COMPACT SEMIGROUPS OF POSITIVE MATRICES

By T.T. West, M.R.I.A.*
School of Mathematics, Trinity College, Dublin

[Received 24 May 2001. Read 16 March 2003. Published 31 December 2003.]

Abstract

The spectral theory of matrices generating compact semigroups is combined with results on block positive matrices to obtain the Frobenius representation for connected non-negative matrices.

1. Introduction

The spectral theory of compact monothetic semigroups of linear operators examined by Kaashoek and West in [1; 2], together with two block matrix theorems where the blocks are either strictly positive or zero, is used to give an exposition of Perron–Frobenius theory of positive matrices. This approach is based on ideas of Smyth and West developed in [4; 5].

We consider a linear operator T with finite dimensions that has a matrix representation $[T]$ relative to a given basis. Where there is no ambiguity we often write the matrix as T. $T \geq 0$ if $[T]_{ij} \geq 0$ $(\forall i, j)$ while $T > 0$ if $[T]_{ij} > 0$ $(\forall i, j)$. The spectrum and spectral radius of T will be denoted by $\sigma(T)$ and $r(T)$, respectively. The trace of T (the sum of its diagonal entries) will be written as $tr(T)$, and the peripheral spectrum will be denoted by $\pi(T) = \{\lambda \in \sigma(T);\ |\lambda| = r(T)\}$. The i^{th} row and j^{th} column of T relative to the given basis will be written $row_i(T)$ and $col_j(T)$, and the diagonal of T will be denoted $diag(T)$. The spectral projection of T relative to $\pi(T)$ will be written P_π.

Smyth [5] has introduced a hierarchy of subsets of matrices $T \geq 0$.

Definitions. The following attributes are appropriate to any matrix $T \geq 0$:
(i) T is *positive* if $T > 0$;
(ii) T is *primitive* if $T^k > 0$ for some positive integer k;
(iii) T is *connected* if $\forall i, j \; \exists$ a positive integer k such that $[T^k]_{ij} > 0$;
(iv) T is *potent* if $diag(T^k) > 0$ for some positive integer k;
(v) T is *zero-free* if no row or column is zero;
(vi) T has positive spectral radius.

Remarks. The above sets are strictly ordered by inclusion. T is connected, if and only if there exists a positive integer p such that $T + T^2 + \ldots + T^p > 0$. It is also connected if no basis permutation results in a block representation

$$T = \begin{pmatrix} U & 0 \\ V & W \end{pmatrix},$$

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* E-mail: twest@tcd.ie

where U and W are square blocks. If S and T are zero-free then so is ST. It follows that if T is zero-free $r(T)>0$. Note also that all these subsets are invariant under a basis permutation and under transposition, and that if $S\geq T$ and T is contained in any one of these sets then so is S.

The following upgrading result will be important.

Proposition 1. *If* S, $T\geq 0$, $T\neq 0$ *and* S *is connected, then* $ST=TS$ *implies that* T *is potent.*

Proof. Observe that by replacing S with $S+S^2+\ldots+S^p$ for sufficiently large p we may assume that $S>0$. First we show that under these conditions T is zero-free. As $T\neq 0$, $[T]_{ij}>0$ for some i,j, then $[ST]_{kj}\geq[S]_{ki}[T]_{ij}>0$ $(\forall k)$, and therefore $[TS]_{kj}>0$ $(\forall k)$. Thus $row_k(T)$ is non-zero $(\forall k)$ and taking transposes gives the same result for columns.

We prove that T is potent by induction on the size of the matrix. The result is trivially true for 1×1 matrices, so assume that it holds for $k\times k$ matrices $(k=1,\ldots,n-1)$.

