ON TIME-LIKE LINE CONGRUENCES AND DUAL SPHERICAL AREAS ON LORENTZIAN SPHERES

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Abstract

By considering the $E_1(X_1, G, N)$ Darboux, $E_2(X_1, X_2, X_3)$ Frenet and $E_3(R_1, R_2, R_3)$ Blaschke trihedrons of a time-like ruled surface R_1 in a time-like line congruence and by using dual quantities, we obtain some relations between the instantaneous velocities of the motions (E_3/E_1), (E_2/E_1) and (E_3/E_2) in ID_1^3. Moreover, by using dual instantaneous vectors, we obtain some theorems on the relations between the dual Steiner vectors of these trihedrons. Then we give some theorems on the relations between the dual angles of pitch of space-like and time-like ruled surfaces and dual Lorentzian spherical areas.

1. Introduction

Research on geometry reached new heights with the definition of dual numbers by W.K. Clifford (1849–79). This improvement showed that the set D of dual numbers is a ring and led to the consideration of the module $D^3 = D \times D \times D$ on that ring. Then E. Study established the Study theory, which shows one-to-one correspondence between the directed lines in R^3 and the dual points of the dual unit sphere. Hence, ruled surfaces and congruences obtained by the motion of a line, depending on one or two parameters, were examined more easily.

It is well known that the geometry of ruled surfaces is very important in the study of kinematics or spatial mechanisms in R^3 [1; 3]. By using the Lorentzian inner product, many researchers have defined theorems and definitions in R^3 for the Minkowski space [5; 7]. These studies have many applications in engineering and architecture, in the study of gears, toothed wheels, modern buildings etc.

In this study we consider various relative motions of Blaschke trihedrons $E_3\{R_1, R_2, R_3\}$, dual Frenet trihedrons $E_2\{X_1, X_2, X_3\}$ and dual Darboux trihedrons $E_1\{X_1, G, N\}$ depending on a generator of a time-like ruled surface $R_1 = R_1(s)$. Then we obtain some relations between the instantaneous rotation axes of these motions in the

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Minkowski space. Moreover, in Theorems 2 and 3 we obtain relations between the dual Lorentzian spherical area, the dual angle of pitch and the total dual geodesic curvature of the dual Lorentzian spherical indicator curve of the ruled surface. Then we obtain some relations between the dual Steiner vectors of the relative motions of the trihedrons E_1, E_2, E_3 and the dual spherical areas of the regions drawn by the edges of these trihedrons on the dual Lorentzian sphere.

In the Minkowski 3-space $R_1^3=[R^3, (+, +, -)]$ the Lorentzian inner product of $a=(a_1, a_2, a_3)$ and $b=(b_1, b_2, b_3)\in R^3$ is given as $\langle a, b\rangle = a_1b_1+a_2b_2-a_3b_3$. Under this condition, if $x=a=(a_1, a_2, a_3)\in R_1^3$ is a non-zero vector, then, for the situations $\langle x, x\rangle > 0$, $\langle x, x\rangle < 0$ and $\langle x, x\rangle = 0$, x is called the *space-like vector*, the *time-like vector* and the *light-like* (*null*) *vector*, respectively [4].

We hope that these results will be helpful to physicists and those studying general relativity theory.

2. The real integral invariants of the closed time-like ruled surfaces

Let a moving space-like line space H be represented by the moving frame $\{O; r_1, r_2, r_3\}$ and let H' be represented by the fixed frame $\{O'; e_1, e_2, e_3\}$. We know that any space-like line in H draws a closed time-like ruled surface in H' along the motion denoted by H/H'. Thus, the equation of the closed time-like ruled surface can be written as

$$x(t, v)=r(t)+vr_1(t), \quad x(t+2\pi, v)=x(t, v), \quad \|r_1\|=1, \tag{1}$$

where $r(t)$ is the base curve. During the motion, we assume that r_1 and r_2 are space-like vectors and that r_3 is a time-like vector. This closed time-like ruled surface is generated by the axis r_1. By differentiating (1), we may write the differential equation of the orthogonal trajectory of the r_1-closed time-like ruled surface as follows:

$$\langle dx, r_1\rangle = 0, \quad \|r_1\|=1.$$

Definition 1. Given a space-like curve (r), the trihedron $\{t, n, b\}$ generated by the tangent t, the principal normal n and the binormal vector b at the parameter value $t=t_0$ is called the *Frenet trihedron* of the curve.

Definition 2. Given a time-like surface $x=x(t, v)$ and a space-like curve (r) on that surface, the trihedron $\{t, n, g\}$ generated by the tangent t, the surface normal n and the vector $g=t\wedge n$ is called the *Darboux trihedron*.

