CALCULATING THE TRACE OF PROJECTIONS IN CERTAIN CROSSED PRODUCT C^*-ALGEBRAS

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Abstract

This paper studies the range of the trace on projections in certain crossed product C^*-algebras without explicitly embedding them in an A.F. algebra.

In [10], Pimsner and Voiculescu showed how to embed the irrational relation C^*-algebra $\mathscr{A}_\theta$ in a particular approximately finite C^*-algebra that had been constructed by Effros and Shen [4]. One of the important consequences of the embedding was that it limited the possible values of the trace on projections in $\mathscr{A}_\theta$, and taken with Rieffel's results [12], this gave a classification of the $\mathscr{A}_\theta$.

In [3], Davidson gave another version of the embedding and in [9], Pimsner gave a general theorem with necessary and sufficient conditions for a crossed product C^*-algebra to be embeddable in an A.F. algebra, based on the notion of pseudo-orbits. Loring [5] gave a version of this general embedding in terms of pseudo-actions.

What worked in [10] was that Pimsner and Voiculescu were able to determine the range of the trace on the projections in the A.F. algebra, and hence limit the possible values on $\mathscr{A}_\theta$. Here we put forward a method for trying to determine or limit the values of a trace on certain algebras. The method we give avoids mention of any embedding, but we point out that it can be understood as constructing an A.F. algebra, with easily computed tracial range, into which the algebra may be embedded.

Our progress is in two steps. Theorem 1 shows that the norms of certain operators in the crossed product can be approximated by the norm of operators in which ϕ is replaced by an approximating sequence of periodic maps ϕ_n. Later in Theorem 2 it is shown that with a further connection between each ϕ_n and ϕ_{n+1} we are lead to the desired results on the range of a trace.

Let ϕ be a homeomorphism on a compact metric space X, and let $\mathscr{A} = C^*(X, \phi)$ be the crossed product C^*-algebra [6]. For all $n \in Z$ there are maps $F_n : C(X) \to C^*(X, \phi)$, where $F_n(f) \in l^1(X, \phi)$ is defined by

$$F_n(f)(k) = \begin{cases} f & \text{if } k \neq n \\ 0 & \text{if } k = n \end{cases}$$

and maps $E_n : C^*(X, \phi) \to C(X)$ such that $E_n \circ F_n = id_{C(X)}$.

If τ is any trace on $\mathscr{A}$, then we can define a linear functional τ_n on $C(X)$ by $\tau_n(f) = \tau(F_n(f))$. Each $\tau_n = \int f d\mu_n$ for some Borel measure on X, and since τ is a trace,

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$\tau_n(f)=\tau_n(f\circ\phi)$ implies that each μ_n is ϕ invariant. If A is an idempotent in $\mathscr{A}$, then it can be shown that $\tau(A)=\int E_0(A)d\mu_0$ (see [2]), and so for idempotents we can ignore any other μ_n. We do this and put $\mu_0=\mu$.

We consider the covariant representation π_μ of $\mathscr{A}$ on $L^2(X,\mu)$ as follows:

$$\pi_\mu(\mathscr{A})=C^*\{U_\phi,\ M_f: f\in C(X)\},$$

where $M_f g=f\cdot g$ and $U_\phi g=g\circ\phi$, for all $g\in L^2(X,\mu)$. For every $A\in\mathscr{A}$, $\pi_\mu(A)$ has a Fourier series $\pi_\mu(A)\sim\sum_{-\infty}^{\infty}M_{f_k}U_\phi^k$. Analogously to the continous functions on the unit circle, it is the Caesaro sums of the series that converge, and $\tau(A)=\int f_0\,d\mu$, so for calculating τ on $\mathscr{A}$ we can replace $\mathscr{A}$ by $\pi_\mu(\mathscr{A})$. We consider this done and drop the π_μ.