If T $(n\times n)$ is connected the result is trivially true; so assume that T is not connected. Then by a basis permutation T has lower triangular block form

$$T=\begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix},$$

where T_{11} and T_{22} are square blocks which must be non-zero as T is zero-free. Corresponding to this decomposition

$$S=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$$

with $S_{ij}>0$ $(\forall i,j)$. Since S and T commute we have $T_{11}S_{11}=S_{11}T_{11}+S_{12}T_{21}$. But $tr(T_{11}S_{11})=tr(S_{11}T_{11})$, and therefore $tr(S_{12}T_{21})=0$. By positivity $T_{21}=0$, but now T_{11} commutes with $S_{11}>0$ and T_{22} commutes with $S_{22}>0$. By our induction hypothesis, these blocks are both potent and hence so is T. ■

2. Compact semigroups

Let T be connected. Then $r(T)>0$, and so, without loss of generality, we take $r(T)=1$.

Proposition 2. *If* T *is connected and* $r(T)=1$, *then* $\|T^n\|\leq M(n=1,2,\ldots)$.

Proof. $\mathcal{S}(T)=\overline{\{T^n;n\geq 1\}}$ is a closed monothetic (singly generated) semigroup; $\mathcal{W}=\Re^+\mathcal{S}(T)$ is also a semigroup and $\mathcal{W}_1=\{W\in\mathcal{W}:\|W\|=1\}$ is a closed, bounded, non-empty subset of $\mathcal{W}$, which is therefore compact. If $W\in\mathcal{W}_1$, then W is potent by Proposition 1, hence $r(W)>0$ for each $W\in\mathcal{W}_1$. Furthermore, the spectral radius is norm-continuous and therefore attains its minimum μ on the compact set $\mathcal{W}_1$. Then $r(W)\geq\mu>0$ $(W\in\mathcal{W}_1)$, and so $r(S)\|S\|^{-1}\geq\mu(S\in\mathcal{S}(T))$. But $r(S)=1$ for $S\in\mathcal{S}(T)$, hence $\|S\|\leq\mu^{-1}$ $(S\in\mathcal{S}(T))$ and the monothetic semigroup $\mathcal{S}(T)$ is closed and bounded, and is therefore compact. ■

The structure of such compact monothetic semigroups of matrices has been determined in [1, theorem 1, corollary 2 and theorem 3] and [2, chapter 1 §1 and §2]. Under the hypotheses of Proposition 2, $\mathscr{S}(T)$ contains a unique idempotent which is P_π. Furthermore,

$$\mathscr{G}(T)=P_\pi\mathscr{S}(T)=\mathscr{S}(P_\pi T)$$

is a compact monothetic group with unit P_π that consists of all cluster points of $\mathscr{S}(T)$.

Proposition 3. *Under the hypotheses of Proposition 2,* $P_\pi \geq 0$ *and* $diag(P_\pi)>0$.

Proof. Since P_π is a limit of powers of T it is ≥ 0. Furthermore, since P_π commutes with T it is potent by Proposition 1, and as it is an idempotent $diag(P_\pi)>0$. ■

Suppose that $\pi(T)=\{\lambda_1, \lambda_2,\ldots, \lambda_h\}$. Then $\mathscr{G}(T)$ is isomorphic to the compact monothetic semigroup of $\mathscr{C}^h$ given by

$$\overline{\{\lambda_1^n, \lambda_2^n,\ldots, \lambda_h^n; n\geq 1\}}.$$

Proposition 4. *Under the hypotheses of Proposition 2,* $\mathscr{G}(T)$ *is a finite cyclic group.*

Proof. T is potent hence $diag(T^p)>0$ for some positive integer p. It follows from [4, proposition 2] that $\pi(T^p)=\{1\}$, and thus all the peripheral eigenvalues of T are p^{th} roots of unity. Hence $\mathscr{G}(T)$ is finite and obviously cyclic. ■

