BANACH ALGEBRAS WITH A DISCRETE STRUCTURE SPACE

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Abstract

Let A be a semi-simple Banach algebra and Γ be its closed two-sided ideals. For $K \in \Gamma$ let K^a denote its annihilator in A. The mapping $K \to K^a$ is shown to be an injective map of Γ into Γ if and only if it is surjective and if and only if both the structure space of A is discrete and A/K is semi-simple for every $K \in \Gamma$. If A has these properties so does K and A/K for each $K \in \Gamma$. Such A is called bidual. Every $K \in \Gamma$ has a dense socle if and only if A is a bidual modular annihilator algebra. Applications are made to the theory of annihilator algebras.

1. Introduction

Throughout A denotes a semi-simple Banach algebra. By an ideal we always mean a two-sided ideal unless otherwise specified. For a subset W of A let $L(W)$ ($R(W)$) denote the left (right) annihilator of W in A. If W is an ideal in A, by [4, p. 462], $L(W) = R(W)$. We then set $W^a = L(W) = R(W)$. Let $\mathfrak{P}(A)$ denote the structure space of A (space of its primitive ideals). It is known [17, theorem 3.14] that $\mathfrak{P}(A)$ is discrete if and only if $P^a \neq 0$ for all $P \in \mathfrak{P}(A)$. Thus the annihilation of ideals plays a central role whenever $\mathfrak{P}(A)$ is discrete.

Let $\Gamma = \Gamma(A)$ be the set of closed ideals of A and let ϕ denote the mapping $K \to K^a$ of Γ into Γ. We show that ϕ is injective if and only if ϕ is surjective and if and only if both $\mathfrak{P}(A)$ is discrete and A/K is semi-simple for each $K \in \Gamma$. That ϕ is surjective is equivalent to the statement that $K = K^{aa}$ for all $K \in \Gamma$. By analogy with the more special dual algebras of Kaplansky [7] we say that A is bidual in that case. The definitions for dual algebras is entirely in terms of one-sided ideals while that for bidual algebras are entirely in terms of two-sided ideals. It is shown that a C^*-algebra A is bidual if and only if $\mathfrak{P}(A)$ is discrete.

A basic property of a bidual algebra A is that K and A/K are also bidual algebras of each $K \in \Gamma$. This is used in applying our results to modular annihilator algebras and annihilator algebras that are defined in terms of one-sided ideals. A careful treatment for these algebras is given in [8, chapter 8]. For an annihilator algebra A the following are equivalent: (1) every $K \in \Gamma$ is an annihilator algebra; (2) every A/K, $K \in \Gamma$, is a semi-simple annihilator algebra; and (3) A is a bidual algebra.

Thus bidual algebras A have strong properties not possessed by those not bidual. Of course every dual algebra A is bidual. For examples see [7]. Every topologically simple algebra A is automatically bidual. We give examples of algebras A that are not topologically simple and bidual but are not dual.

A semi-simple annihilator algebra that is not bidual was fashioned in [6].

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2. Bi-annihilator algebras

Throughout A denotes a semi-simple Banach algebra. By an ideal we always mean a two-sided ideal unless otherwise specified. We use Γ to mean the set of all closed ideals of A. The structure space (space of primitive ideals) of A is denoted by $\mathfrak{P}(A)$. We say that A is a *bi-annihilator algebra* if $K^a \neq 0$ for all $K \neq A$, $K \in \Gamma$ and that A is a *bidual algebra* if $K = K^{aa}$ for all $K \in \Gamma$. These notions are connected by [14, corollary 2.9], which states that if A is a bi-annihilator algebra, then A is a bidual algebra if and only if every $K \in \Gamma$ is a bi-annihilator algebra. This result is a consequence also of Theorem 2.2 and Theorem 4.5 below.

Lemma 2.1. *Suppose that* $\mathfrak{P}(A)$ *is discrete and* $K \in \Gamma$. *Then*:

(1) $K^a = (0)$ *if and only if* A/K *is a radical algebra*;
(2) K^{aa}/K *is a radical algebra*;
(3) A/K *is semi-simple if and only if* $K = K^{aa}$.

Proof. (1) Suppose that $K^a = (0)$. We show that $P \supset K$ for no $P \in \mathfrak{P}(A)$. For if $P \supset K$ then $P^a \subset K^a = (0)$. But $P^a \neq (0)$ by [17, theorem 3.14]. But then, by [5, proposition 1, p. 205], A/K has no primitive ideal and so it is a radical algebra.