Next we want to restrict to maps ϕ that can be approximated in a certain way by periodic maps. Specifically we suppose that for each n, X can be partitioned into a finite collection of μ-measurable, preferably connected, disjoint sets as follows:

$$\{C^n_{i,j}\},\quad i=1,\ h(j,n),\ j=1,\ w(n)$$

with

$$\phi(C^n_{i,j})=C^n_{i+1,j}\quad\text{for all } j,\ n,\ i\neq h(j,n).$$

This can be represented diagrammatically as follows:

$$\begin{array}{lcccc}
X= & C^n_{1,1} & C^n_{1,2} & \cdots & C^n_{1,w(n)}\\
 & \downarrow\phi & \downarrow\phi & & \downarrow\phi\\
 & C^n_{2,1} & C^n_{2,2} & & C^n_{2,w(n)}\\
 & \downarrow\phi & \vdots & & \downarrow\phi\\
 & \vdots & \downarrow\phi & & \vdots\\
 & \downarrow\phi & C^n_{h(2,n),2} & & \downarrow\phi\\
 & C^n_{h(1,n),1} & & & C^n_{h(w(n),n),w(n)}
\end{array}$$

The above type of partitions can occur, as shown in the two examples that follow.

First let ϕ be an irrational rotation on S^1 [7]. Suppose the angle is $2\pi\theta$, with $\theta<\frac{1}{2}$. Choose any point x in S^1 and take a maximal connected subset A containing x, such that A and $\phi(A)$ are disjoint. A will be a half-open half-closed interval of length θ. Keep iterating $A,\ \phi(A),\ldots,\ \phi^{n_1}(A)$, until $\phi^{n_1+1}(A)\cap A\neq\phi$. Switching to $[0,1)$, and by translation with θ, we may take A to be $[0,\theta)$. Then if $n_1=3$, we have the following picture:

$$\overline{\quad\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$0\qquad\theta\qquad 2\theta\qquad 3\theta\qquad 1\ .$$

This gives the partition

$$\begin{array}{llcl}
X= & 0\,\underline{\qquad}\,\theta & & 3\theta\,\underline{\qquad}\,1\\
 & \qquad\downarrow & & \\
 & \theta\,\underline{\qquad}\,2\theta & & \\
 & \qquad\downarrow & & \\
 & 2\theta\,\underline{\qquad}\,3\theta & &
\end{array}$$

So $C^1_{1,1}=[0,\theta)$ and $C^1_{1,2}=[3\theta,1)$.

If $\theta>\frac{1}{2}$, then the maximal set will be of length $1-\theta$, and the decomposition is into two pieces.

The next step is to consider the bottom piece of each stack. ϕ will map the union of these to the union of the two top pieces. We have the following:

$$\phi([3\theta,1))\subset[0,\theta),\quad \phi^2([3\theta,1))\subset[\theta,2\theta),\quad \phi^3([3\theta,1))\subset[2\theta,3\theta),$$

by chopping off a substack of the first stack, and then either

$$\phi^4([3\theta,1))\cap[3\theta,1)\neq\emptyset \quad\text{or}\quad \phi^4([3\theta,1))\subset[0,\theta).$$

If it is the former we stop; if it is the latter then

$$\phi^5([3\theta,1))\subset[\theta,2\theta] \quad\text{and}\quad \phi^6([3\theta,1))\subset[2\theta,3\theta),$$

and we have chopped off another complete substack of the first stack.

Next $\phi^7([3\theta,1))\cap[3\theta,1)\neq\theta$, and we stop, or again we get another complete substack of the first stack. Suppose the former so that this stage stops. Then if $C^2_{1,1}=[3\theta,1)$ and $C^2_{1,2}=[0,\theta)\backslash(\phi([3\theta,1))\cup\phi^4(3\theta,1))=[0,7\theta)$, we have a partition into two stacks again, where the second is a substack of the previous first stack.

$$\begin{array}{rcc} X= & C^2_{1,1} & C^2_{1,2} \\ & \downarrow & \downarrow \\ & C^2_{2,1} & C^2_{2,2} \\ & \downarrow & \downarrow \\ & \vdots & C^2_{3,2} \\ & C^2_{6,1} & \end{array}$$

