. If $(f,\varphi) \in A(X) \times Exp(Y)$ and $f\varphi \in F$, then $B(f,\varphi) \in F^2$ and $\|f\varphi\| \geq \|B(f,\varphi)\|$, with equality if either φ or f depends only on a finite number of variables.*

PROOF. Note that $\phi \in \mathfrak{A}$ belongs to F if and only if the restriction $\phi^{[n]}$ of ϕ to the first n variables belongs to F_n for all n and $\lim \|\phi^{[n]}\|_n = \sup \|\phi^{[n]}\|_n < \infty$. Assume that $f\varphi \in F$. By the (n-dimensional) analogue of this theorem in [17], we obtain that

$$\|f\varphi\| = \lim \|(f\varphi)^{[n]}\|_n = \lim \|f^{[n]}\varphi^{[n]}\|_n = \lim \|B_n(f^{[n]},\varphi^{[n]})\|_n \equiv N.$$

If φ or f depends only on a finite number of variables, we have by inspection that $B_n(f^{[n]},\varphi^{[n]}) = B(f,\varphi)^{[n]}$ for large n. This proves the second part. To complete the proof we must prove that $N \geq \|B(f,\varphi)\|$. By (24), $B(f,\varphi)(x,y) = \sum u_{\alpha\beta}x^\alpha y^\beta$, $u_{\alpha\beta} \equiv \langle f_\alpha, \varphi_\beta\rangle/\alpha!\beta!$. Since

$$u^{(n)}_{\alpha\beta} \equiv \langle (f^{[n]})_\alpha, (\varphi^{[n]})_\beta\rangle/\alpha!\beta! = \langle (f_\alpha)^{[n]}, (\varphi_\beta)^{[n]}\rangle/\alpha!\beta!, \quad l(\alpha), l(\beta) \leq n, \tag{25}$$

we also have that $B_n(f^{[n]},\varphi^{[n]})(x,y) = \sum_{l(\alpha),l(\beta)\leq n} u^{(n)}_{\alpha\beta}x^\alpha y^\beta$, where $l(\alpha) = \max_{\alpha_k \neq 0} k$. Let $v_{\alpha\beta} \equiv \alpha!\beta!|u_{\alpha\beta}|^2$ and $v^{(n)}_{\alpha\beta} = \alpha!\beta!|u^{(n)}_{\alpha\beta}|^2$ if $l(\alpha), l(\beta) \leq n$, and $v^{(n)}_{\alpha\beta} = 0$ otherwise. Then $\lim \sum_{\alpha\beta} v^{(n)}_{\alpha\beta} = N^2$. By (25), $\lim v^{(n)}_{\alpha\beta} = v_{\alpha\beta}$ for all α, β. Fatou's lemma implies that $\sum v_{\alpha\beta} \leq \lim \sum v^{(n)}_{\alpha\beta} = N^2$. ■

Corollary 4.9. *Let X be an ℓ_1-closed sequence space and let $Y \subseteq X^s$. Let $(f,\varphi) \in A(X) \times Exp(Y)$ and let $f_\alpha = \partial^\alpha f$, $\varphi_\beta = \partial^\beta \varphi$. If $f\varphi \in F$, then $f_\alpha(D)\varphi$, $\varphi_\beta(D)f \in F$ for all α, β and*

$$\|f\varphi\|^2 \geq \|B(f,\varphi)\|^2 = \sum_\alpha \|f_\alpha(D)\varphi\|^2/\alpha! = \sum_\beta \|\varphi_\beta(D)f\|^2/\beta!. \tag{26}$$

In particular, if $\partial^\beta\varphi = \varphi_\beta$ is a non-zero constant for some β, then $f \in F$ and $\|f\varphi\| \geq \|B(f,\varphi)\| \geq C_0\|f\|$ for some $C_0 > 0$, and analogously if $\partial^\alpha f = f_\alpha$ is a constant different from zero for some α.

PROOF. We know that $B(f,\varphi) \in F^2$ and $\|f\varphi\| \geq \|B(f,\varphi)\|$. If $u_{\alpha\beta} = \langle f_\alpha, \varphi_\beta\rangle/\alpha!\beta!$, then, by Theorem 4.7 and (24),

$$\|f\varphi\|^2 \geq \|B(f,\varphi)\|^2 = \sum_{\alpha\beta} \alpha!\beta!|u_{\alpha\beta}|^2 = \sum_\beta \beta! \sum_\alpha \alpha!|u_{\alpha\beta}|^2. \tag{27}$$

By Theorem 4.7 we have that $\varphi_\beta(D)f(x)/\beta! = \sum_\alpha u_{\alpha\beta}x^\alpha$. Hence, (27) shows that $\varphi_\beta(D)f \in F$ for all β, and $\|B\|^2$ coincides with the expression on the right in (26). The other equality follows similarly. ■

The hypothesis that $\partial^\alpha P$ is a non-zero constant for some α is weaker than that P is a proper polynomial with respect to some subset.