Suppose A/K is a radical algebra and let π be the natural homomorphism of A onto A/K. Then $\pi(K^a)$ is an isomorphic copy of K^a and is a semi-simple ideal in the radical algebra A/K. Therefore $\pi(K^a) = (0)$ so that $K^a = (0)$.

(2) By [5, proposition 2, p. 206]. $\mathfrak{P}(K^{aa})$ is discrete. Since $K^a \cap K^{aa} = (0)$ we see that K^{aa}/K is a radical algebra by (1).

(3) If $K = K^{aa}$ then A/K is semi-simple by [19, lemma 2.5]. Suppose that A/K is semi-simple. Now $K^{aa}K$ is an ideal of A/K and so, by (2), we see that $K = K^{aa}$. ■

Theorem 2.2. *The following statements are equivalent*:

(1) $\mathfrak{P}(A)$ *is discrete and* A/K *is a radical algebra for no* $K \in \Gamma$;
(2) A *is a bi-annihilator algebra*;
(3) A *is the direct topological sum of its minimal closed ideals.*

Proof. That (1) and (2) are equivalent follows from [17, theorem 3.14] and (1) of Lemma 2.1. That (2) and (3) are equivalent was shown in [14, corollary 2.7]. ■

Let Δ denote the set of minimal closed ideals of A.

Theorem 2.3. *Suppose that* A *is a bi-annihilator algebra. Then*

$$\mathfrak{P}(A) = \{K^a : K \in \Delta\}$$

for $K \in \Delta$, $K = P^a$ where *for some* $P \in \mathfrak{P}(A)$ *if and only if* $K = K^{aa}$.

Proof. By [14, lemma 5.1] each K^a, for $K \in \Delta$, is a primitive ideal. By Theorem 2.2 we have $\cap\{K^a : K \in \Delta\} = (0)$. Hence the set of K^a for $K \in \Delta$ is a dense subset of $\mathfrak{P}(A)$. Since: $\mathfrak{P}(A)$ is discrete, we have $\mathfrak{P}(A) = \{K^a : K \in \Delta\}$.

If $K=P^a$ for some $P\in\mathfrak{P}(A)$ then that P is unique. For let $K=P_1^a=P_2^a$ where each $P_j\in\mathfrak{P}(A)$. Then (see [17, lemma 3.1], $P_1=P_1^{aa}=P_2^{aa}=P_2$. Moreover, $K^{aa}=P^{aaa}=P^a=K$.

Conversely suppose $K=K^{aa}$, where $K\in\Delta$. As noted above $K^a=P$ for some $P\in\mathfrak{P}(A)$ so that $K=K^{aa}=P^a$. ■

Theorem 2.4. *Let A be a bi-annihilator algebra. Let Δ_1 be the set of minimal closed ideals of A with an identity. Then $P\in\mathfrak{P}(A)$ is a modular maximal ideal if and only if $P=K^a$ for some $K\in\Delta_1$.*

PROOF. By [14, theorem 5.5] if $P=K^a$, where $K\in\Delta$, then P is a modular maximal ideal. Then, as $P^a\neq(0)$, we have $P^a=K\in\Delta_1$ and $P=P^{aa}=K^a$. ■

As in [9, p. 59] by the strong radical $\mathfrak{R}(A)$ of A we mean the intersection of its modular maximal ideals.

A is said to be *strongly semi-simple* if $\mathfrak{R}(A)=(0)$.

Theorem 2.5. *Let A be a bi-annihilator algebra. The following statements are equivalent*:

(1) *A is strongly semi-simple*;
(2) *Every $K\in\Delta$ has an identity*;
(3) *Every primitive ideal is a modular maximal ideal.*

PROOF. As $\mathfrak{P}(A)$ is discrete (1) implies (3) by [17, theorem 4.4]. As A is semi-simple (3) implies (1). By Theorem 2.4, (2) and (3) are equivalent. ■

The situation of Theorem 2.5 holds, for example, for the group algebra of a compact group.

3. On the mapping $K\to K^a$

Let ϕ denote the mapping $K\to K^a$ of Γ into Γ. We show ϕ is injective if and only if ϕ is surjective and if and only if A is a bidual algebra.

