

SPECTRAL MAPPING THEOREMS FOR HYPONORMAL OPERATORS

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ABSTRACT

Let $T = H + iK$ be hyponormal and φ be a strictly monotone increasing continuous function on $\sigma(H)$. We define $\tilde{\varphi}(T)$ by $\tilde{\varphi}(T) = \varphi(H) + iK$. In this paper, we show that if z is an isolated eigenvalue of $\tilde{\varphi}(T)$, then the corresponding Riesz projection is self-adjoint. Also we introduce Xia spectrum and study the existence of an invariant subspace of an operator $\tilde{\varphi}(T)$.

1. Introduction

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space \mathcal{H} . An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$.

For an operator T , we denote the positive part of T by $|T|$ and decompose $T = H + iK$, with $H = \operatorname{Re}T = \frac{1}{2}(T + T^*)$ and $K = \operatorname{Im}T = \frac{1}{2i}(T - T^*)$. Using this decomposition, several authors have given generalisations of hyponormal operators (see, e.g. [8; 9; 11; 13; 16]). Following Istrăţescu [8; 9], in [3] we studied spectral properties of the operator $\tilde{\varphi}(T) = \varphi(H) + iK$ associated with a strictly monotone increasing continuous function φ on the spectrum $\sigma(H)$. In the present paper, first we show that the Riesz projection for an isolated eigenvalue of $\tilde{\varphi}(T)$ is self-adjoint. In [1] we gave a condition for the existence of invariant subspaces in the case of p -hyponormal operations. Next we introduce Xia spectrum of $\tilde{\varphi}(T)$ and study the existence of invariant subspaces of an operator $\tilde{\varphi}(T)$. Also we study structural properties of the spectrum of $\tilde{\varphi}(T)$ for a pure hyponormal operator T .

Let $T \in B(\mathcal{H})$. We denote the spectrum, the point spectrum and the approximate point spectrum of T by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ respectively.

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A point $z \in \mathbf{C}$ is in the essential spectrum $\sigma_e(T)$ of T if there exists a sequence $\{x_n\}$ of unit vectors in \mathcal{H} such that $x_n \rightarrow 0$ (weakly) and $(T - z)x_n \rightarrow 0$. It is well known that

$$\sigma(T) = \sigma_e(T) \cup \sigma_p(T) \cup \{z : \bar{z} \in \sigma_p(T^*)\}, \tag{1}$$

(see [7]). Throughout this paper, for a strictly monotone increasing continuous function $\varphi(\cdot)$ on \mathbf{R} we define $\tilde{\varphi}(\cdot)$ on \mathbf{C} by $\tilde{\varphi}(a + ib) = \varphi(a) + ib$ for $a, b \in \mathbf{R}$.

For a pair of operators (T_1, T_2) , $(z_1, z_2) \in \mathbf{C}^2$ is in the joint approximate point spectrum $\sigma_a(T_1, T_2)$ if there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(T_j - z_j)x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (j = 1, 2).$$

We denote the (Taylor) joint spectrum of a commuting pair of operators (T_1, T_2) by $\sigma(T_1, T_2)$ (see [14]).

2. Theorems

In this paper we sometimes use the following results: let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$.

- (1) If $(\tilde{\varphi}(T) - (a + ib))x_n \rightarrow 0$, then there exists a_0 such that $\varphi(a_0) = a$, $(H - a_0)x_n \rightarrow 0$, $(K - b)x_n \rightarrow 0$ and $(\tilde{\varphi}(T) - (a + ib))^*x_n \rightarrow 0$ [3, lemma 2].
- (2) $\sigma(\tilde{\varphi}(T)) = \tilde{\varphi}(\sigma(T))$ and $\sigma_a(\tilde{\varphi}(T)) = \tilde{\varphi}(\sigma_a(T))$ [3, theorem 1].

First we show the following:

Theorem 1. *Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. We define $\tilde{\varphi}(\cdot)$ by $\tilde{\varphi}(a + ib) = \varphi(a) + ib$ and $\tilde{\varphi}(T) = \varphi(H) + iK$. Let z be an isolated eigenvalue of $\tilde{\varphi}(T)$ and $r > 0$ be $\{\lambda : |\lambda - z| \leq r\} \cap \sigma(\tilde{\varphi}(T)) = \{z\}$. Put $E = \frac{1}{2\pi i} \int_{|\lambda - z|=r} (\lambda - \tilde{\varphi}(T))^{-1} d\lambda$. Then the following statements hold:*

- (1) $E\mathcal{H} = \ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$.
- (2) E is self-adjoint.