Theorem 4.10. *Let X be an ℓ_1-closed sequence space contained in ℓ_2 and let $Y \subseteq X^s$. Let $P \in Exp(Y)$ and assume that $\partial^\beta P$ is a non-zero constant for some β. Then $P(D)$ is a surjection on F. In particular $P(D)$ is surjective if P is a proper polynomial with respect to some $S \subseteq N$.*

Proof. We must prove that the adjoint $P(D)^*$ is one-to-one and has a closed image. Since $P(D)^* = P^*$, $P(D)^*$ is one-to-one. Moreover, by the assumption on P, $\partial^\beta P^*$ is a non-zero constant for some β. Corollary 4.9 implies that $\|P^* f\| \geq C_0 \|f\|$, for some $C_0 > 0$ and for all f in the domain of definition of P^*. This shows that P^* has a closed image and hence that $P(D)$ is a surjection. ∎

5. Approximation

We have studied the equations $f(D)\varphi = \psi$ and $\varphi(D)f = g$ in $\mathrm{Exp}(Y)$ and $A(X)$ respectively. The purpose of this section is to show that every homogeneous solution is the limit of homogeneous solutions consisting of so-called exponential (finitely supported) polynomials (Theorem 5.4) (see also [2; 7]). To prove this analogue of Malgrange's result, we apply Theorem 4.5 and a technique of Treves.

In this section, Z denotes the direct sum $C^{(N)}$. Consider the spaces $\mathfrak{B}^{(Z)} \equiv \oplus_{z \in Z} \mathfrak{B}$ and $\mathfrak{A}^Z \equiv \prod_{z \in Z} \mathfrak{A}$, endowed with the direct sum and product topologies respectively. The pairing (9) between $\mathfrak{A}$ and $\mathfrak{B}$ puts $\mathfrak{B}^{(Z)}$ and $\mathfrak{A}^Z$ in duality in a natural way. Moreover,

$$(\mathfrak{B}^{(Z)})' = (\mathfrak{B}^{(Z)})^* = \mathfrak{A}^Z, \quad (\mathfrak{A}^Z)' = \mathfrak{B}^{(Z)}.$$

Let X, Y be sequence spaces such that $Y \subseteq X^s$. Consider the map $i : \mathfrak{B}^{(Z)} \to A_\Theta(X)$ defined by $i((b_z)) \equiv \sum_{z \in Z} b_z e_z$. The image is the space of exponential (finitely supported) polynomials, $E\mathfrak{B}(X)$. The map $j : \mathfrak{B}^{(Z)} \to \mathrm{Exp}_\Theta(Y)$ and the space $E\mathfrak{B}(Y)$ are defined in the same way. For any vector space E, Gâteaux holomorphic function $g \in H_G(E)$ and $z \in E$, we put $M_g f \equiv fg$, $\tau_z f \equiv f(\cdot - z)$, $f \in H_G(E)$.

Lemma 5.1. *The transposes ${}^t i : Exp_\Theta(Y) \to \mathfrak{A}^Z$, ${}^t j : A_\Theta(X) \to \mathfrak{A}^Z$ of the maps i and j are given by*

$${}^t i \varphi = (\tau_{-z}\varphi)_{z \in Z}, \quad {}^t j f = (\tau_{-z} f)_{z \in Z}.$$

Proof. For $y \in Y$ we have ${}^t M_{e_y} = \tau_{-y}$ on $\mathrm{Exp}_\Theta(Y)$ where the transpose is the transpose for the duality between $A_\Theta(X)$ and $\mathrm{Exp}_\Theta(Y)$. Since $Z \subseteq Y$

$$\langle ib, \varphi \rangle = \sum_z \langle b_z e_z, \varphi \rangle = \sum_z \langle b_z, \tau_{-z}\varphi \rangle = \langle b, (\tau_{-z}\varphi) \rangle.$$

This proves the assertion for i. As ${}^t M_{e_x} = \tau_{-x}$, $x \in X$, on $A_\Theta(X)$ and $Z \subseteq X$, the expression for ${}^t j$ follows. ∎