For the next step, observe that $\phi(C^2_{3,2})\subset C^2_{1,1}$, i.e. $[2\theta,9\theta)\to[3\theta,1)$, and again we keep chopping off substacks of the first stack for as long as we can. We have to stop when the translate of $\phi(C^2_{3,2})$ is not disjoint from $C^2_{1,2}$. Again there is a substack of the first stack that has not been chopped off. The subset of $C^2_{1,1}$ with which this starts is $C^3_{1,2}$, and the subset of $C^3_{1,1}$ is $C^2_{1,2}$. Then again we get two stacks in the partition, as follows:

$$\begin{array}{rcc} X= & C^3_{1,1} & C^3_{1,2} \\ & \downarrow & \downarrow \\ & C^3_{2,1} & C^3_{2,2} \\ & \vdots & \vdots \\ & \vdots & \downarrow \\ & \vdots & C^3_{6,2} \\ & \downarrow & \\ & C^3_{h(1,3),1} & \end{array}$$

We can continue in this way. At the nth stage we have a partition with two stacks. So $w(n)=2$, and the heights $h(1,n)$ and $h(2,n)$ are q_n and q_{n-1}, the denominators of the sequence of best rational approximants for θ, i.e. the continued fraction expansion for θ [7].

Next we consider the Khakutani adding machine [8]. Here $X=\prod_1^\infty\{0,\ 1\}$, and ϕ takes any sequence and changes all the elements up to and including the first zero, from 1 to 0 or 0 to 1. Then for $n=1$ we take

$$\begin{array}{c} X=\{0,\ *,\ *,\ \ldots\} \\ \downarrow \\ \{1,\ *,\ *,\ \ldots\}. \end{array}$$

So $h(1,1)=2$ and $w(1)=1$.

For $n=2$ we can take

$$\begin{array}{c} X=\ \{0,\ 0,\ *,\ *,\ \ldots\} \\ \downarrow \\ \{1,\ 0,\ *,\ *,\ \ldots\} \\ \downarrow \\ \{0,\ 1,\ *,\ *,\ \ldots\} \\ \downarrow \\ \{1,\ 1,\ *,\ *,\ \ldots\}. \end{array}$$

So $h(1,2)=4$ and $w(2)=1$.

At the nth stage we take

$$\begin{array}{cc} X=C^n_{1,1}=\{0,\ldots,0, & *,\ *,\ \ldots\} \\ \downarrow & \downarrow \\ \vdots & \vdots \\ \downarrow & \downarrow \\ C^n_{2^n,1}=\{1,\ldots,1, & *,\ *,\ \ldots\}. \end{array}$$

So $h(1,n)=2^n$ and $w(n)=1$.

Returning to the general case, let $E_n=\cup_j C^n_{h(j,n),j}$ be the union of all the bottom pieces. Then define ϕ_n as follows:

$$\phi_n(x)=\begin{cases} \phi(x) & \text{if } x\notin E_n \\ \phi^{-h(j,n)+1}(x) & \text{if } x\in C^n_{h(j,n),j} \end{cases},$$

and define $U_{\phi_n}f=f\circ\phi_n$ for all $f\in L^2(X,\mu)$. Then U_{ϕ_n} is unitary since ϕ_n is μ-invariant. Let $\mathscr{A}_n=C^*\{U_{\phi_n},\ M_f; f\in C(X)\}$. Let $h(n)=\min_j\{h(j,n)\}-1$. Then $I,\ U_{\phi_n},\ldots,\ U_{\phi_n}^{h(n)}$ are linearly independent as elements of a $C(X)$-module, i.e.

$$T=\sum_0^{h(n)} M_{f_k}U^k_{\phi_n}=0 \Longleftrightarrow M_{f_k}=0 \quad \text{for all } k,$$

and $U_{\phi_n}^{*[h(n)/2]},\ldots,I,\ldots,U_{\phi_n}^{[h(n)/2]}$ are also linearly independent, where $[e]$ denotes the integer part of e. For ϕ_n to approximate ϕ we require $h(n)\to\infty$ and diam $E_n\to 0$ as $n\to\infty$. These are similar to the conditions that Loring used in [5] for his embedding, but he and Pimsner [9] dealt with open covers, not partitions. Open covers do not lead to results on the range of the trace and do not lead from the irrational rotation C^*-algebras to the Effros–Shen A.F. C^*-algebras. Also we have no conditions here relating C^n_{ij} to C^{n+1}_{ij}, and thus no embedding of $\mathscr{A}_n$ into $\mathscr{A}_{n+1}$.