Lemma 3.1. *ϕ is surjective if and only if A is bidual. In this case ϕ is injective.*

PROOF. Suppose that ϕ is surjective. Let $K\in\Gamma$. Then $K=W^a$ for some $W\in\Gamma$; therefore $K^{aa}=W^{aaa}=W^a=K$. Hence A is bidual. To see that ϕ is then injective, suppose $K_1^a=K_2^a$ for $K_1, K_2\in\Gamma$. Then $K_1^{aa}=K_2^{aa}$ or $K_1=K_2$. ■

Theorem 3.2. *ϕ is injective if and only if ϕ is surjective.*

PROOF. Suppose ϕ is injective. Then $K^a\neq 0$ for all $K\in\Gamma$, $K\neq A$ as $A^a=(0)$. Thus A is a bi-annihilator algebra.

Let $K\in\Gamma$. As pointed out in [14, p. 311], any closed ideal in K is an ideal in A. We show that $K\neq A$ is a bi-annihilator algebra. For let W be a closed ideal in K, $W\neq K$. Then $W^a\neq(0)$ in A. We must show that $W^a\cap K\neq(0)$. Suppose that $W^a\cap K=(0)$; then $W^aK=(0)$

so that $W^a \subset K^a$. But $W \subset K^a$ so that $W=K$. Hence K is a bi-annihilator algebra. It then follows by [14, corollary 2.9] that A is bidual. Then ϕ is surjective by Lemma 3.1.

For a bidual algebra A the mapping ϕ is a one-to-one mapping $\mathfrak{P}(A)$ onto Δ, the set of minimal closed ideals of A. That $\phi(\mathfrak{P}(A))=\Delta$ is given by Theorem 2.3. ■

4. Bidual algebras

We undertake a closer examination of bidual algebras.

Theorem 4.1. *A is a bidual algebra if and only if* $\mathfrak{P}(A)$ *is discrete and* A/K *is semi-simple for each* $K \in \Gamma$.

Proof. Suppose A is a bidual algebra. Then $P^a \neq (0)$ for all $P \in \mathfrak{P}(A)$ and $\mathfrak{P}(A)$ is discrete by [17, theorem 3.14]. That A/K is semi-simple for each $K \in \Gamma$ is given by Lemma 2.1. The converse also follows from Lemma 2.1. ■

Corollary 4.2. *For a C*-algebra A, A is a bidual algebra if and only if* $\mathfrak{P}(A)$ *is discrete.*

Proof. As is well known A/K is a C^*-algebra and so is semi-simple for each $K \in \Gamma$. Then we apply Theorem 4.1. ■

This result also follows from Lemma 2.1 where it is shown that K^{aa}/K is a radical algebra for each $K \in \Gamma$ if $\mathfrak{P}(A)$ is discrete.

Lemma 4.3. *Let A be a bidual algebra and* $K \in \Gamma$. *If* $x \in A$ and $xA \subset K$, *then* $x \in K$.

Proof. Let π be the natural homomorphism of A onto A/K. Then $\pi(x)\pi(A)=(0)$ in A/K and A/K is semi-simple by Theorem 4.1. Hence $\pi(x)=0$. ■

Our next result is basic for the sequel.

Theorem 4.4. *Let A be a bidual algebra and* $K \in \Gamma$. *Then K and A/K are bidual algebras.*

Proof. First of all $\mathfrak{P}(K)$ and $\mathfrak{P}(A/K)$ are discrete by Theorem 4.1 and [5, pp 205–6]. Let W be a closed ideal in K. As noted above W is an ideal in A. Then A/W is semi-simple by Theorem 4.1. But K/W is an ideal of A/W and so is semi-simple. Theorem 4.1 then assures us that K is a bidual algebra.

We turn to a discussion of the semi-simple algebra A/K. For an ideal β in A/K let $\beta^{\#}$ denote its annihilator in A/K. We must show that $\beta^{\#\#}=\beta$ for a closed ideal β.

Let π denote the natural homomorphism of A onto A/K. Let $V=\pi^{-1}(\beta)$ and suppose that for $y \in A$, $\pi(y) \in \beta^{\#\#}$. We must show that $\pi(y) \in \beta$, or equivalently, that $y \in V$.