Translation by vectors in Z is well defined in $\mathfrak{A}$ for vectors $a \in \mathfrak{B}$. Indeed, let $\tau_{-z}a = \sum a_\alpha(x+z)^\alpha = \sum_\gamma c_\gamma x^\gamma$ where

$$c_\gamma = c_\gamma(z) = \sum_{\alpha \geq \gamma} a_\alpha \frac{\alpha!}{(\alpha-\gamma)!\gamma!} z^{\alpha-\gamma}.$$

However, if $a \in \mathfrak{A}$, the sum for c_γ may not now be convergent. The set of elements $a \in \mathfrak{A}$ such that $c_\gamma(z)$ is convergent for every $z \in Z$ and γ is denoted by $\tau_Z^{-1}\mathfrak{A}$. In particular we have that $A_\Theta(X), \mathrm{Exp}_\Theta(Y) \subseteq \tau_Z^{-1}\mathfrak{A}$. For $a \in \tau_Z^{-1}\mathfrak{A}$ we define $a(D) : \mathfrak{B}^{(Z)} \to \mathfrak{B}^{(Z)}$ by $a(D)(b_z) \equiv (\tau_{-z}a(D)b_z)$. In view of Theorem 4.1, $a(D)\mathfrak{B}^{(Z)} = \mathfrak{B}^{(Z)}$ for every (non-zero) $a \in \tau_Z^{-1}\mathfrak{A}$ (see also [14, lemma 6.4]).

Lemma 5.2. *Let* $a \in \tau_Z^{-1}\mathfrak{A}$. *Then* ${}^t a(D) : \mathfrak{A}^Z \to \mathfrak{A}^Z$ *has weakly closed range.*

PROOF. Theorem 4.1 implies that $a(D)$, $a \neq 0$, is surjective. Hence the transpose is a weak injective strict morphism. This means that ${}^t a(D) : \mathfrak{A}^Z \to \mathrm{Im}\,{}^t a(D)$ is an isomorphism for the weak topologies $\sigma(\mathfrak{A}^Z, \mathfrak{B}^{(Z)})$ and $\sigma(\mathrm{Im}\,{}^t a(D), \mathfrak{B}^{(Z)})$. For any vector space E, the algebraic dual E^* is complete for the weak topology $\sigma(E^*, E)$. Since $\mathfrak{A}^Z = (\mathfrak{B}^{(Z)})^*$, $\mathfrak{A}^Z$ is complete with respect to the topology $\sigma(\mathfrak{A}^Z, \mathfrak{B}^{(Z)})$. This implies that $\mathrm{Im}\,{}^t a(D)$ is complete and therefore weakly closed. ■

Lemma 5.3. *If* $\varphi \in \mathrm{Exp}_\Theta(Y)$, *then* $i \circ \varphi(D) = \varphi(D) \circ i$ *where* i *is the embedding* $\mathfrak{B}^{(Z)} \to A_\Theta(X)$. *In the same way,* $j \circ f(D) = f(D) \circ j$ *if* $f \in A_\Theta(X)$ *for the embedding* $j : \mathfrak{B}^{(Z)} \to \mathrm{Exp}_\Theta(Y)$.

PROOF. We first prove that $\varphi(D)(fe_y) = e_y(\tau_{-y}\varphi)(D)f$ if $\varphi \in \mathrm{Exp}_\Theta(Y)$ and $f \in A_\Theta(X)$. Since ${}^t M_{e_y} = \tau_{-y}$, $y \in Y$,

$$\begin{aligned}\varphi(D)(e_y f)(x) &= \langle fe_y, \varphi e_x\rangle = \langle f, \tau_{-y}(\varphi e_x)\rangle \\ &= e^{\langle x,y\rangle}\langle f, e_x \tau_{-y}\varphi\rangle = e^{\langle x,y\rangle}(\tau_{-y}\varphi)(D)f(x).\end{aligned}$$

Hence, if $b = (b_z) \in \mathfrak{B}^{(Z)}$, we obtain that

$$\begin{aligned}i(\varphi(D)b) &= i[((\tau_{-z}\varphi)(D)b_z)] = \sum_z e_z(\tau_{-z}\varphi)(D)b_z \\ &= \sum_z \varphi(D)(b_z e_z) = \varphi(D)\sum_z b_z e_z = \varphi(D)(ib).\end{aligned}$$

This completes the first part of the proof, and the second part follows in the same way since ${}^t M_{e_x} = \tau_{-x}$ on $A_\Theta(X)$. ■

We are now ready to prove the main result in this section. One of the keys to the proof is the following ('division') result in Treves [20, p. 36]. Let P be a polynomial in n variables, and assume that $g_z \in \mathfrak{A}_n$ for all $z \in C^n$ and that there is an entire function $h \in A(C^n)$ such that $(\tau_{-z}P)g_z = \tau_{-z}h$ in $\mathfrak{A}_n$ for all $z \in C^n$. Then there is an entire function $f \in A(C^n)$ such that $g_z = \tau_{-z}f$ for all $z \in C^n$, and hence $Pf = h$.