PROOF. (1) We show $\ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$. Let $\tilde{\varphi}(z_0) = z$. Then z_0 is an isolated eigenvalue of T and by [3, corollary 4] we have $\ker(\tilde{\varphi}(T) - z) = \ker(T - z_0)$. Hence $\ker(T - z_0)$ reduces for T and $\ker(\tilde{\varphi}(T) - z)$ reduces for $\tilde{\varphi}(T)$. Therefore, we have the decomposition

$$\tilde{\varphi}(T) - z = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \text{ on } \ker(\tilde{\varphi}(T) - z) \oplus \ker(\tilde{\varphi}(T) - z)^\perp.$$

We may consider that this decomposition is on $\ker(T - z_0) \oplus \ker(T - z_0)^\perp$. Let $\mathcal{N} = \ker(T - z_0)$. Since $T|_{\mathcal{N}} = H|_{\mathcal{N}} + iK|_{\mathcal{N}}$ (hyponormal operator) and $T|_{\mathcal{N}^\perp} = H|_{\mathcal{N}^\perp} + iK|_{\mathcal{N}^\perp}$ (hyponormal operator) are Cartesian decompositions, hence we have

$$\tilde{\varphi}(T|_{\mathcal{N}}) = \varphi(H|_{\mathcal{N}}) + iK|_{\mathcal{N}} \quad \text{and} \quad \tilde{\varphi}(T|_{\mathcal{N}^\perp}) = \varphi(H|_{\mathcal{N}^\perp}) + iK|_{\mathcal{N}^\perp}.$$

Also we have $\tilde{\varphi}(T|_{\mathcal{N}^\perp}) - z = A$. If $0 \in \sigma(A)$, then $z \in \sigma(\tilde{\varphi}(T|_{\mathcal{N}^\perp}))$, and hence z is an isolated point of $\sigma(\tilde{\varphi}(T|_{\mathcal{N}^\perp}))$. Therefore z is an eigenvalue of $\tilde{\varphi}(T|_{\mathcal{N}^\perp})$. This is a contradiction. Since A is invertible, we have $\ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$. Hence we have $E\mathcal{H} = \ker(\tilde{\varphi}(T) - z) = \ker((\tilde{\varphi}(T) - z)^*)$.

(2) Since $\{\lambda : |\lambda - z| \leq r\} \cap \sigma(A) = \emptyset$, it holds that $\frac{1}{2\pi i} \int_{|\lambda - z|=r} (\lambda - z - A)^{-1} d\lambda = 0$. Hence we have

$$\begin{aligned}
 E &= \frac{1}{2\pi i} \int_{|\lambda - z|=r} (\lambda - \tilde{\varphi}(T))^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda - x|=r} \begin{pmatrix} (\lambda - z)^{-1} & 0 \\ 0 & (\lambda - z - A)^{-1} \end{pmatrix} d\lambda \\
 &= \begin{pmatrix} \frac{1}{2\pi i} \int_{|\lambda - z|=r} (\lambda - z)^{-1} d\lambda & 0 \\ 0 & \frac{1}{2\pi i} \int_{|\lambda - z|=r} (\lambda - z - A)^{-1} d\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
 \end{aligned}$$

Therefore, E is self-adjoint. So the proof is complete. ■

In [2] we introduced Xia spectrum for a p -hyponormal operator. For the existence of invariant subspaces of $\tilde{\varphi}(T)$, we introduce Xia spectrum of $\tilde{\varphi}(T)$. Let $T = H + iK$ be hyponormal. Define

$$\mathcal{S}^+(K) = \lim_{t \rightarrow \infty} e^{itH} K e^{-itH} \quad \text{and} \quad \mathcal{S}^-(K) = \lim_{t \rightarrow \infty} e^{-itH} K e^{itH}.$$

If $T = H + iK$ is hyponormal, then there exist operators $\mathcal{S}^\pm(K)$ (see [15]). Define

$$K_k = k\mathcal{S}^+(K) + (1 - k)\mathcal{S}^-(K),$$

for k ($0 \leq k \leq 1$). Then we have the following:

Theorem A. [4, lemma 2.1]. *Under the above notations, if $T = H + iK$ is hyponormal, then $H\mathcal{S}^\pm(K) = \mathcal{S}^\pm(K)H$.*

Therefore, it is clear that $HK_k = K_kH$ and $\varphi(H)K_k = K_k\varphi(H)$ for every k ($0 \leq k \leq 1$). We define

$$T_k = H + iK_k \quad \text{and} \quad \tilde{\varphi}(T_k) = \varphi(H) + iK_k \quad \text{for } k \in [0, 1].$$

Then, for $k \in [0, 1]$, T_k and $\tilde{\varphi}(T_k)$ are normal and the following theorem holds.