Theorem 1. *If ϕ_n approximates ϕ as above and $A=\sum_{-N}^{N} M_{f_k} U_\phi^k \in \mathscr{A}$, let $A_n=\sum_{-N}^{N} M_{f_k} U_{\phi_n}^k \in \mathscr{A}_n$, and then $\|A_n\| \to \|A\|$. In fact we will prove more. If $V_n=U_\phi$ on $L^2(X \backslash E_n, \mu)$, V_n unitary, then*

$$\left\| \sum_{-N}^{N} M_{f_k} V_n^k \right\| \to \|A\|.$$

PROOF. If $L<h(n)$ we can write $L^2(X, \mu)=H_0 \oplus \bigoplus_{i=1}^{L-1} H_i$, where

$$H_i=L^2(\phi^{-L+i+1}(E_n), \mu), \quad i=1, L-1.$$

Then $V_n^* H_i = U_\phi^* H_i = H_{i+1}$, $i=1, L=-1$ and $V_n^*|_{H_i}=U_\phi^*|_{H_i}$, $i \neq L-1$.

Then Loring's lemma [5, lemma 2.1] says that there exists a unitary W such that:

(i) $\|W V_n^* W^* - U_\phi^*\| < \pi/L$;
(ii) $W H_i = H_i$, for all i;
(iii) $W|_{H_0} = id_{H_0}$.

So given $\varepsilon>0$, let $L>\varepsilon/(4\pi N^2)$. Then for all n with $h(n)>L$, say $n \geq N_1$, we have W_n, each using this fixed L, such that

$$\|W_n V_n^k W_n^* - U_\phi^k\| < \frac{\varepsilon}{4N} \quad \text{for all } k=-N, N,$$

where $\|A^k - B^k\| \leq \|A-B\||k|$ for unitary A and B.

Next consider the continuous functions $\{f_k \circ \phi^j\}$, $k=-N, N$, $j=1, L-1$. Since $\operatorname{diam}(E_n) \to 0$ the functions become approximately constant on E_n, so we can find N_2 such that $n \geq N_2$ implies that for each k, j there exists a constant d_{kj} such that $|f_k \circ \phi^j(x) - d_{kj}| < \varepsilon/(8N)$, for all $x \in E_n$. Then on each H_j, $j=1, L-1$, we have $\|M_{f_k} - d_{kj} I\| < \varepsilon(8N)$, for all k.

Since $W_n H_j \subset H_j$, $j=1, L-1$, and $W_n H_0 = Id_{H_0}$, it follows that $\|W_n M_{f_k} W_n^* - M_{f_k}\| < \varepsilon/(4N)$.

Combining this with $\|W_n V_n^{*k} W_n^* - U_\phi^{*k}\| < \varepsilon/(4N)$, and if $n > \max(N_1, N_2)$, we obtain

$$\left\| W_n \sum_{-N}^{N} M_{f_k} V_n^k W_n^* - \sum_{-N}^{N} M_{f_k} U_\phi^k \right\| < \varepsilon,$$

which gives

$$\left\| \sum_{-N}^{N} M_{f_k} V_n^k \right\| - \left\| \sum_{-N}^{N} M_{f_k} U_\phi^k \right\| < \varepsilon,$$

and hence the desired result. ■

Note: In the sequel we need

$$\left\| \left(\sum_{-N}^{N} M_{f_k} V_n^k \right)^2 - \sum_{-N}^{N} M_{f_k} V_n^k \right\| \to \left\| \left(\sum_{-N}^{N} M_{f_k} U_\phi^k \right)^2 - \sum_{-N}^{N} M_{f_k} U_\phi^k \right\|.$$

The theorem does not immediately apply as we will have to compare terms like

$$M_{f_n} V_n^N M_{f_n} V_n^{*N} V_n^{2N}$$

with the term $M_{f_N} M_{f_N \circ \phi^{-N}} U_\phi^{2N}$.

However, to achieve the desired results, we need only take $L > \varepsilon/(8\pi N^2)$ above and choose N_2 such that

$$\|f_i \circ \phi^j(x) - C_{ij}\| < \frac{\varepsilon}{16N},$$

for some C_{ij}, for all $i = -N, N$ and all $j = -N - L, N$.