Definition 3. The pitch (öffnungsstrecke) of the closed time-like ruled surface $r_1(t)$ is defined by

$$l_{r_1} = \oint d\lambda = -\oint \langle dr, r_1\rangle.$$

This definition means that, after one period, an orthogonal trajectory of a closed time-like ruled surface $r_1(t)$ intersects the axis r_1 at a point P_1, as well as the point P_0. Thus, $l_{r_1} = \overline{P_0P_1}$ is obtained.

Let us consider the unit time-like vector n_2 and the space-like vector n_3 in the plane (r_2, r_3), which are defined by

$$n_2 = sh\theta\, r_2 + ch\theta\, r_3, \quad n_3 = ch\theta\, r_2 + sh\theta\, r_3. \tag{2}$$

The time-like unit vector n_2 generates a time-like ruled surface along the orthogonal trajectory of the r_1-closed time-like ruled surface during the closed motion, where here θ is the real angle between the unit time-like vectors n_2 and r_3. Thus, the equation of the time-like ruled surface is

$$T = x + wn_2, \quad w \in IR.$$

Then, using equation (2), we obtain the following relations:

$$dr_2 = -r_3 d\theta, \quad dr_3 = r_2 d\theta. \tag{3}$$

So, from (3), $d\theta$ is calculated as

$$d\theta = \langle dr_2, r_3\rangle = -\langle dr_3, r_2\rangle. \tag{4}$$

If we take an integral, from (4), then during the one-parameter spatial motion H/H' the real integral λ_{r_1} is obtained as

$$\lambda_{r_1} = \oint_{(r)} d\theta$$

or

$$\lambda_{r_1} = \oint_{(r)} d\theta = \oint \langle dr_2, r_3\rangle = -\oint \langle dr_3, r_2\rangle$$

[6].

3. The dual Lorentzian space D_1^3

Let $ID = \{(c, c_0) | C = c + \varepsilon c_0, c, c_0 \in R\}$ be a set of dual numbers. Thus, the set of dual numbers ID is a commutative ring with unit element $\varepsilon = (0, 1)$. Then $(ID^3, +)$ is a module on the dual numbers ring. We call it the ID-module, and dual vectors are the elements of this module. We write the dual unit vector as $A = (a, a_0) = a + \varepsilon a_0$, $A^2 = 1$, $\varepsilon^2 = 0$, where a, $a_0 \in IR^3$.

By considering the Lorentzian inner product, we may write the inner product of A and B as follows:

$$\langle A, B\rangle = \langle a, b\rangle + \varepsilon(\langle b, a_0\rangle + \langle a, b_0\rangle, \ \varepsilon^2 = 0, \quad a, a_0, b \text{ and } b_0 \in IR_1^3.$$

The pair $(ID_1^3, \langle\ \rangle)$ is called the dual Lorentzian space [7]. The dual Lorentzian sphere and the dual hyperbolic sphere of radius 1 in R_1^3 are defined by

$$S_1^2 = \{A = a + \varepsilon a_0 |\ \|A\| = (1, 0);\ a, a_0 \in R_1^3, \text{ and } a \text{ is space-like}\}$$

$$H_0^2 = \{A = a + \varepsilon a_0 |\ \|A\| = (1, 0);\ a, a_0 \in R_1^3, \text{and } a \text{ is time-like}\}$$

respectively [7].

Definition 4. Let $A=(a, a_0)=a+\varepsilon a_0, \in ID_1^3$. The dual vector A is said to be *space-like* if the vector a is space-like, *time-like* if the vector a is time-like, and *light-like* (or *null*) if the vector a is light-like [7].

Definition 5. Let $A, B \in ID_1^3$. We define the Lorentzian cross-product of A and B by

$$A \wedge B = -\begin{vmatrix} E_1 & E_2 & -E_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}, \quad E_i = e_i + \varepsilon e_{i0} \quad i=1, 2, 3,$$

where $A=(A_1, A_2, A_3)$, $B=(B_1, B_2, B_3)$, and $E_1 \wedge E_2 = E_3$, $E_2 \wedge E_3 = -E_1$, $E_3 \wedge E_1 = -E_2$, [7].

Definition 6. Let M be a matrix with dual coefficients. M is said to be a *dual Lorentzian orthogonal matrix* if

$$M^{-1} = SM^T S, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where S is the signature matrix in ID_1^3 [8].

4. The closed dual Lorentzian spherical motions in ID_1^3

The closed dual spherical motions and the real integral invariants of the closed ruled surfaces are investigated in [1; 2; 3]. A motion of a rigid body about a fixed point O uniquely defines a dual motion K/K' of the moving dual Lorentzian unit sphere K with the fixed centre O over the fixed Lorentzian unit sphere K' of the same centre.

Let $\{R_1, R_2, R_3\}$ and $\{E_1, E_2, E_3\}$ be two right-handed sets of orthogonal unit vectors that are rigidly linked to the unit dual spheres K and K', respectively, and denoted by

$$R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}.$$

We can write the relation

$$R = ME$$

between these dual orthogonal systems, where

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

is a dual orthogonal matrix and the elements M_{ij} of the matrix M are regarded as functions of a real single parameter t, and we write $M=M(t)$ to indicate that we restrict the discussion to one-parameter motions. In this study we assume that R_1, R_2, E_1 and E_2 are space-like dual vectors and that R_3 and E_3 are time-like dual vectors.