Theorem 5.4. *Let X, Y be sequence spaces where X is ℓ_1-closed and $Y \subseteq X^s$.*

Let $f \in A(X)$ and let $P \in Exp(Y)$ be a polynomial. If $P(D)f = 0$, then f is the limit (in $A(X)$) of homogeneous solutions in $E\mathfrak{B}(X)$.

Similarly, if $P(D)\varphi = 0$, $\varphi \in Exp(Y)$, for some polynomial $P \in A(X)$, then φ is the limit (in $Exp(Y)$) of homogeneous solutions in $E\mathfrak{B}(Y)$.

Proof. We apply Theorem 4.5 with

$$u = P(D) : E \equiv A_\Theta(X) \to F \equiv A_\Theta(X),$$
$$u_0 = P(D) : E_0 \equiv \mathfrak{B}^{(Z)} \to F_0 \equiv \mathfrak{B}^{(Z)},$$

and the pairings $(A_\Theta(X), \mathrm{Exp}_\Theta(Y))$ and $(\mathfrak{B}^{(Z)}, \mathfrak{A}^Z)$. With the notation of Theorem 4.5, let $a = b = i : \mathfrak{B}^{(Z)} \to A_\Theta(X)$. By Lemmas 5.2 and 5.3 it suffices to establish property 2 in Theorem 4.5. In view of Lemma 5.1 we must prove that, if

$$(\tau_{-z}P)b_z = \tau_{-z}\varphi, \quad \text{in } \mathfrak{A} \quad \forall z \in Z = C^{(N)}, \tag{28}$$

for some $\varphi \in \mathrm{Exp}_\Theta(Y)$ and family $b_z \in \mathfrak{A}$, $z \in Z$, then there is a $\psi \in \mathrm{Exp}_\Theta(Y)$ such that $b_z = \tau_{-z}\psi$ for all $z \in Z$. If (28) holds, it holds for all indexes α such that $l(\alpha) \le n$ and all $z \in Z^{[n]} \equiv \{z \in Z : z_k = 0, \forall k > n\} \simeq C^n$. Thus

$$(\tau_{-z}P^{[n]})b_z^{[n]} = \tau_{-z}\varphi^{[n]}, \quad \text{in } \mathfrak{A}_n \quad \forall z \in Z^{[n]}, \tag{29}$$

for all n, where $P^{[n]}$ is the function in n variables obtained by restricting P to $Z^{[n]} = Y^{[n]}$ and $\varphi^{[n]}$ and $b_z^{[n]}$ are defined analogously. By the result in [20, p. 36], there exists for each n an entire function $\psi_n \in A(C^n)$ such that

$$b_z^{[n]} = \tau_{-z}\psi_n, \quad z \in Z^{[n]} \simeq C^n, \tag{30}$$

$$P^{[n]}\psi_n = \varphi^{[n]}. \tag{31}$$

By (30), $\psi_n = \psi_m|_{Z^{[n]}}$ when $m > n$. Hence the system (ψ_n) defines a function ψ on $Z = C^{(N)}$. We prove that ψ extends by its Taylor expansion about the origin to define a function in $\mathrm{Exp}_\Theta(Y)$. Let P_m denote the principal part for P. Choose $c \in Y$ such that $C \equiv P_m(c) \neq 0$. As $P_m \in \mathrm{Exp}_\Theta(Y)$, P_m is continuous on Y_Θ, and hence there exists a $\delta > 0$ such that $0 < C - \delta \le |P_m^{[n]}(c^{[n]})|$ for all n sufficiently large ($n \ge n_\delta$). (Recall that $c^{[n]} \equiv \sum_1^n c_k e_k \in Z^{[n]}$.) If $|\varphi(y)| \le Me^{\|y\|_r}$, we have, by (31) and Lemma 4.2, for arbitrary $\varepsilon > 0$