For $x \in A$, $\pi(x) \in \beta^{\#}$ if and only if $\pi(x)\pi(V)=(0)$ or if and only if $xV \subset K$. Hence, if $xV \subset K$, then $\pi(x)\pi(y)=0$ or $xy \in K$. Now consider $x \in V^a$. Then $xy \in K$ and $xyw=0$ for all $w \in K^a$. Therefore $yK^a \subset V^{aa}=V$. But $yK \subset K \subset V$ so that $y(K \oplus K^a) \subset V$. However, $K \oplus K^a$ is dense in A so that $yA \subset V$. Thus $y \in V$ by Lemma 4.3. ■

We provide a criterion for a bi-annihilator algebra A to be bidual. Let A be such an algebra and $K \in \Gamma$. As noted above, each closed ideal of K is an ideal of A. Therefore each minimal closed ideal of K is a minimal closed ideal of A.

Theorem 4.5. *A bi-annihilator algebra A is bidual if and only if each $K \in \Gamma$ is the direct topological sum of this minimal closed ideals.*

Proof. Suppose that A is bidual and $K \in \Gamma$. Then K is bidual by Theorem 4.4 and is a direct topological sum of its minimal closed ideals by Theorem 2.2. The converse follows from the following argument.

Let $K \in \Gamma$. Let W be a minimal closed ideal of K^{aa}. Then $W \cap K^a = (0)$. Either $W \cap K = (0)$ or $W \cap K = W$. If $W \cap K = (0)$ then $WK = (0)$ and $W \subset K^a$ contrary to $W \cap K^a = (0)$. Thus $W \subset K$ so that $K^{aa} \subset K$. Hence A is bidual. ■

5. Applications

We apply the theory of bidual algebras to algebras A whose main features are given in terms of one-sided ideals. In particular we consider algebras with a dense socle, modular annihilator algebras and annihilator algebras. A careful treatment on this topic is given in [8, chapter 8].

Lemma 5.1. *Suppose that A has a dense socle and that $K \in \Gamma$. Then $K \supset AK^{aa}$ and $K \supset (K^{aa})A$.*

Proof. Set $W = K \oplus K^a$. If eA, $e^2 = e$, is a minimal right ideal of A then by [14, lemma 5.1] either $e \in K$ or $e \in K^a$. Therefore the socle of A is contained in W so that W is dense in A.

Let $x \in W \cap \mathrm{K}^{aa}$. We have that $x = u + v$ for some $u \in K$, $v \in K^a$ and simultaneously $x \in K^{aa}$. Thus $v = x - u \in K^a \cap K^{aa} = (0)$ so that $W \cap K^{aa} \subset K$. But then $WK^{aa} \subset K$ and $K^{aa}W \subset K$. As W is dense in A, $K \supset AK^{aa}$ and $K \supset (K^{aa})A$. ■

Theorem 5.2. *Suppose that A has a dense socle and that, for each $x \in A$, either x lies in the closure of xA or x lies in the closure of Ax. Then A is a bidual algebra.*

Proof. By Lemma 5.1 we see that $K^{aa} = K$ for each $K \in \Gamma$ so that A is a bidual algebra. ■

Theorem 5.3. *The following statements are equivalent*:

(1) *Each $K \in \Gamma$ has a dense socle*;

(2) *A is a modular annihilator bidual algebra.*

Proof. Assume (1) is true. As A has a dense socle then A is a modular annihilator algebra by [13, lemma 3.11]. Let $K \in \Gamma$. The socle S of K^{aa} is contained in AK^{aa}. But $AK^{aa} \subset K$ by Lemma 5.1 and S is dense in K^{aa} by (1). Therefore $K = K^{aa}$ so that A is a bidual algebra.

Next assume (2) is true. Let $K \in \Gamma$ and let S denote the socle of K. Now S is an ideal of A as well as of K by [14, p. 311]. Let $S^{\#}$ denote the set of annihilators of S in K so that $S^{\#} = S^a \cap K$. As A is a modular annihilator algebra then so is K by [13, theorem 3.7]. As noted in [13, p. 39] and also [2, theorem 4.2], $S^{\#}$ is the radical of K. But K is semi-simple so that $S^a \cap K = (0)$. Let T denote the closure of S. We have $T^aK = (0)$ so that $T^a \subset K^a$. But then $K^{aa} \subset \mathrm{T}^{aa}$. As A is a bidual algebra we see that $K \subset \mathrm{T} \subset K$ and K has a dense socle. ■

Corollary 5.4. *In a bidual Banach algebra A with a dense socle, every $K \in \Gamma$ has a dense socle.*

Proof. A is a modular annihilator algebra by [13, lemma 3.1]. We apply Theorem 5.2. ■

This situation is illuminated by the example of Johnson in [6] of a commutative semi-simple annihilator Banach algebra A that is not a dual algebra. Here dual and bidual have the same meaning. His analysis produces a closed ideal K of A that fails to have a dense socle. See also [3, p. 663].