Proposition 5. *Under the above hypotheses* $R=P_\pi T\geq 0$ *and connected.*

Proof. Trivially $R\geq 0$. Since T is connected $T_p=\sum_{n=1}^{p} T^n>0$ for sufficiently large p; furthermore, $R_p=\sum_{n=1}^{p} R^n=P_\pi T_p$. Now since $diag(P_\pi)>0$ it follows that $[R_p]_{ij}=[P_\pi T_p]_{ij}\geq [P_\pi]_{ii}[T_p]_{ij}>0$ $(\forall i,j)$, and so $R_p>0$ and R is connected. ■

Consider the simple case in which $\pi(T)=\{1\}$. Then by the isomorphism $\mathscr{G}(T)$ consists of one element P_π, so $T^n\to P_\pi$ $(n\to\infty)$. Thus for a general connected T as we have seen $\pi(T^p)=\{1\}$ for a positive integer p, and so $\mathscr{G}(T)$ is a finite cyclic group and $(T^p)^n=T^{pn}\to P_\pi$ $(n\to\infty)$.

We now use these results to characterise primitive matrices. Note that if $T^k>0$, then $T^{k+1}=T^kT$ is the product of a positive with a zero-free matrix that is therefore positive. Hence $T^{k+n}>0$ for all positive integers n.

Proposition 6. *Let* $T\geq 0$ *with* $r(T)=1$. *Then the following are equivalent*:

(i) *T is primitive*;
(ii) *T is connected and* $\pi(T)=\{1\}$;
(iii) *T is connected and* $T^n\to P_\pi$;
(iv) *T is connected and* $\mathscr{G}(T)=\{P_\pi\}$.

Proof. (*i*)$\Leftrightarrow$(*ii*). Assume that T is primitive. Then T is connected. Now $T^k>0$ for some k hence $diag(T^k)>0$. Therefore $T^{kn}\to P_\pi$ $(n\to\infty)$, and so $T^kP_\pi=P_\pi$.

Next we show that $P_\pi>0$. Suppose not; then $[P_\pi]_{ij}=0$ for some i,j, and hence $[T^k P_\pi]_{ij}=\sum_{m=1}^{n}[T^k]_{im}[P_\pi]_{mj}=0$. Thus, using positivity, $[P_\pi]_{mj}=0$ $(\forall m)$, that is

$col_j(P_\pi)=0$, which contradicts the fact that $diag(P_\pi)>0$. Hence $P_\pi>0$ and hence $rank(P_\pi)=1$, and therefore $\pi(T)=\{1\}$.

Conversely let T be connected with $\pi(T)=\{1\}$. Then $T^n\to P_\pi$ $(n\to\infty)$, and so $T^nP_\pi=P_\pi T^n=P_\pi$ $(\forall n)$. Suppose $[P_\pi]_{ij}=0$ for some i,j. Since T is connected $[T^k]_{ij}>0$ for some k, and therefore $0=[P_\pi]_{ij}\geq[T^k]_{ij}[\mathrm{P}_\pi]_{jj}$ and so $[P_\pi]_{jj}=0$, which contradicts the fact that $diag(P_\pi)>0$. Hence $P_\pi>0$, and because $T^n\to P_\pi$ $(n\to\infty)$, it now follows that $T^m>0$ for some m.

$(ii)\Leftrightarrow(iii)$. The preceding remarks show that if $\pi(T)=\{1\}$ then $T^n\to P_\pi$. Conversely if $T^n\to P_\pi$ then $\pi(T^n)\to\pi(P_\pi)=\{1\}$ $(n\to\infty)$, which implies $\pi(T)=\{1\}$.

$(ii)\Leftrightarrow(iv)$. It is clear from our remarks prior to Proposition 6 that if T is connected and $r(T)=1$ then $\pi(T)=\{1\}\Leftrightarrow\mathcal{G}(T)=P_\pi$. ■

3. Block matrix representations

The following block representation of a zero-free idempotent $P\geq0$ is well known [3, lemma 5.1.9].