Theorem B. [16, theorem IV.3.1]. *Let $T = H + iK$ be hyponormal. Then*

$$\sigma(T) = \bigcup_{0 \leq k \leq 1} \sigma(T_k).$$

Definition 1. We define Xia spectrum $\sigma_X(\tilde{\varphi}(T))$ of $\tilde{\varphi}(T)$ by

$$\sigma_X(\tilde{\varphi}(T)) = \bigcup_{0 \leq k \leq 1} \sigma_a(\varphi(H), K_k).$$

Remark. We cannot define Xia spectrum for an arbitrary operator. We prepare the following results.

Theorem C. [6]. *Let (T_1, T_2) be a commuting pair of normal operators. Then*

$$\sigma(T_1, T_2) = \sigma_a(T_1, T_2).$$

Theorem D. [5; 12; 14]. Let (T_1, T_2) be a commuting pair and $f(z_1, z_2)$ be a polynomial of two variables. Then

$$f(\sigma(T_1, T_2)) = \sigma(f(T_1, T_2)) \quad \text{and} \quad f(\sigma_a(T_1, T_2)) = \sigma_a(f(T_1, T_2)).$$

Theorem 2. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. Then

$$a + ib \in \sigma(\tilde{\varphi}(T)) \quad \text{if and only if} \quad (a, b) \in \sigma_X(\tilde{\varphi}(T)).$$

PROOF. Let $a + ib \in \sigma(\tilde{\varphi}(T))$. Then there exists a_0 such that $a_0 + ib \in \sigma(T)$ and $\varphi(a_0) = a$. Since $T = H + iK$ is hyponormal, there exists k ($0 \leq k \leq 1$) such that $(a_0, b) \in \sigma_a(H, K_k)$. Hence we have $(a, b) \in \sigma_a(\varphi(H), K_k)$ and $(a, b) \in \sigma_X(\tilde{\varphi}(T))$. Next assume that $(a, b) \in \sigma_X(\tilde{\varphi}(T))$. Then there exists k ($0 \leq k \leq 1$) such that $(a, b) \in \sigma_a(\varphi(H), K_k)$. Since $T_k = H + iK_k$ is a normal operator, $\tilde{\varphi}(T_k) = \varphi(H) + iK_k$ and $a + ib \in \sigma_a(\tilde{\varphi}(T_k))$, and by lemma 2 of [3] there exists $(a_0, b) \in \sigma_a(H, K_k)$ such that $\varphi(a_0) = a$. Since $a_0 + ib \in \sigma(T)$, by theorem 1 of [3] we have $a + ib \in \sigma(\tilde{\varphi}(T))$. So the proof is complete. ■

Corollary 3. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. Then

$$\sigma(\tilde{\varphi}(T)) = \bigcup_{0 \leq k \leq 1} \sigma(\tilde{\varphi}(T_k)).$$

PROOF. Let $a + ib \in \sigma(\tilde{\varphi}(T))$. Then by Theorem 2 there exists k ($0 \leq k \leq 1$) such that $(a, b) \in \sigma_a(\varphi(H), K_k)$. Hence by Theorem D it is clear that $a + ib \in \sigma(\tilde{\varphi}(T_k))$. Hence $a + ib \in \bigcup_{0 \leq k \leq 1} \sigma(\tilde{\varphi}(T_k))$. Next let $a + ib \in \sigma(\tilde{\varphi}(T_k))$ for some k ($0 \leq k \leq 1$). Since $\tilde{\varphi}(T_k)$ is a normal operator, we have $a + ib \in \sigma_a(\tilde{\varphi}(T_k))$. Since $(\varphi(H), K_k)$ is a commuting pair of hermitian operators, by Theorem C we have $(a, b) \in \sigma_a(\varphi(H), K_k)$. Hence by Theorem 2 we have $a + ib \in \sigma(\tilde{\varphi}(T))$. ■

We introduce the generalised spectrum $G(|\tilde{\varphi}(T)|)$ of $|\tilde{\varphi}(T)|$.