Theorem 5.5. *Let A be a bidual modular annihilator algebra and $K \in \Gamma$. Then K and A/K are bidual modular annihilator algebras.*

Proof. That K and A/K are semi-simple bidual algebras is given by Theorem 4.4. By [13, theorem 3.7] K is a modular annihilator algebra. Now A has a dense socle by Theorem 5.3 and $K = K^{aa}$. Then A/K has a dense socle by [12, theorem 3.7] and so is a modular annihilator algebra by [13, lemma 3.1]. ■

Theorem 5.6. *Let A be an annihilator algebra. The following statements are equivalent:*

(1) *A is a bidual algebra.*

(2) *Every $K \in \Gamma$ is an annihilator algebra.*

(3) *For every $K \in \Gamma$, A/K is a semi-simple annihilator algebra.*

Proof. Suppose (1) is true and consider $K \in \Gamma$. Note that $x \in (AK)^a$ if and only if $x \in K^a$. Let T denote the closure of AK. Then $T = T^{aa} = K^{aa} = K$. By [9, theorem 2.8.12] we see that K is an annihilator algebra.

Next suppose (2) is true. Then each $K \in \Gamma$ has a dense socle by [9, corollary 2.8.16]. Theorem 5.3 then shows that A is a bidual algebra. Thus (1) and (2) are equivalent. We assume that A is a bidual annihilator algebra and wish to show that each A/K is an annihilator algebra. Let π be the natural homomorphism of A onto A/K and β be a proper closed right ideal in A/K. Then $\pi^{-1}(\beta)$ is a proper closed right ideal in the annihilator algebra A, where $\pi^{-1}(\beta) \supset K$. Let W denote the left annihilator of $\pi^{-1}(\beta)$ in A, where $W \neq 0$. As $\pi^{-1}(\beta) \supset K$ then $W \subset K^a$. Set $v \neq 0$ in W. As $v \in K^a$ we have $v \notin K$ so that $\pi(v) \neq 0$. Since $v\pi^{-1}(\beta) = (0)$ we get $\pi(v)\beta = (0)$. Likewise the right annihilator of a proper closed left ideal in A/K is non-zero. Thus (1) and (2) imply (3).

Assume (3) is true. As each A/K is semi-simple, A is bidual by Theorem 4.1. ■

For each $x \in A$ let $L_x(R_x)$ be the operator on A defined by $L_x(y) = xy$ $(R_x(y) = yx)$. Set

$$\mathfrak{R}_L = \{x \in A : \ L_x \text{ is a compact operator}\},$$

and

$$\mathfrak{R}_R = \{x \in A : \ R_x \text{ is a compact operator}\}.$$

These are closed ideals in A. In [15, theorem 4.3] it was shown that if A has a dense socle then $\mathfrak{N}_L$ if and only if $\mathfrak{N}_R=A$. In [10] Smyth gave an example of a semi-simple modular annihilator algebra where $\mathfrak{N}_L=A$ and $\mathfrak{N}_R\neq A$. We find two classes of algebras A for which $\mathfrak{N}_L=\mathfrak{N}_R$.

It is shown in [10] that for $x\in A$, xA is finite dimensional if and only if Ax is finite dimensional. Let $\mathfrak{F}$ be the closure of the set of $x\in A$ for which xA is finite-dimensional. $\mathfrak{F}$ is a closed ideal of A.