Proposition 7. *Let $P\geq0$ be a zero-free idempotent matrix of rank h; then via a basis permutation P has the block matrix representation*

$$P=\begin{pmatrix} P_{11} & 0 & \dots & 0 \\ 0 & P_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & P_{hh} \end{pmatrix},$$

where all the off-diagonal blocks are zero, the diagonal blocks are square and $P_{ii}>0$ is an idempotent of rank one $(i=1,\dots,h)$.

Proof. If P is connected, since P is an idempotent, $P>0$, $h=1$ and the result holds. Assume then that via a basis rearrangement P has the block representation

$$P=\begin{pmatrix} U & 0 \\ V & W \end{pmatrix}.$$

As $P^2=P\geq0$, then $U^2=U\geq0$ and $W^2=W\geq0$ and $VU+WV=V$. Thus $WVU+WV=WV$, and so $WVU=0$. Now as P is zero-free U has no zero rows and W has no zero columns, thus $V=0$. Then U and W are both idempotents ≥0.

$$P=\begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$$

is zero-free, hence so are U and W. The result follows by further reduction until the diagonal blocks are all connected idempotents and are therefore >0. ■

(For a general block representation of ≥0 idempotent matrices see [3, lemma 5.1.9]).

Now let $T\geq0$ be connected with $r(T)=1$. If $P_\pi\ T=R$, then $\mathcal{G}(T)=\mathcal{S}(R)$ is a finite cyclic group, and we can find $S\in\mathcal{S}(R)$ such that $P_\pi R=RP_\pi=R$, $P_\pi S=SP_\pi=S$, $SR=P_\pi=RS$. Furthermore, R, $S\geq0$ and R is connected. Also R, S are zero-free since P_π is, so each block row of the block matrix R (and S) corresponding to the block representation of P_π will have at least one non-zero block. From now on we drop the subscript and write P for P_π.

Suppose that R_{ij} is a block that is not zero. Then there exist s, t such that $[R_{ij}]_{st}>0$. Now $R=PRP$ and the blocks $P_{ii}, P_{jj}>0$ $(\forall i, j)$, so, for every compatible pair m, n, $[R_{ij}]_{mn}\geq[P_{ii}]_{ms}[R_{ij}]_{st}[P_{jj}]_{tn}>0$, and thus the block $R_{ij}>0$.

We now generalise a well known result for ≥ 0 invertible matrices.

Proposition 8. Let R, S, P *be as above and such that* $RS=P=SR$. *Then* R (*and* S) *have exactly one block in each row or column that is* >0, *and the remaining blocks are zero.*

Proof. Since P is zero-free so are R and S, and so they both have (at least) one non-zero block in each row or column. Suppose $R_{1k}>0$, then $RS=P$, and so the block $P_{1i}=(RS)_{1i}=\sum_{j=1}^{h}R_{1j}S_{ji}=0$ $(\forall i>1)$. Taking $j=k$ gives $S_{ki}=0$ $(\forall i>1)$ by positivity, that is *blockrow*$_k(S)$ has exactly one non-zero block S_{k1} and $S_{k1}>0$. Reversing the order and taking transposes gives the required result. ■

Replacing each positive block of R with the number one and each zero block with the number zero gives an $h\times h$ permutation matrix, which, since R is connected, must be a single cycle. A basis permutation then ensures that R has an $h\times h$ block representation of the form

$$R=\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & R_{1h} \\ R_{21} & \cdot & \cdot & \cdot & \cdot \\ \cdot & R_{32} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & R_{h,h-1} & \cdot \end{pmatrix},$$

where R_{1h} and all blocks $\mathrm{R}_{i,i-1}>0$; all others are zero.

Consider the equivalent $h\times h$ block representation of T. For each i, j block $R_{ij}=(TP)_{ij}\geq T_{ij}P_{jj}$. Since $P_{jj}>0$ we deduce that $R_{ij}=0$ implies that $T_{ij}=0$. Thus the block representation T is subservient to that of R, in the sense that its non-zero blocks can only occur in positions i, j in which $R_{ij}>0$.