Remark. If ϕ has a point whose orbit is dense in support of μ, then one can take a faithful representation of $\mathscr{A}$ on $L^2(X, \nu)$, where ν is a point mass at each point of the orbit. Then $\sum_{-N}^{N} M_{f_k} U_\phi^k$ is represented as a doubly infinite matrix with bandwidth $2N+1$. Each $\sum_{-N}^{N} M_{f_k} U_{\phi_n}^k$ will be represented as larger and larger block diagonals. It is easy to see what the norm statement in the theorem is saying, and one might imagine being able to prove it without the Berg technique [1]. That would be interesting but not impossible.

Now let us consider $\mathscr{A}_n$ more closely. We can consider a matrix decomposition corresponding to the sets $C_{i,j}^n$, i.e. write $L^2(X) = \bigoplus_{i,j} L^2(C_{ij}^n)$. With respect to this decomposition, M_f is just a diagonal operator $\bigoplus_{i,j} M_{f|_{C_{i,j}}}$, and U_{ϕ_n} is a direct sum of square blocks of sizes $h(1, n)$ to $h(w(n), n)$. The algebra generated is a direct sum of algebras with $\sum_0^{h(j,n)} M_{f_k} U_{\phi_n}^k$, where the f_k are no longer continuous on X, but only on the disjoint union of $C_{i,j}^n$, as the entries in the j-block.

Now if A is a subset of X, then $\{f \in C(A) : f = g|_A$ and $g \in C(X)\}$ is a C^*-algebra, and since these all extend to the closure of A, then $\bar{A}$, and any $h \in C(\bar{A})$ can be extended to X by Tietze, and this algebra is (isomorphic to) $C(\bar{A})$.

Since $\phi(C_{i,j}^n) = C_{i+1,j}^n$, if $i < h(j, n)$, and ϕ is μ invariant, the map $f_{ij} \to f_{1,j} \circ \phi^{i-1}$ is a unitary map $L^2(C_{ij}^n) \to L^2(C_{i,j}^n)$. With this map on each piece we get a unitary map $V: \bigoplus_{ij} L^2(C_{ij}^n) \to \bigoplus_j (\bigoplus (h(j, n)$ copies of $L^2(C_{1,j}^n)))$. Under this V, U_{ϕ_n} is unitarily equivalent to a direct sum of scalar shift matrices, and

$$\mathscr{A}_n \cong C(\overline{C_{1,1}^n}) \otimes M_{h(1,\ n)} \oplus \cdots \oplus C(\overline{C_{1,w(n)}^n}) \otimes M_{h(w(n),\ n)},$$

where M_n denotes the $n \times n$ matrices.

We define a trace on $\mathscr{A}_n$ by

$$t_n(A) = \sum \int g_{ii}\, d\mu,$$

where g_{ii} are the functions on the diagonal that comes with the original matrix decomposition. Note that we get the same result if we use the diagonal in the last representation.

Let us return to our trace on $\mathscr{A}$. Suppose $\mathscr{A}$ is a self-adjoint idempotent $\mathscr{A} \sim \sum_{-\infty}^{\infty} M_{f_k} U_\phi^k$. The Caesaro means of A converge to A, i.e. if

$$\sigma_N(A) = \sum_{-N}^{N} M_{f_k} \frac{N - |k|}{N} U_\phi^k,$$

then $\sigma_N(A) \to A$.

So $\forall i$, choose $N_{1,i}$ such that $\|\sigma_{N_{1,i}}(A)-A\|<1/i$.

Now if A is self-adjoint, then $\sigma_N(A)$ is also, but $\sigma_N(A_n)$, i.e. $\sigma_N(A)$ with ϕ replaced by ϕ_n is not necessarily so. However, the proof of Theorem 1 shows that $\|(\frac{1}{2})(\sigma_N(A_n)+\sigma_N(A_n)^*)\|\to\|A\|$. So we let $\sigma_N^R(A_n)=(\frac{1}{2})(\sigma_N(A_n)+\sigma_N(A_n)^*)$.