Since the matrix M is a dual Lorentzian orthogonal matrix, by Definition 6 we may write

$$SM^TSM=I, \tag{5}$$

where I is the unit matrix and S is the signature matrix. By differentiating w.r.t. t, equation (5) yields

$$(S\,dM^TS\,M)=-(S\,M^TS\,dM).$$

This relation shows that the matrix

$$\Omega=(S\,dM^TS\,M)$$

has the form

$$\Omega=\begin{bmatrix} 0 & \Omega_1^2 & \Omega_1^3 \\ -\Omega_1^2 & 0 & \Omega_2^3 \\ \Omega_1^3 & \Omega_2^3 & 0 \end{bmatrix}.$$

Differentiating the orthonormal system $\{R_1, R_2, R_3\}$ w.r.t. t yields

$$dR_1=\Omega_1^2R_2+R_3\Omega_1^3 \quad dR_2=-\Omega_1^2R_1+R_3\Omega_2^3 \quad dR_3=\Omega_1^3R_1+R_2\Omega_2^3. \tag{6}$$

Thus, we define a dual vector Ψ by the following equation:

$$\Psi=\Omega_2^3R_1-\Omega_1^3R_2-\Omega_1^2R_3.$$

Here, Ω_1^2, Ω_1^3, Ω_2^3 are non-zero elements of the matrix Ω, and Ψ is called the *dual instantaneous Pfaffian vector* of the dual motion K/K'. The Pfaffian vector Ψ, at a given instant t, of a one-parameter motion on a dual Lorentzian sphere is an analogue to the Darboux vector in the differential geometry of space curves. Hence, (6) can be written as

$$dR_i=\Psi\wedge R_i, \quad i=1, 2, 3.$$

Definition 7. Let Ψ be a dual Pfaffian vector of the dual motion K/K'. The dual vector W, which is defined as

$$W=\oint\Psi=\oint\Omega_2^3R_1-\Omega_1^3R_2-\Omega_1^2R_3=d+\varepsilon d_0=\oint\Psi+\varepsilon\oint\Psi_0,$$

is called the *dual Steiner vector of the dual motion* [6].

5. Time-like congruence in the dual Lorentzian space ID_1^3

A time-like line congruence in the line space R_1^3 can be represented by a unit space-like dual vector that depends on two real parameters u and v as follows:

$$R(u, v)=r(u, v)+\varepsilon r_0(u, v), \quad R^2=1.$$

Definition 8. The time-like ruled surfaces $u=$constant and $v=$constant of a time-like line congruence are called the time-like parameter ruled surfaces.

Definition 9. Let $R_1=R_1(t)$ be a time-like ruled surface in a time-like congruence $R=R(u,v)$. At the generator R_1 of this ruled surface the dual trihedron $\{R_1, R_2, R_3\}$, consisting of the generator R_1, the normal R_2 of the surface at the striction point belonging to that generator, and the vector $R_1 \wedge R_2 = R_3$, is called the *Blaschke trihedron*.

Let us consider a time-like ruled surface $R_1 = R_1(t)$ of the time-like congruence $R = R(u, v)$, where u and v are functions of t.

We write the time-like parameter ruled surfaces as

$$R_{11} = R_{11}(u, v_0) \quad \text{and} \quad R_{21} = R_{21}(u_0, v).$$

The time-like ruled surfaces R_1, R_{11} and R_{21} have a common space-like line, which is defined by the relation

$$R_0 = R(u_0, v_0) = R_1(u_0, v_0) = R_{11}(u_0, v_0) = R_{21}(u_0, v_0).$$

Blaschke trihedrons of these time-like ruled surfaces are of the form

$$(R_0 = R_1, R_2, R_3), \quad (R_0 = R_{11}, R_{12}, R_{13}), \quad (R_0 = R_{21}, R_{22}, R_{23}), \tag{7}$$

where R_3, R_{13} and R_{23} are time-like and R_2, R_{12}, R_{22} and R_0 are space-like. Thus, we can write

$$\begin{aligned}
&R_1^2 = R_2^2 = 1, \quad R_3^2 = -1, \quad R_3 \wedge R_1 = -R_2, \quad R_2 \wedge R_3 = -R_1, \quad R_1 \wedge R_2 = R_3,\\
&R_{11}^2 = R_{12}^2 = 1, \quad R_{13}^2 = -1, \quad R_{13} \wedge R_{11} = -R_{12}, \quad R_{12} \wedge R_{13} = -R_{11}, \quad R_{11} \wedge R_{12} = R_{13},\\
&R_{21}^2 = R_{22}^2 = 1, \quad R_{23}^2 = -1, \quad R_{23} \wedge R_{21} = -R_{22}, \quad R_{22} \wedge R_{23} = -R_{21}, \quad R_{21} \wedge R_{22} = R_{23}.
\end{aligned}$$

On the other hand, if we choose the time-like parameter ruled surfaces as time-like principal ruled surfaces, we may write $f = 0$ and $f_0 = 0$. Thus,

$$F = R_{\mathrm{u}} \,.\, R_{\mathrm{v}} = f + \varepsilon f_0$$

can be written.