Finally observe that by [1; 2] if $|\lambda|=1$ and $P(\lambda; R)$ denotes the spectral projection of the point λ associated with the linear operator R, then $n^{-1}\sum_{k=1}^{n}\lambda^{-k}R^{k}\rightarrow P(\lambda; R)$ $(n\rightarrow\infty)$, where $P(\lambda; R)\neq 0$ if and only if $\lambda\in\pi(R)$. But since $R^{h+1}=R$ choosing λ such that $\lambda^{h}=1$ gives $h^{-1}\sum_{k=1}^{h}\lambda^{-k}R^{k}=P(\lambda; R)$. To show that every h^{th} root of unity is an eigenvalue of R observe that, from our $h\times h$ block representation of R, $diag(R^{k})=0$ $(k=1,\ldots, h-1)$, but that $diag(R^{h})=diag(P)>0$. Thus $P(\lambda; R)\neq 0$ if and only if $\lambda^{h}=1$.

With this block matrix representation for T let D be the block diagonal matrix $D=diag(e^{i\omega}I_1, e^{2i\omega}I_2,\ldots, e^{hi\omega}I_h)$, where $\omega=2\pi/h$. Then $DTD^{-1}=e^{i\omega}T$ and the whole spectral theory of T is invariant under rotations by multiples of ω.

Now recall that the trace of T^{n} is given by $tr(T^{n})=\sum_{i=1}^{p}\lambda_i^{n}$, where $\sigma(T)=\{\lambda_i;\ 1\leq i\leq p\}$.

Proposition 9. *If* $T\geq 0$ *and* $r(T)=1$ *then* T *is primitive* $\Leftrightarrow$ T *is connected and* $tr(T^{n})\rightarrow 1$ $(n\rightarrow\infty)$.

PROOF. Let T be primitive. Then, by Proposition 3(iii), $T^n \to P_\pi$, and so $tr(T^n) \to tr(P_\pi) = 1$. Conversely let T be connected and, as before, set $P_\pi T = R$. The above discussion shows that the eigenvalues of R are precisely the h^{th} roots of unity for some positive integer h, and therefore $tr(R^n) = h$ whenever n is divisible by h; otherwise $tr(R^n) = 0$. However, $\pi(T) = \pi(R)$ and as $n \to \infty$ the n^{th} powers of $\sigma(T) \backslash \pi(T)$ go to zero. Hence $tr(T^n) - tr(R^n) \to 0$ as $n \to \infty$ so $tr(T^n)$ is convergent if and only if $h = 1$ and the limit in this particular case is always 1. ■

Corollary. *If* $T \geq 0$ *and* $r(T) = 1$, *then* T *is primitive* $\Leftrightarrow$ T *is connected and* $\{tr(T^n)\}_1^\infty$ *is a convergent sequence.*

ACKNOWLEDGEMENTS

The author is indebted to Dr Roger Smyth for his many constructive comments during the preparation of this paper. The support of the Trinity College Maintenance Grants Scheme is gratefully acknowledged.

REFERENCES

[1] M.A. Kaashoek and T.T. West, Locally compact monothetic semi-algebras, *Proceedings of the London Mathematical Society* **18** (1968), 428–38.

[2] M.A. Kaashoek and T.T. West, *Locally compact semi-algebras with applications to spectral theory of positive operators*, North-Holland Mathematics Studies 9, North-Holland, Amsterdam, 1974.

[3] H. Radjavi and P. Rosenthal, *Simultaneous Triangularization*, Springer-Verlag, New York, 2000.

[4] M.R.F. Smyth and T.T. West, The minimum diagonal element of a positive matrix, *Studia Mathematica* **131** (1) (1998), 95–99.

[5] M.R.F. Smyth, A spectral theoretic proof of Perron–Frobenius, *Mathematical Proceedings of the Royal Irish Academy* **102**A (1) (2002), 29–35.