The usual method of producing self-adjoint idempotents is now used. If $\|T^2-T\|<\frac{1}{4}$. and T is self-adjoint, then $\frac{1}{2}$ does not belong to $sp(T)$, the spectrum of T. So we have an idempodent P in $C^*(T)$ corresponding to the continuous function on the spectrum

$$g(x)=\begin{cases}0 & \text{if } x<\frac{1}{2}\\ 1 & \text{if } x>\frac{1}{2}\end{cases}.$$

If $\|T\|$ is close to 1 then $P\neq 0$. The smaller $\|T^2-T\|$ is, the closer $Sp(T)$ must be to a subset of $\{0, 1\}$, and hence the smaller $\|P-T\|=sup_x\|g(x)-x\|$ is.

Now for every $i\geq 1$, by Theorem 1 choose $N_{2,i}$ such that if $n\geq N_{2,i}$, we have:

(i) $h(n)>2N_{1,i}$;
(ii) $\|\sigma^R_{N_{1,i}}(A_n)^2-\sigma^R_{N_{1,i}}(A_n)\|-\|\sigma_{N_{1,i}}(A)^2-\sigma_{N_{1,i}}(A)\|<\dfrac{1}{i}$;

and if $i>1$

(iii) $\|\sigma^R_{N_{1,i-1}}(A_n)-\sigma^R_{N_{1,i}}(A_n)\|-\|\sigma_{N_{1,i-1}}(A)-\sigma_{N_{1,i}}(A)\|<\dfrac{1}{i}$.

It follows that if i is large enough, say $i>I$, we have idempotents $P_{N_{1,i},n}$ for all $n\geq N_{2,i}$, with the property that $\|P_{N_{1,i},N_{2,i}}-\sigma^R_{N_{1,i}}(A_{N_{2,i}})\|\to 0$ as $i\to\infty$. By (i), we have the linear independence that was mentioned earlier, so for t_n on $\mathscr{A}_n$,

$$t_n(\sigma^R_{N_{1,i}}(A_n))=t_n(\sigma_{N_{1,i}}(A_n))=\int f_0\, d\mu=\tau(A).$$

Hence $t_{N_{2,i}}(P_{N_{1,i},N_{2,i}})\to\tau(A)$, as $i\to\infty$, $n\geq N_{2,i}$. Henceforth we will assume $i>I$.

Now if the $\overline{C^n_{1,j}}$ are all connected, then the trace of any idempotent in $\mathscr{A}_n$ is of the form

$$n_1\mu(C^n_{1,1})+n_2\mu(C^n_{1,2})+\ \dots\ +n_{w(n)}\mu(C^n_{1,w(n)})$$

for some integers $n_1,\dots, n_{w(n)}$. If the $\overline{C^n_{1,j}}$ are not connected, one must use the components of each. Then we have terms of the form $\mu(O\cap C^n_{1,j})$, if O is a component $\overline{C^n_{1,j}}$.

Now just being a limit of such numbers will probably not yield much information. We need to relate $t_{N_{2,i}}(P_{N_{1,i}N_{2,i}})$ to $t_{N_{2,i+1}}(P_{N_{1,i+1}N_{2,i+1}})$. This would only seem possible in some C^*-algebra containing both projections, onto which one has the traces on $\mathscr{A}_{N_{2,i}}$ and $\mathscr{A}_{N_{2,i+1}}$ extending. The easiest, and maybe the only way, is if we have an embedding $\mathscr{A}_{N_{2,i}}\subset\mathscr{A}_{N_{2,i+1}}$. This of course gives an A.F. algebra. At this stage we have a choice. We can prove that the Crossed Product can be embedded in the A.F. algebra, and with the $\overline{C^n_{ij}}$ connected, we have the range of the trace on idempotents as above [11]. Or we can avoid any explicit mention of the A.F. algebra and show that eventually the sequence of $\tau(P_{N_{1,i},N_{2,i}})$ becomes constant. We proceed with the latter.