The derivative formulas of these Blaschke trihedrons, defined by (7), are

$$\begin{aligned}
R_1' &= PR_2, & R_2' &= -PR_1 + QR_3, & R_3' &= QR_2\\
R_{11}' &= P_1R_{12}, & R_{12}' &= -P_1R_{12} + Q_1R_{13}, & R_{13}' &= Q_1R_{12}\\
R_{21}' &= P_2R_{22}, & R_{22}' &= -P_2R_{21} + Q_2R_{23}, & R_{23}' &= Q_2R_{22}
\end{aligned}$$

Hence, the Blaschke vectors of the Blaschke trihedrons can be given by the relations

$$B = QR_0 - PR_3, \quad B_1 = Q_1R_0 - P_1R_{13}, \quad B_2 = Q_2R_0 - P_2R_{23}.$$

respectively [5].

Theorem 1. *Let a time-like ruled surface R_1 and time-like parameter ruled surfaces R_{11} and R_{21} of a time-like congruence $R(u, v)$ pass through a space-like line R_0 of the congruence. Let B, B_1 and B_2 be Blaschke vectors of these ruled surfaces, respectively, and let Θ be the dual angle between the edges R_2 and R_{12} of the Blaschke trihedrons of the time-like ruled surfaces R_1 and R_{11}, respectively. The following relation holds between the Blaschke vectors and the common line R_0 of these three time-like ruled surfaces*:

$$B = P\left(\frac{\cos\Theta}{P_1}B_1 + \frac{\sin\Theta}{P_2}B_2 + \frac{d\Theta}{dS}R_0\right), \tag{8}$$

where P, P_1 and P_2 are dual curvatures of the time-like ruled surfaces R_1, R_{11} and R_{21}, respectively, and S is the dual arc length of the time-like ruled surface R_1 [5].

6. Some new relations and theorems

Let $R_1 = R_1(s)$ be a time-like ruled surface where s represents the arc length of the striction curve. Let us denote the dual Darboux and Frenet trihedrons of a striction curve by $E_1(X; X_1, G, N)$ and $E_2(X; X_1, X_2, X_3)$, respectively, and the Blaschke trihedron of the ruled surface by $E_3(X; R_1, R_2, R_3)$. By considering these orthonormal systems, we find that the instantaneous velocities of the Darboux, Frenet and Blaschke trihedrons that are moving w.r.t. a fixed trihedron E_0 are

$$D = -(\tau_\rho + \varepsilon)X_1 - \rho_n G + \rho_g N, \quad F = -(\tau + \varepsilon)X_1 + \rho X_3, \quad B = -(\tau_g + \varepsilon)X_1 - \rho_n G, \tag{9}$$

respectively. The relations

$$B = D - \rho_g N, \quad B = F + (\tau - \tau_g)X_1 - \rho_g N, \quad D = F + (\tau - \tau_g)X_1, \tag{10}$$

hold between these instantaneous velocities [7]. If we consider the time-like ruled surface R_1 in the composite motion $(E_3/E_0) = (E_3/E_1)(E_1/E_0)$, we may write the relation

$$B = \Gamma_{3/1} + D \tag{11}$$

between the instantaneous velocities of this motion, and, similarly, if the time-like ruled surfaces R_{11} and R_{21} are considered, the relations

$$B_1 = \Gamma^1_{3/1} + D_1, \quad B_2 = \Gamma^2_{3/1} + D_2, \tag{12}$$

are obtained. Here, $\Gamma_{3/1}$, $\Gamma^1_{3/1}$ and $\Gamma^2_{3/1}$ are the instantaneous velocities in the motion E_3 relative to E_1 for the time-like ruled surfaces R_1, R_{11} and R_{21}, respectively. If (11) and (12) are replaced in (8), the relation

$$\Gamma_{3/1} = \frac{P}{P_1}\cos\Theta\Gamma^1_{3/1} + \frac{P}{P_2}\sin\Theta\Gamma^2_{3/1} + \frac{d\Theta}{dS}R_0 - \left[D - \frac{P}{P_1}\cos\Theta D_1 - \frac{P}{P_2}\sin\Theta D_2\right] \tag{13}$$

is obtained between the instantaneous velocities $\Gamma_{3/1}$, $\Gamma^1_{3/1}$ and $\Gamma^2_{3/1}$.

