A SCHWARZ LEMMA FOR BOUNDED SYMMETRIC DOMAINS IN HILBERT SPACES

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Abstract

Let D be an irreducible bounded symmetric domain in a complex Hilbert space and let $f : D \to D$ be a holomorphic map with $f(0) = 0$. We give conditions under which f is linear, injective or isometric.

1. Introduction

We first recall the Schwarz lemma for the open unit disc Δ in the complex plane $\mathbf{C}$, which states that, *given a holomorphic map* $f : \Delta \to \Delta$ *with* $f(0) = 0$, *the following conditions hold:*

(i) $|f(z)| \le |z|$ *for any* $z \in \Delta$;

(ii) *if there exists* $z_0 \in \Delta \backslash \{0\}$ *such that* $|f(z_0)| = |z_0|$ (*or if* $|f'(0)| = 1$), *then* $f(z) = \lambda z$ *for some complex number* λ *of unit modulus*; *equivalently*, f *is a linear isometry.*

The Schwarz lemma has been extended in many directions (see, for example, [3; 4; 8; 9; 14; 15; 16; 19; 20]). Condition (i) in the lemma can be extended to several complex variables and infinite dimension by the Hahn–Banach theorem, namely, if $f : B \to B$ is a holomorphic map on the open unit ball B of any complex Banach space with $f(0) = 0$, then $\|f(z)\| \le \|z\|$ for all $z \in B$. However, condition (ii) above no longer holds, even for the bidisc $\Delta \times \Delta$. In fact, one can easily construct a holomorphic self-map on $\Delta \times \Delta$ such that $f(0) = 0$ and $\|f(z)\| = \|z\|$ for z in an open subset of $\Delta \times \Delta$, but f is not an isometry (cf. [19]). Nevertheless, Vigué [19; 20] (see also [16]) has shown that, for *irreducible* bounded symmetric domains in $\mathbf{C}^n$, the condition that $f(0) = 0$ and $\|f(z)\| = \|z\|$ on a non-empty open subset does indeed imply that f is a linear isometry. It is natural to ask whether this result can be extended to infinite dimensions. In this paper we consider the irreducible bounded symmetric domains in infinite-dimensional Hilbert spaces. Kaup [12] has classified the bounded symmetric domains in Hilbert spaces. He showed that the *irreducible* ones are of the type $\triangle_q$ and D_{pq}. We prove a generalisation of the Schwarz lemma for a class of bounded symmetric domains including those of the type D_{pq}. In particular, Vigué's result holds for type D_{pq} domains.

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We note that the Schwarz lemma has already been extended to the open unit balls in Hilbert spaces (e.g. [6; 9]) and, more generally, to the open unit balls in strictly convex Banach spaces (cf. [8; 19]). One crucial property of these balls is used to deduce the Schwarz lemma, namely, their boundary points are exactly the complex extreme points. In infinite dimensions, domains of the type D_{pq} have the weaker property that their extreme points are only weakly dense in the boundary. Therefore we begin by proving a version of the Schwarz lemma for domains in complex Banach spaces with weakly dense extreme points in the boundary. Domains of the type D_{pq} are biholomorphically equivalent to the open unit balls of the Banach spaces $\mathcal{B}(H, K)$ of bounded linear operators between Hilbert spaces H and K with $\dim H < \aleph_0 \leq \dim K$ [12]. This enables us to apply Morita's algebraic geometric arguments in [16] to the finite-rank operators to obtain a Schwarz lemma for these domains.

2. Domains in Hilbert spaces

Throughout, all Banach spaces are over the complex field. By Kaup's Riemann mapping theorem [13], the bounded symmetric domains in Banach spaces are biholomorphically equivalent to the open unit balls of a special class of Banach spaces, called JB^*-triples. A *JB^*-triple* is a complex Banach space E equipped with a Jordan triple product

$$\{\cdot, \cdot, \cdot\} : E \times E \times E \to E$$

that is symmetric linear in the outer variables and conjugate linear in the middle variable such that, for $a, b, x, y, z \in E$,

(i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$;

(ii) the linear map $v \in E \mapsto \{z, z, v\} \in E$ is Hermitian with non-negative spectrum;

(iii) $\|z^{(3)}\| = \|z\|^3$, where $z^{(3)} = \{z, z, z\}$.