Theorem 2. *If ϕ_n approximate ϕ as before and $E_{n+1}\subset E_n$, then each $P_{N_{1,i},N_{2,i}}$ is connected to $P_{N_{1,i},N_{2,i+1}}$ by a path of idempotents, if i is large enough.*

PROOF. First we want to see that $E_{n+1}\subset E_n$ implies $\mathscr{A}_n\subset\mathscr{A}_{n+1}$. Recall that E_n is the union of the bottom pieces of all the stacks in the n-partition. Let I_n be the union of all the top pieces. Then $\phi(E_n)=I_n$. So $\phi(E_{n+1})=I_{n+1}\subset I_n=\phi(E_n)$. This means that each stack of the $(n+1)$-partition has at its top a piece of the top of a stack of the n-partition, and as its bottom a piece of a stack of the n-partition, and hence that each stack of the $(n+1)$-partition consists of complete substacks of the n-partition. By a substack of $C^n_{1,1}\to\ldots\to C^n_{h(1,n),1}$ we mean $A_1\to\ldots\to A_{h(1,n)}$, where $A_1\subset C^n_{1,1}$. Now one can see that $\mathscr{A}_n\subset\mathscr{A}_{n+1}$, by looking at the L^2 decompositions corresponding to the n-partition and its refinement the $(n+1)$-partition.

If for each $L^2(C^n_{h(j,n),j})$ we choose an orthonormal basis $\{f_{jk}\}_{k=1}^{\infty}$, then $\{f_{jk}\circ\phi^i\}$ is an orthonormal basis for $L^2(C^{n+1}_{h(j,n)-i,j})$, and taking these for all $j=i$, $w(n)$, we get an orthonormal basis for $L^2(X)$. Using these with the $\bigoplus_{i,j}L^2(C^{n+1}_{i,j})$ decomposition, both U_{ϕ_n} and $U_{\phi_{n+1}}$ are 'scalar' matrices that agree, except on that part of the matrix corresponding to mapping $L^2(E_n)$ onto $L^2(I_n)$. Hence they can be joined by a path of unitaries that agree with U_{ϕ_n} and $U_{\phi_{n+1}}$, except on that piece. This path is in $\mathscr{A}_{n+1}$. Similarly $\mathscr{A}_{N_{1,i},N_{2,i}}\subset\mathscr{A}_{N_{1,i},N_{2,i+1}}$, and the corresponding U_{ϕ_n} are connected by a path of unitaries, say U_t, in the latter.

We had

$$\sigma_{N_{1,i}}(A_{N_{2,i}})=\sum_{-N_{1,i}}^{N_{1,i}}M_{f_k}\frac{N_{1,i}-|k|}{N_{1,i}}U^k_{\phi_{N_{2,i}}}.$$

If we replace $U_{\phi_{N_{2,i}}}$ by U_t, and make them self-adjoint, taking σ^R as before, we get a path of self-adjoints in $\mathscr{A}_{N_{2,i+1}}$ joining $\sigma^R_{N_{1,i}}(A_{N_{2,i}})$ and $\sigma^R_{N_{1,i}}(A_{N_{2,i+1}})$. But from Theorem 1 it follows that all the elements of this path have spectra disconnected at $\frac{1}{2}$, so we get idempotents P_t. Since these can be had by integrating along a fixed path around the point 1, it follows that we have a continuous path of idempotents in $\mathscr{A}_{N_{2,i+1}}$ connecting $P_{N_{1,i},N_{2,i}}$ and $P_{N_{1,i},N_{2,i+1}}$ as desired.

On $\mathscr{A}_{N_{2,i+1}}$ we have $t_{N_{2,i+1}}$, and the restriction of this to $\mathscr{A}_{N_{2,i}}$ is $t_{N_{2,i}}$. Then since they are connected we have $t_{N_{2,i}}(P_{N_{1,i},N_{2,i}})=t_{N_{2,i+1}}(P_{N_{1,i},N_{2,i+1}})$. And by (*iii*) of the choice of $N_{2,i}$, $\|P_{N_{1,i},N_{2,i+1}}-P_{N_{1,i+1},N_{2,i+1}}\|\to 0$. Once it is less than one $t_{N_{2,i+1}}(P_{N_{1,i},N_{2,i+1}})=t_{N_{2,i+1}}(P_{N_{1,i+1},N_{2,i+1}})$. Hence we have a constant subsequence converging to $\tau(A)$. So $\tau(A)=n_1\mu(C^n_{11})+\ldots+n_{w(n)}\mu(C^n_{1w(n)})$ for some n, again with the connectivity assumptions. ■

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