In the composite motion $(E_3/E_0) = (E_3/E_2)(E_2/E_0)$ of trihedrons, the instantaneous velocities for the time-like ruled surfaces R_1, R_{11} and R_{21} are

$$B = \Gamma_{3/2} + F, \quad B_1 = \Gamma^1_{3/2} + F_1, \quad B_2 = \Gamma^2_{3/2} + F_2, \tag{14}$$

respectively. If these are used in (8), the relation

$$\Gamma_{3/2} = \frac{P}{P_1}\cos\Theta\Gamma^1_{3/2} + \frac{P}{P_2}\sin\Theta\Gamma^2_{3/2} + \frac{d\Theta}{dS}R_0 - \left[F - \frac{P}{P_1}\cos\Theta F_1 - \frac{P}{P_2}\sin\Theta F_2\right] \tag{15}$$

between the instantaneous velocities $\Gamma_{3/2}$, $\Gamma^1_{3/2}$ and $\Gamma^2_{3/2}$ of the motion of E_3 w.r.t. E_2 is obtained. The first relation of (10) written for the time-like ruled surface R_1 is in the form

$$B_1 = D_1 - \rho_{1g} N_1, \quad B_2 = D_2 - \rho_{2g} N_2, \tag{16}$$

for the time-like ruled surfaces R_{11} and R_{21}. If the first relations of (10) and (16) are replaced in (8),

$$D-\rho_g N=\frac{P}{P_1}\cos\Theta(D_1-\rho_{1g}N_1)+\frac{P}{P_2}\sin\Theta(D_2-\rho_{2g}N_2)+\frac{d\Theta}{dS}R_0 \tag{17}$$

is obtained.

By considering the composite motion $(E_2/E_0)=(E_2/E_1)(E_1/E_0)$ of the trihedrons, for the instantaneous velocities of the time-like ruled surfaces R_1, R_{11} and R_{21}, we obtain the relations

$$D=F-\Gamma_{2/1},\quad D_1=F_1-\Gamma^1_{2/1},\quad D_2=F_2-\Gamma^2_{2/1}. \tag{18}$$

If relation (18) is replaced in (17), in the motion of E_2 w.r.t. E_1, for the instantaneous velocities $\Gamma_{2/1}$, $\Gamma^1_{2/1}$ and $\Gamma^2_{2/1}$, the equation

$$\begin{aligned}\Gamma_{2/1}=&\frac{P}{P_1}\cos\Theta\Gamma^1_{2/1}+\frac{P}{P_2}\sin\Theta\Gamma^2_{2/1}-\frac{d\Theta}{dS}R_0+\left[F-\frac{P}{P_1}\cos\Theta F_1-\frac{P}{P_2}\sin\Theta F_2\right]\\ &-\rho_g N+\frac{P}{P_1}\cos\Theta\rho_{1g}N_1+\frac{P}{P_2}\sin\Theta\rho_{2g}N_2\end{aligned} \tag{19}$$

is obtained.

Now let us replace (18) in (13). Here, between the instantaneous velocities in the motions of E_3 w.r.t. E_1 and E_2 w.r.t. E_1, we obtain the relation

$$\begin{aligned}\Gamma_{3/1}-\Gamma_{2/1}=&\frac{P}{P_1}\cos\Theta(\Gamma^1_{3/1}-\Gamma^1_{2/1})+\frac{P}{P_2}\sin\Theta(\Gamma^2_{3/1}-\Gamma^2_{2/1})\\ &+\frac{d\Theta}{dS}R_0-\left[F-\frac{P}{P_1}\cos\Theta F_1-\frac{P}{P_2}\sin\Theta F_2\right].\end{aligned} \tag{20}$$

If (18) is replaced in (15), the relation

$$\begin{aligned}\Gamma_{3/2}+\Gamma_{2/1}=&\frac{P}{P_1}\cos\Theta(\Gamma^1_{3/2}+\Gamma^1_{2/1})+\frac{P}{P_2}\sin\Theta(\Gamma^2_{3/2}+\Gamma^2_{2/1})\\ &+\frac{d\Theta}{dS}R_0-\left[D-\frac{P}{P_1}\cos\Theta D_1-\frac{P}{P_2}\sin\Theta D_2\right]\end{aligned} \tag{21}$$

is obtained for the instantaneous velocities of the motions E_3 w.r.t. E_2 and E_2 w.r.t. E_1.