JB^*-triples include C^*-algebras, spaces of rectangular matrices, and the spin factors. Let H, K be Hilbert spaces and let $\mathcal{B}(H, K)$ be the Banach space of bounded linear operators between H and K. Then $\mathcal{B}(H, K)$ is a JB^*-triple with the triple product

$$\{a, b, c\} = \tfrac{1}{2}(ab^*c + cb^*a).$$

A Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ is a JB^*-triple in the triple product

$$2\{x, y, z\} = \langle x, y \rangle z + \langle z, y \rangle x.$$

We write $\mathcal{B}(H) = \mathcal{B}(H, H)$ and denote by $\mathcal{S}(H, K)$ the Banach space of Hilbert–Schmidt operators between H and K, with the Hilbert–Schmidt norm

$$\|z\|_2 = \left(\sum_{h \in \mathcal{O}} \|z(h)\|^2 \right)^{\frac{1}{2}}$$

where $\mathcal{O}$ is an orthonormal basis of H. We note that $\|z\| \leq \|z\|_2$ for all $z \in \mathcal{S}(H, K)$ where $\|\cdot\|$ is the norm in $\mathcal{B}(H, K)$. We also note that $\mathcal{S}(H, K)$ is $\|\cdot\|$-dense in the Banach space $\mathcal{K}(H, K)$ of compact operators between H and K and is weak*-dense in $\mathcal{B}(H, K)$.

A Hilbert space H of dimension at least two with a conjugation $h \mapsto \bar{h}$ gives rise to a *spin factor* $Z = \mathbf{C} \oplus H$ that is a JB^*-triple in the following norm and triple product

$$\|z\|^2 = \langle z, z \rangle + \sqrt{\langle z, z \rangle - |\langle z, \bar{z} \rangle|^2}$$

$$\{x, y, z\} = \langle x, y\rangle z + \langle z, y\rangle x - \langle x, \bar{z}\rangle \bar{y}$$

where $\langle\cdot, \cdot\rangle$ is the natural inner product on Z and $\overline{\alpha + h} = \bar{\alpha} + \bar{h}$ for $\alpha \in \mathbf{C}$ and $h \in H$.

A surjective linear map $f : E \to F$ between two JB^*-triples E and F is an isometry if and only if it is a triple isomorphism, in other words, it is a linear isomorphism and preserves the triple product. We refer to [2] for a recent brief introduction to JB^*-triples and symmetric domains. A detailed exposition and relevant references can be found in [17].

The bounded symmetric domains in a Hilbert space can be classified. Kaup [12] has shown that there are only two types of irreducible domain: the ones of type Δ_q are biholomorphically equivalent to the open unit balls of spin factors, and the ones of type D_{pq} are biholomorphically equivalent to the open unit balls of the JB^*-triples $\mathcal{B}(H, K)$ with $1 \le p = \dim H < \aleph_0 \le q = \dim K$. We will prove a generalisation of the Schwarz lemma for the open unit balls of $\mathcal{B}(H, K)$, for arbitrary H and K, which include the domains of type D_{pq}.

Let $\Delta = \{\zeta \in \mathbf{C} : |\zeta| < 1\}$ be the unit disc in the complex plane $\mathbf{C}$. The Poincaré distance ρ on Δ is defined by

$$\rho(\zeta, \eta) = \tfrac{1}{2}\log \frac{1 + |(\zeta - \eta)/(1 - \zeta\bar{\eta})|}{1 - |(\zeta - \eta)/(1 - \zeta\bar{\eta})|} \quad (\zeta, \eta \in \Delta).$$

For a domain D in a Banach space E, the *Carathéodory distance* C_D on D is defined by

$$C_D(z, w) = \sup\{\rho(f(z), f(w)) : f \text{ is a holomorphic map from } D \text{ to } \Delta\} \ (z, w \in D).$$

We note that $C_\Delta(\zeta, \eta) = \rho(\zeta, \eta)$. A holomorphic map $\varphi: \Delta \to D$ is called a *complex geodesic on D* if it preserves the Carathéodory distance, that is,

$$C_D(\varphi(\zeta), \varphi(\eta)) = \rho(\zeta, \eta) \quad (\zeta, \eta \in \Delta).$$

By [18, proposition 3.3], φ is a complex geodesic if $C_D(\varphi(0), \varphi(\zeta)) = C_\Delta(0, \zeta)$ for some $\zeta \in \Delta\backslash\{0\}$. We also note that, for a convex domain, the Carathéodory distance coincides with the Kobayashi distance (cf. [7, p. 173]), and, in particular, for the open unit ball B of a Banach space, we have $C_B(0, x) = C_\Delta(0, \|x\|) = \tanh^{-1} \|x\|$ for all $x \in B$ (cf. [5, p. 85]).

Let D be a convex subset of a Banach space E. A point $z \in D$ is called a *real extreme point of D* if $z \pm w \in D$ implies $w = 0$; more generally, it is called a *complex extreme point of D* if $z + \Delta w \subset D$ implies $w = 0$.

3. Schwarz lemma

By a *dual Banach space* we mean a Banach space E that is the dual of a Banach space E_*, and in this case the *weak*-topology* on E is induced by the pair (E, E_*). A function between two dual Banach spaces is *weak*-continuous* if it is continuous with respect to the weak*-topologies of the two spaces. We denote the norm closure of a set $S \subset E$ by $\overline{S}$, and the weak*-closure by $\overline{\overline{S}}$. We begin by proving a general result for domains in dual Banach spaces with weak*-dense extreme points in the boundary. We prove two lemmas first.