Definition 2. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. For $k \in [0, 1]$, let T_k be the operator defined following Theorem A. For an operator $\tilde{\varphi}(T)$, the generalised spectrum $G(|\tilde{\varphi}(T)|)$ of $|\tilde{\varphi}(T)|$ is defined by

$$G(|\tilde{\varphi}(T)|) = \bigcup_{0 \leq k \leq 1} \sigma(|\tilde{\varphi}(T_k)|).$$

Theorem 4. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$.

- (1) If $r \in G(|\tilde{\varphi}(T)|)$, then there exists $a + ib \in \sigma(\tilde{\varphi}(T))$ such that $|a + ib| = r$.
- (2) If $z = a + ib \in \sigma(\tilde{\varphi}(T))$, then $|z| \in G(|\tilde{\varphi}(T)|)$.

PROOF. (1) Let $r \in G(|\tilde{\varphi}(T)|)$. Then there exists k ($0 \leq k \leq 1$) such that $r \in \sigma(|\tilde{\varphi}(T_k)|)$. Since $r^2 \in \sigma(\varphi(H)^2 + K_k^2)$, by Theorem D there exists $(a, b) \in \sigma_{ja}(\varphi(H), K_k)$ such that $a^2 + b^2 = r^2$. Since $(a, b) \in \sigma_X(\tilde{\varphi}(T))$, by Theorem 2 we have $a + ib \in \sigma(\tilde{\varphi}(T))$.

(2) Let $z \in \sigma(\tilde{\varphi}(T))$. Then by Corollary 3 there exists k ($0 \leq k \leq 1$) such that $z \in \sigma(\tilde{\varphi}(T_k))$. Since $\tilde{\varphi}(T_k) = \varphi(H) + iK_k$ is normal, we have $|z|^2 \in \sigma(|\tilde{\varphi}(T_k)|^2)$ and hence $|z| \in \sigma(|\tilde{\varphi}(T_k)|) \subset G(|\tilde{\varphi}(T)|)$. So the proof is complete. ■

Theorem 5. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. If $G(|\tilde{\varphi}(T)|)$ is not interval, then $\tilde{\varphi}(T)$ has a non-trivial invariant subspace.

PROOF. By the assumption, there exist non-negative numbers r_1 and r_2 such that $r_1 < r_2$ and $(r_1, r_2) \cap G(|\tilde{\varphi}(T)|) = \emptyset$. By theorem 4(1) there exist z_1 and z_2 in $\sigma(\tilde{\varphi}(T))$ such that $|z_i| = r_i$ ($i = 1, 2$). By Theorem 4(2), we have

$$\sigma(\tilde{\varphi}(T)) \cap \{z : r_1 < |z| < r_2\} = \emptyset.$$

Thus $\sigma(\tilde{\varphi}(T))$ is not connected and $\tilde{\varphi}(T)$ has a non-trivial invariant subspace. ■

The boundary of a set $E \subset \mathbb{C}$ is denoted by ∂E . An operator T is called pure hyponormal if it has no non-trivial reducing subspace on which it is normal.

Theorem 6. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. If $z \in \partial\sigma(\tilde{\varphi}(T))$, then

$$|z| \in \sigma(|\tilde{\varphi}(T)|) \cap \sigma(|\tilde{\varphi}(T)^*|).$$

Moreover, if T is pure, then

$$|z| \in \sigma_e(|\tilde{\varphi}(T)|) \cap \sigma_e(|\tilde{\varphi}(T)^*|).$$

PROOF. Since $z \in \partial\sigma(\tilde{\varphi}(T)) \subset \sigma_a(\tilde{\varphi}(T))$, we have $\bar{z} \in \sigma_a(\tilde{\varphi}(T)^*)$ by lemma 2 of [3]. Hence we have $|z| \in \sigma(|\tilde{\varphi}(T)|) \cap \sigma(|\tilde{\varphi}(T)^*|)$.