By using (11), (12) and (18), relations between the instantaneous velocities of the motions E_3 w.r.t. E_1 and E_2 w.r.t. E_1, as well as (20), we also obtain the relations

$$B-F=\Gamma_{3/1}-\Gamma_{2/1},\quad B_1-F_1=\Gamma^1_{3/1}-\Gamma^1_{2/1},\quad B_2-F_2=\Gamma^2_{3/1}-\Gamma^2_{2/1}. \tag{22}$$

Similarly, from (14) and (18), relations between the instantaneous velocities of E_3 w.r.t. E_2 and E_2 w.r.t. E_1, as well as (21), we obtain the relations

$$B-D=\Gamma_{3/2}+\Gamma_{2/1},\quad B_1-D_1=\Gamma^1_{3/2}+\Gamma^1_{2/1},\quad B_2-D_2=\Gamma^2_{3/2}+\Gamma^2_{2/1}. \tag{23}$$

From (11), (12) and (14), relations between the instantaneous velocities of the motions E_3 w.r.t. E_1 and E_2, we obtain the relations

$$D-F=\Gamma_{3/2}-\Gamma_{3/1},\quad D_1-F_1=\Gamma^1_{3/2}-\Gamma^1_{3/1},\quad D_2-F_2=\Gamma^2_{3/2}-\Gamma^2_{3/1}. \tag{24}$$

Thus, from (22), (23) and (24), the relation

$$\Gamma_{3/1} - \Gamma_{2/1} = \Gamma_{3/2} \tag{25}$$

is obtained. Similarly, the relations

$$\Gamma^1_{3/1} - \Gamma^1_{2/1} = \Gamma^1_{3/2} \quad \text{and} \quad \Gamma^2_{3/1} - \Gamma^2_{2/1} = \Gamma^2_{3/2} \tag{26}$$

are obtained. These give us the relations between the instantaneous velocities of the motions E_3 w.r.t. E_1, E_2 w.r.t. E_3 and E_2 w.r.t. E_1, respectively, for the trihedrons of the time-like ruled surfaces R_1, R_{11} and R_{21}. Finally, when (25) and (26) are replaced in (19), for the instantaneous velocities of the motions of E_3 w.r.t. E_1 and E_2, the relation

$$\begin{aligned}\Gamma_{3/1} - \Gamma_{3/2} = {} & \frac{P}{P_1}\cos\Theta(\Gamma^1_{3/1} - \Gamma^1_{3/2}) - \frac{P}{P_2}\sin\Theta(\Gamma^2_{3/1} - \Gamma^2_{3/2}) - \frac{d\Theta}{dS}R_0 \\ & + \left[F - \frac{P}{P_1}\cos\Theta F_1 - \frac{P}{P_2}\sin\Theta F_2\right] - \rho_g N + \frac{P}{P_1}\cos\Theta\rho_{1g}N_1 \\ & + \frac{P}{P_2}\sin\Theta\rho_{2g}N_2\end{aligned}$$

is obtained.

Let (C*) be a dual Lorentzian spherical closed indicatrix curve of the closed time-like ruled surface (R_1). Then, using the Gauss–Bonnet theorem, for the dual Lorentzian spherical image of the ruled surface $R_1(t)$, we may write

$$-\oint G_g ds^* = 2\pi + \iint dA^*, \tag{27}$$

where G_g is the geodesic curvature of (C*).

Theorem 2. *Let S^2_1 be a dual Lorentzian sphere and (C*) a dual Lorentzian spherical indicatrix curve of the closed time-like ruled surface (r_1). Then*

$$\oint G_{g_1} ds^* = -\Lambda_{r_1},$$

where G_{g_1} is the dual geodesic curvature of (C) and Λ_{r_1} is the dual angle of pitch.*

PROOF. By considering the total geodesic curvature along the closed indicatrix curve (C*), we may write

$$\oint G_{g_1} ds^* = \oint \frac{(R_1, R''_1, R''_1)}{\|R'_1\|^3} ds^* = -\oint \langle R_1 \wedge R_2,\, dR_2\rangle = -\oint \langle R_3,\, dR_2\rangle = -\Lambda_{r_1} \tag{28}$$

Thus, we have the following theorem.

Theorem 3. *Let S^2_1 be a dual Lorentzian sphere and (C*) a dual Lorentzian spherical closed indicatrix curve of the time-like ruled surface (R_1). Then, there is the relation*

$$-\Lambda_{r_1} = 2\pi + \mathrm{A}_{r_1} \tag{29}$$

between the dual angle of pitch of the closed time-like ruled surface (R_1) and the real Lorentzian spherical area A_{r_1}.

Proof. From (27),

$$-\oint G_g ds^* = 2\pi + \iint dA^*$$

holds for the dual Lorentzian sphere S_1^2, where $\iint dA^*$ is the area of the closed space-like curve (C^*) on S_1^2. Thus, if we consider relation (28), the relation (29) is easily derived.

Theorem 4. *During the closed motion K/K′ the dual space-like vector* X_1, *which is fixed in the trihedron* $\{E_1, E_2, E_3\}$, *draws a closed time-like ruled surface. The dual angle of pitch of this time-like ruled surface is*

$$\Lambda_{X_1} = -\langle W, X_1\rangle = \lambda_{X_1} + \varepsilon l_{X_1},$$

where W is the dual Steiner vector of the motion K/K′, λ_{X_1} *is the real angle of pitch, and* l_{X_1} *is the pitch of the time-like ruled surface* R_1 [6].