$$\begin{aligned} |\psi_n(z)| &\le A(\delta, c, \varepsilon) \sup_{|\xi| \le \varepsilon} |P^{[n]}(z + \xi c^{[n]})\psi^{[n]}(z + \xi c^{[n]})| \\ &\le A \sup_{|\xi| \le \varepsilon} |\varphi^{[n]}(z + \xi c^{[n]})| \le AMe^{\varepsilon\|c\|_r}e^{\|z\|_r} = Be^{\|z\|_r}, \quad z \in Z^{[n]}, \end{aligned} \tag{32}$$

where $A = A(\delta, c, \varepsilon) = \varepsilon^{-m}/(C - \delta)$ and $B = AMe^{\varepsilon\|c\|_r}$. This shows that $\psi \in \mathrm{Exp}_\Theta(Z)$, and hence ψ extends to $\mathrm{Exp}_\Theta(Y)$ (Theorem 3.4). By (30) we conclude that $b_z = \tau_{-z}\psi$. This completes the first part of the proof, and the second part follows in the same way by using the embedding $j : \mathfrak{B}^{(Z)} \to \mathrm{Exp}_\Theta(Y)$. ■

REFERENCES

[1] R. Aron, Tensor products of holomorphic functions, *Indagationes Mathematicae* **76** (1973), 192–202.

[2] P.J. Boland, Malgrange theorem for entire functions on nuclear spaces, in T.L. Hayden and T.J. Suffridge (eds), *Proceedings on infinite dimensional holomorphy*, Lecture Notes in Mathematics 364, Springer, New York, 1974, pp 135–44.

[3] S. Dineen, Holomorphy types on a Banach space, *Studia Mathematica* **39** (1971), 241–88.

[4] S. Dineen, *Complex analysis in locally convex spaces*, North Holland Mathematics Studies 57, Amsterdam, 1981.

[5] S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer-Verlag Monographs in Mathematics, London, 1999.

[6] A.W. Dwyer, Partial differential equations in Fischer–Fock spaces for the Hilbert–Schmidt holomorphy type, *Bulletin of the American Mathematical Society* **77** (1971), 725–30.

[7] A.W. Dwyer, Convolution equations for vector-valued entire functions of nuclear bounded type, *Transactions of the American Mathematical Society* **217** (1976), 105–19.

[8] A.W. Dwyer, Vector-valued convolution equations for the nuclear holomorphy type, *Proceedings of the Royal Irish Academy* **76**A (1976), 101–10.

[9] C. Gupta, *Convolution operators and holomorphic mappings on a Banach space*, Semanarie D'Analyse Moderne 2, Université de Sherbrooke, Quebec, 1969.

[10] J. Horvath, *Topological vector spaces and distributions, vol. 1*, Addison-Wesley, Reading, Mass., 1966.

[11] A. Khrennikov, Infinite dimensional pseudo-differential operators, *Mathematics of the USSR: Izvestiya* **31** (1988), 575–601.

[12] A. Khrennikov, The Feynamn in phase space and symbols of infinite dimensional pseudo-differential operators, *Matematicheskie Zametki* **37** (1995), 734–42 [in Russian].

[13] B. Lascar, Une condition nécessaire et suffisante d'éllipticité pour une classe d'opérateurs différentiels en dimension infinie, *Communications in Partial Differential Equations* **2** (1) (1977), 31–67.

[14] H. Petersson, *Infinite dimensional holomorphy in the ring of formal power series*, Acta Wexionensia 8, Växjö University, Sweden, 2001.

[15] M. Reed and B. Simon, *Methods of modern mathematical physics, vol. 1. Functional analysis*, Academic Press, New York, 1980.

[16] H.S. Shapiro, An algebraic theorem of E. Fischer, and the holomorphic Goursat problem, *Bulletin of the London Mathematical Society* **21** (1989), 513–37.

[17] H.S. Shapiro and D.J. Newman, Fischer spaces of entire functions, in J. Korevaar, S.-S. Chern, L. Ehrenpreis and W.H.J. Fuchs (eds), *Entire functions and related parts of analysis*, Proceedings of Symposia in Pure Mathematics 11, American Mathematical Society, Providence, RI, 1968, pp 360–9.

[18] O.G. Smolyanov, Infinite-dimensional pseudo-differential operators and Schrödinger quantization, *Doklady Akademii Nauk SSSR* **263** (1982), 558–62 [in Russian].

[19] F. Treves, *Locally convex spaces and linear partial differential equations*, Springer-Verlag, New York, 1967.

[20] F. Treves, *Linear partial differential equations with constant coefficients*, Gordon and Breach, New York, 1970.