Next we assume that T is pure. By theorem 1 of [3], let z_0 be $\tilde{\varphi}(z_0) = z$. Since φ is strictly monotone increasing, we have $z_0 \in \partial\sigma(T)$. Hence by theorem 1 of [11] there exists a sequence $\{x_n\}$ of unit vectors such that $x_n \rightarrow 0$ (weakly) and $(T - z)x_n \rightarrow 0$. Since T is hyponormal, it is easy to see that $(\tilde{\varphi}(T) - z)x_n \rightarrow 0$ and $(\tilde{\varphi}(T) - z)^*x_n \rightarrow 0$. Therefore, we have

$$|z| \in \sigma_e(|\tilde{\varphi}(T)|) \cap \sigma_e(|\tilde{\varphi}(T)^*|).$$

So the proof is complete. ■

Theorem 7. Let $T = H + iK$ be hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. If $z \in \sigma(\tilde{\varphi}(T))$ and $\bar{z} \notin \sigma_p(\tilde{\varphi}(T)^*)$, then $|z| \in \sigma_e(|\tilde{\varphi}(T)|) \cap \sigma_e(|\tilde{\varphi}(T)^*|)$.

PROOF. If $z \in \sigma_p(\tilde{\varphi}(T))$, then by lemma 2 of [3] we have $\bar{z} \in \sigma_p(\tilde{\varphi}(T)^*)$. Hence we have $z \notin \sigma_p(\tilde{\varphi}(T))$. By Equation (1) we have $z \in \sigma_e(\tilde{\varphi}(T))$. Hence there exists a sequence $\{x_n\}$ of

unit vectors such that

$$x_n \rightharpoonup 0 \text{ (weakly) and } (\tilde{\varphi}(T) - z)x_n \rightarrow 0.$$

By lemma 2 of [3] it holds that $(\tilde{\varphi}(T) - z)^*x_n \rightarrow 0$. Hence we have

$$(|\tilde{\varphi}(T)| - |z|)x_n \rightarrow 0 \text{ and } (|\tilde{\varphi}(T)^*| - |z|)x_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $x_n \rightharpoonup 0$ (weakly), we have $|z| \in \sigma_e(|\tilde{\varphi}(T)|) \cap \sigma_e(|\tilde{\varphi}(T)^*|)$. So the proof is complete. ■

In order to give a proof of Theorem 8, we need the following simple lemma. We contain its proof for the sake of completeness. Let m_1 and m_2 be the Lebesgue measure on \mathbf{R} and the planar Lebesgue measure, respectively.

Lemma 1. *Let $\varphi(\cdot)$ be a strictly monotone increasing continuous function on \mathbf{R} such that φ^{-1} is absolutely continuous. Let A be a Lebesgue measurable set in \mathbf{R} . If $m_2(\tilde{\varphi}^{-1}(A)) = 0$, then $m_2(A) = 0$.*

PROOF. Recall $\tilde{\varphi}(x, y) = (\varphi(x), y)$. For $E \subset \mathbf{R}^2$ and $y \in \mathbf{R}$, we define $E_y = \{x : (x, y) \in E\}$. We have $\tilde{\varphi}^{-1}(A)_y = \varphi^{-1}(A_y)$, so that

$$m_2(\tilde{\varphi}^{-1}(A)) = \int_{\mathbf{R}} m_1(\tilde{\varphi}^{-1}(A)_y) dm_1(y) = \int_{\mathbf{R}} m_1(\varphi^{-1}(A_y)) dm_1(y) = 0.$$

Then

$$m_1(\varphi^{-1}(A_y)) = 0 \text{ for a.e. } y.$$

Since φ^{-1} is absolutely continuous,

$$m_1(A_y) = 0 \text{ for a.e. } y.$$

Therefore,

$$m_2(A) = \int_{\mathbf{R}} m_1(A_y) dm_1(y) = 0.$$

So the proof is complete. ■

Theorem 8. *Let $T = H + iK$ be pure hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on \mathbf{R} such that φ^{-1} is absolutely continuous. If G is any open disk of the plane, then $m_2(\sigma(\tilde{\varphi}(T)) \cap G) > 0$ whenever $\sigma(\tilde{\varphi}(T)) \cap G \neq \emptyset$.*