Theorem 5. *Let the dual Darboux, dual Frenet and fixed trihedrons belonging to the striction curve of the closed time-like ruled surface be given by the unit dual vector* $R_1(s) = r_1(s) + \varepsilon r_{10}(s)$ *be* $E_1(X_1, G, N)$, $E_2(x_1 = X_1, X_2, X_3)$ *and* E_0, *respectively. Let the unit dual Lorentzian spheres corresponding to these trihedrons be* K_1, K_2 *and* K_0, *respectively. Since the instantaneous rotation vectors and the dual Steiner vectors of the motions* K_1/K_0, K_2/K_0 *and* K_2/K_1 *are D, F,* Γ_{21} *and* W_d, W_f, *W, respectively, between the Steiner vectors belonging to the composite motion*

$$K_2/K_0 = (K_2/K_1)(K_1/K_0)$$

the relation

$$W = W_f - W_d \tag{30}$$

holds.

Proof. By integrating equation $(18)_1$, which holds between the instantaneous dual rotation vectors of the composite motion $(K_2/K_0) = (K_2/K_1)(K_1/K_0)$, we obtain

$$\oint \Gamma_{2/1} = \oint F - \oint D.$$

By considering Definition 7, we obtain (30).

Theorem 6. *Since the first edge of the trihedron* E_1 (*the dual Darboux trihedron*) *and* E_2 (*the dual Frenet trihedron*) *considered in Theorem 5 is common, then in the composite motion considered between the dual angles of pitch of the ruled surfaces drawn by the common edge in the line space the relation*

$$\Lambda_{x_1} = \Lambda_{x_{1f}} - \Lambda_{x_{1d}} \tag{31}$$

holds.

Proof. From equation (30),

$$\langle W, X_1\rangle = \langle W_f, X_1\rangle - \langle W_d, X_1\rangle$$

is obtained. Then, by using Theorem 4, we obtain (31).

Theorem 7. *Let W_d be the dual Steiner vector in the motion of the dual Darboux trihedron E_1 w.r.t. the fixed trihedron E_0. Between the areas drawn by the edges of the trihedron E_1 on the E_0 dual fixed sphere and W_d the following relation holds*

$$W_d=(A_{x_1}+2\pi)X_1+(A_g+2\pi)G-(A_n+2\pi)N. \tag{32}$$

Also, between the dual spherical areas A_{x_1}, A_g and A_n and the dual angles of pitch Λ_{x_1}, Λ_g and Λ_n the relations

$$A_{x_1}-A_g=\Lambda_g-\Lambda_{x_1},\quad A_{x_1}-A_n=\Lambda_n-\Lambda_{x_1},\quad A_g-A_n=\Lambda_n-\Lambda_g, \tag{33}$$

hold.

PROOF. By considering the first relation of (9) in the motion E_1/E_0, we obtain the dual Steiner vector as

$$W_d=\oint D=-\oint(\tau_g+\varepsilon)X_1-\oint\rho_n G-\oint\rho_g N, \tag{34}$$

and the dual angles of pitch of the ruled surfaces drawn by the edges of the trihedron (X_1, G, N) are obtained as

$$\begin{aligned}&\Lambda_{x_1}=-\langle W_d, X_1\rangle=-\oint(\tau_g+\varepsilon),\quad \Lambda_g=-\langle W_d, G\rangle=\oint\rho_n,\\ &\Lambda_n=-\langle W_d, N\rangle=-\oint\rho_g.\end{aligned} \tag{35}$$

Moreover, by considering (29), we obtain the areas drawn by the moving trihedron (X_1, G, N) on the fixed unit dual Lorentzian sphere as

$$-\Lambda_{x_1}=2\pi+A_{x_1},\quad -\Lambda_g=2\pi+A_g,\quad -\Lambda_n=2\pi+A_n, \tag{36}$$

or

$$2\pi+A_{x_1}=\oint(\tau_g+\varepsilon),\quad 2\pi+A_g=-\oint\rho_n,\quad 2\pi+A_n=\oint\rho_g. \tag{37}$$

If (35) is replaced in (36), then (37) is obtained. Then, by considering (34) and (37), we derive (32). Moreover, from (36), (33) can be obtained.

Theorem 8. *Let A_{r_1}, A_{r_2} and A_{r_3} be the areas of the region drawn by the edges of the Blaschke trihedron (R_1, R_2, R_3) on the fixed unit dual Lorentzian sphere, respectively, and let Λ_{r_1}, Λ_{r_2} and Λ_{r_3} be the dual angles of pitch of the ruled surfaces drawn by these edges. Between the dual Steiner vector W_b and the dual spherical areas the relation*

$$W_b=(A_{r_1}+2\pi)R_1+(A_{r_3}+2\pi)R_3 \tag{38}$$

holds. Also, between the dual angles of pitch of the edges R_1 and R_3 and the dual spherical areas enclosed by these edges on the fixed unit dual Lorentzian sphere the relations

$$A_{r_1}-A_{r_3}=\Lambda_{r_3}-\Lambda_{r_1},\quad -A_{r_1}=\Lambda_{r_1}+A_{r_2},\quad A_{r_2}+\Lambda_{r_1}=-A_{r_3} \tag{39}$$

hold.