Theorem 5.7. *Let A be a bidual modular annihilator algebra. Then $\mathfrak{N}_L=\mathfrak{N}_R=\mathfrak{F}$.*

Proof. Since A is a modular annihilator algebra then $\mathfrak{N}_L^a=\mathfrak{N}_R^a=\mathfrak{R}$, where $\mathfrak{R}$ is the strong radical of A as shown in [15, theorem 3.3]. As A is a bidual algebra then $\mathfrak{N}_L=\mathfrak{N}_R=\mathfrak{R}^a$. By Theorem 5.2, A has a dense socle. Then $\mathfrak{F}^a=\mathfrak{R}$ by [18, theorem 3]. ■

Theorem 5.8. *Suppose that A has a dense socle and is strongly semi-simple. Then $\mathfrak{N}_L=\mathfrak{N}_R=\mathfrak{F}=A$.*

Proof. A is a modular annihilator algebra by [13, lemma 3.11]. By the argument of Theorem 5.7 we get

$$\mathfrak{N}_L^a=\mathfrak{N}_R^a=\mathfrak{F}^a=\mathfrak{R}=(0).$$

However, it is known that a strongly semi-simple Banach algebra with a dense socle must be an annihilator algebra by [15, theorem 4.1]. Therefore $\mathfrak{N}_L=\mathfrak{N}_R=\mathfrak{F}=A$ in this case. ■

6. Examples

Of course every dual algebra is bidual and every topologically simple algebra A is automatically bidual. We provide examples of a bidual algebra A that is not topologically simple and not dual.

For the algebra A let $c_0(A)$ be the set of all sequences $\{x_n\}$, where each $x_n\in A$ and $\lim x_n=0$. We consider $c_0(A)$ an algebra by defining $\{a_n\}\{b_n\}=\{a_nb_n\}$ and a Banach algebra with norm $\|\{a_n\}\|=\sup\|a_n\|$.

Theorem 6.1. *Let A be a bidual algebra where, for each $x\in A$, x lies in the closure of xA. Then $W=c_0(A)$ is a semi-simple bidual algebra.*

Proof. As in [9, p. 107], it is readily seen that $c_0(A)$ is semi-simple. We show first that, for each $\alpha=\{x_n\}\in W$, the closure of αW contains α. Let $\varepsilon>0$. There exists a positive integer N such that $\|x_n\|<\varepsilon$ for all $n\geq N$. For each positive integer k there exists some $a_k\in A$ such that $\|x_ka_k-x_k\|<\varepsilon$. Let $\beta=\{y_n\}$ where $y_k=a_k$ for $k=1,\ldots,N$ and $y_k=0$ for $k>N$. Then $\beta\in W$ and $\|\alpha\beta-\alpha\|<\varepsilon$. Thus α lies in the closure of αW.

For the remainder of the proof we adapt to our situation the arguments of Rickart in [9, p. 106]. Let K be a closed ideal in W. We must show that $K=K^{aa}$ in W. For each positive integer n, let A_n be the set of $\{x_k\}$ in W, where $x_k=0$ for $k\neq n$. Let $K_n=K\cap A_n$.

K_n is a closed ideal in W and also a closed ideal in A_n. Let $K_n^{\#}$ be the annihilator of K_n in A_n. Then $A_nKK_n^{\#}=(0)$. As A_n is semi-simple being a copy of A and as $KK^{\#}\subset A_n$, we see that $KK_n^{\#}=(0)$. Therefore $K_n^{\#}\subset K^a$. Now let $\alpha\in K^{aa}$. We have $K_n^{\#}\alpha A_n=(0)$ so that $\alpha A_n\subset K_n^{\#\#}=K_n\subset K$. It follows that $\alpha W\subset K$. As α lies in the closure of αW, we see that $\alpha\in K$.

It is readily seen that if A is not a dual algebra neither is $c_0(A)$. ■

We give two examples where $c_0(A)$ is a semi-simple bidual algebra that is not dual and is not topologically simple.

Example 1. Let A be the Banach algebra of all compact linear operators on the Banach space c_0. In [16] it was shown that A is left dual but not dual. Thus A is a bidual algebra. As seen in [16, p. 28] x lies in the closure of xA for each $x\in A$. Therefore, by Theorem 6.1, $c_0(A)$ is a desired example.

Example 2. Consider a semi-simple right-complemented Banach algebra B as introduced in [11]. As shown there every closed ideal in B has a dense socle. Then, by Theorem 5.3, or directly, B is a bidual Banach algebra. As shown in [1, lemma 3], x lies in the closure of xB for each $x\in B$.

In [2, p. 139] Alexander gave an example of a semi-simple complemented Banach algebra A that is not a dual algebra. Then, by Theorem 6.1, $c_0(A)$ is a desired example.

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