PROOF. Assume that, for some open disk G , $\sigma(\tilde{\varphi}(T)) \cap G \neq \emptyset$ and $m_2(\sigma(\tilde{\varphi}(T)) \cap G) = 0$. Let $z \in \sigma(\tilde{\varphi}(T)) \cap G$. Then there exists $z_0 \in \sigma(T)$ such that $\tilde{\varphi}(z_0) = z$. Since $\tilde{\varphi}$ is continuous, there exists an open disk G_0 such that

$$z_0 \in G_0 \text{ and } \tilde{\varphi}(G_0) \subset G.$$

By Lemma 1 it holds that $m_2(\sigma(T) \cap \tilde{\varphi}^{-1}(G)) = 0$. Since $\sigma(T) \cap G_0 \subseteq \sigma(T) \cap \tilde{\varphi}^{-1}(G)$, so that $m_2(\sigma(T) \cap G_0) = 0$. But $z_0 \in \sigma(T) \cap G_0$. Since T is a pure hyponormal operator, this is a contradiction to Putnam's result [10, theorem 4]. Hence the proof is complete. ■

Finally we show the following:

Theorem 9. *Let $T=H+iK$ be pure hyponormal and $\varphi(\cdot)$ be a strictly monotone increasing continuous function on $\sigma(H)$. If $m_1(\sigma(|\tilde{\varphi}^{-1}(T)|))=0$, then there exists a finite or countably infinite number of pairwise disjoint open annuli $A_n = \{z: a_n < |z| < b_n\}$ ($n=1, 2, \dots$) such that $\sigma(\tilde{\varphi}(T))$ is the closure of the set $\cup A_n$ and $\cup A_n \subset \sigma_p(\tilde{\varphi}(T)^*)$.*

PROOF. Let $z_1 \in \sigma(\tilde{\varphi}(T))$. Consider any open disk G containing z_1 . Then G contains a closed disk F satisfying

$$F = \{z : |z - z_1| \leq \delta, \delta > 0\} \subset \sigma(\tilde{\varphi}(T)).$$

Otherwise $\sigma(\tilde{\varphi}(T)) \cap G \subset \partial\sigma(\tilde{\varphi}(T))$. Hence if the half-line $\ell : \theta = \text{constant}$ intersects G , then by Theorem 6 each r satisfying $re^{i\theta} \in \ell \cap (\sigma(\tilde{\varphi}(T)) \cap G)$ belongs to $\sigma(|\tilde{\varphi}(T)|)$. Hence, by this assumption the set of such numbers r has linear measure 0. It follows from Fubini's theorem that $m_2(\sigma(\tilde{\varphi}(T)) \cap G) = 0$. By Theorem 8, this is a contradiction. This proves (2).

Let $a = \inf\{|z| : z \in F\}$ and $b = \sup\{|z| : z \in F\}$. Then $a < b$ and

$$\{z : a \leq |z| \leq b\} \subset \sigma(\tilde{\varphi}(T)).$$

In fact, assume that there exists $z_2 \notin \sigma(\tilde{\varphi}(T))$ such that $a < |z_2| < b$. Since $z_2 \in \sigma(\tilde{\varphi}(T))^C$, there exists $\varepsilon > 0$ such that

$$\{z : |z - z_2| < \varepsilon\} \subset \sigma(\tilde{\varphi}(T))^C.$$

We may assume that $\varepsilon < \min\{|z_2| - a, b - |z_2|\}$. Hence by Theorem 6 we have $[|z_2| - \varepsilon, |z_2|] \subset \sigma(|\tilde{\varphi}(T)|)$ or $[|z_2|, |z_2| + \varepsilon] \subset \sigma(|\tilde{\varphi}(T)|)$. This is a contradiction to $m_1(\sigma(|\tilde{\varphi}(T)|)) = 0$. Hence it follows (3). Therefore, we can see that $\sigma(\tilde{\varphi}(T))$ can be taken as the closure of countably union of disjoint closed annuli, each of the form $\{z : c \leq |z| \leq d\}$ with $c < d$.

For a fixed annulus let $\cup(c_n, d_n)$ denote the canonical decomposition of the linear open set $[c, d] - \{[c, d] \cap \sigma(|\tilde{\varphi}(T)^*|)\}$. In fact for z in this open set, if $|z| = d$, then $z \in \partial\sigma(\tilde{\varphi}(T))$. Hence $|z| \in \sigma(|\tilde{\varphi}(T)^*|)$. Also, if $|z| = c$, then we have $|z| \in \sigma(|\tilde{\varphi}(T)^*|)$. It follows that $\{z : c \leq |z| \leq d\}$ is the closure of $\cup B_n$, where $B_n = \{z : c_n < |z| < d_n\}$. Also by Theorem 7 we have each $B_n \subset \sigma_p(\tilde{\varphi}(T)^*)$. This completes the proof. ■

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