Proof. Since the Blaschke vector belonging to the Blaschke trihedron (R_1, R_2, R_3) is known from (14), the Steiner vector of the motion is

$$W_b = \oint B = \oint QR_1 - \oint PR_3. \tag{40}$$

The areas drawn on the fixed unit dual Lorentzian sphere by the edges of the Blaschke trihedron are obtained as

$$-\Lambda_{r_1} = 2\pi + A_{r_1}, \quad A_{r_2} = 2\pi, \quad -\Lambda_{r_3} = 2\pi + A_{r_3}. \tag{41}$$

Here,

$$\Lambda_{r_1} = -\langle W_b, R_1\rangle = -\oint Q, \quad \Lambda_{r_2} = -\langle W_b, R_2\rangle = 0, \quad \Lambda_{r_3} = -\langle W_b, R_3\rangle = \oint P, \tag{42}$$

can be written. From (41) and (42),

$$\oint Q = 2\pi + A_{r_1}, \quad -\oint P = 2\pi + A_{r_3}, \tag{43}$$

are found. If (41) is replaced in (42), then (43) is obtained. Then, by considering (40) and (43), we obtain (38). Moreover, from (41), (39) can be derived.

Theorem 9. *If Λ_{x_1}, Λ_{x_2} and Λ_{x_3} are the dual angles of pitch of time-like and space-like ruled surfaces drawn by the edges of the Frenet trihedron moving on the closed striction curve of the ruled surface R_1, and A_{x_1}, A_{x_2}, A_{x_3} are the closed dual Lorentzian spherical areas drawn by these edges on the fixed unit dual Lorentzian sphere, then, between the areas drawn by the edges of the trihedron and the dual Steiner vector W_f of the motion, the relation*

$$W_f = -(2\pi + A_{x_1})X_1 + (2\pi + A_{x_3})X_3 \tag{44}$$

holds; and between the dual angles of pitch and the dual Lorentzian spherical areas of the closed regions drawn on the unit dual Lorentzian sphere by the edges X_1 and X_3, the relations

$$A_{x_1} - A_{x_3} = \Lambda_{x_3} - \Lambda_{x_1}, \quad -A_{x_1} = \Lambda_{x_1} - A_{x_2}, \quad -A_{x_3} = \Lambda_{x_3} - A_{x_2}, \tag{45}$$

hold.

Proof. The dual instantaneous velocity of the dual Frenet trihedron is known from the second relation of (9). The Steiner vector of the motion is

$$W_f = \oint F = -\oint (\tau + \varepsilon)X_1 + \oint \rho X_3. \tag{46}$$

The areas on the fixed unit dual Lorentzian sphere drawn by the edges are obtained as

$$-\Lambda_{x_1} = 2\pi + A_{x_1}, \quad A_{x_2} = 2\pi, \quad -\Lambda_{x_3} = 2\pi + A_{x_3}. \tag{47}$$

Here,

$$\Lambda_{x_1} = -\langle W_f, X_1\rangle = -\oint (\tau + \varepsilon), \quad \Lambda_{x_2} = -\langle W_f, X_2\rangle = 0, \quad \Lambda_{x_3} = -\langle W_f, X_3\rangle = -\oint \rho, \tag{48}$$

can be written. From (47) and (48),

$$\oint (\tau+\varepsilon)=2\pi+A_{x_1}, \qquad \oint \rho=2\pi+A_{x_3} \tag{49}$$

are obtained. If (48) is inserted into (47), then (49) is obtained. Then, considering (46) and (49), we obtain (44). Moreover, from (47), (45) is obtained.

Theorem 10. *Let K_f, K_d and K_b be the dual spheres represented by the dual Frenet, dual Darboux and Blaschke trihedrons, respectively, moving on the closed striction curve of the ruled surface R_1, and let K_0 be the fixed unit dual Lorentzian sphere. The instantaneous velocity vectors belonging to the motions K_f/K_0, K_d/K_0 and K_b/K_0 are F, D and B, respectively. Also, between the dual Steiner vectors of these motions the relations*

$$W_d=W_f+\oint(\tau-\tau_g)X_1, \quad W_b=W_f-\oint \rho_g N-(\tau_g-\tau)X_1, \quad W_b=W_d-\oint \rho_g N, \tag{50}$$

hold. Moreover, between the dual Steiner vectors in the motion of the unit Lorentzian spheres K_f, K_d and K_b w.r.t. each other the relation

$$W_{3/1}=W_{3/2}+W_{2/1} \tag{51}$$

holds.

PROOF. By applying a line integral to equation (10), we obtain (50). Moreover, by adding (50) side by side, we obtain (51).

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