Lemma 3.1. *Let B be an open ball in a dual Banach space E and let* $f : B \to \mathbf{C}$ *be a weak*-continuous holomorphic function. Then the derivative* $f'(a) : E \to \mathbf{C}$ *is also weak*-continuous for all* $a \in B$.

Proof. Let (x_α) be a net in E weak*-convergent to x in E. We show that $f'(a)(x_\alpha)\to f'(a)(x)$. Let $\varepsilon>0$. There exists $\delta>0$ such that

$$|f(a+h)-f(a)-f'(a)(h)|<\varepsilon\|h\|$$

whenever $\|h\|<\delta$. By the uniform boundedness principle, there exists $\delta'>0$ such that $\|\delta' x_\alpha\|$, $\|\delta' x\|<\delta$ and $a+\delta' x_\alpha$, $a+\delta' x\in B$. We have

$$\begin{aligned}\delta'|f'(a)(x_\alpha)-f'(a)(x)| &\le |f(a+\delta' x_\alpha)-f(a)-f'(a)(\delta' x_\alpha)|\\ &\quad+|f(a+\delta' x_\alpha)-f(a)-f(a+\delta' x)+f(a)|\\ &\quad+|f(a+\delta' x)-f(a)-f'(a)(\delta' x)|\\ &\le \varepsilon\|\delta' x_\alpha\|+|f(a+\delta' x_\alpha)-f(a+\delta' x)|+\varepsilon\|\delta' x\|,\end{aligned}$$

which completes the proof. ■

Lemma 3.2. *Let B be an open ball in a dual Banach space E and let $f:B\to F$ be a weak*-continuous holomorphic map into a dual Banach space F. Then the derivative $f'(a):E\to F$ is also weak*-continuous for all $a\in B$.*

Proof. This follows immediately from the above lemma by observing that $\varphi\circ f'(a)=(\varphi\circ f)'(a)$ is weak*-continuous for every φ in the predual F_* and $a\in B$. ■

Alternatively, the above result follows from the Cauchy integral formula, which also implies that, for weak*-continuous $f:B\to F$, the polynomial

$$P_k(z)=\frac{1}{k!}f^{(k)}(0)(z,\ldots,z)$$

is weak*-continuous on E, for $k\in\mathbf{N}$.

When we say that a holomorphic map $f:B\to F$ is linear, it is understood that f is the restriction of a linear map on the ambient Banach space $E\supset B$, in which case we also denote this linear map by f.

We give the following simple example to illustrate the conditions in Proposition 3.3.

Example. Let $f:\Delta\times\Delta\to\Delta\times\Delta$ be a holomorphic map on the bidisc $\Delta\times\Delta$ defined by $f(z,w)=(z,0)$. Plainly f is not an isometry. Let $V=\{(z,w)\in\Delta\times\Delta : |z|>|w|\}$, which is open in $\Delta\times\Delta$. We have $\|f(z,w)\|=\|(z,w)\|$ for $(z,w)\in V$, and the image of f does not contain any non-empty open subset of $\Delta\times\Delta$. Also, $U=\{(z,w)\in\Delta\times\Delta : |z|>\frac{1}{2}>|w|\}$ is a proper open subset of V such that $f(V)=f(V\backslash U)$ and $f(\overline{V}\backslash U)=f(\overline{V})$ where $\overline{V}=\{(z,w):|z|\ge|w|\}$. This motivates the following definition.

We call a holomorphic map $f:B\to F$ *exact* if, for any closed subset $D\subset B$ and non-empty open subset $U\subset D$, the condition $D\backslash U\neq\emptyset$ implies $f(D\backslash U)\neq f(D)$.

Let l_2 be the Hilbert space of square summable sequences. The map $f:l_2\to l_2$ given by

$$f(z_1,z_2,z_3,\ldots)=(e^{z_1},\, z_1(z_1+1),\, z_2,\, z_3,\ldots)$$

is exact but not injective.

Proposition 3.3. *Let E be a dual Banach space and let $B=\{z\in E : \|z\|<1\}$ be the open unit ball such that the complex extreme points in its closure $\overline{B}$ are weak*-dense in the boundary ∂B. Let $f:B\to B$ be an exact weak*-continuous holomorphic map with $f(0)=0$. If there is a non-empty relatively weak*-open subset V in B such that $\|f(w)\|=\|w\|$ for all*

$w \in V$, and if $f(V)$ has non-empty interior in the relative weak-topology on B, then f is linear and injective.*

PROOF. Let G be a non-empty relatively weak*-open subset of $f(V)$ and let $G \cap B_r \neq \emptyset$, where $B_r = rB$ and $0 < r < 1$. Since $\|f(w)\| = \|w\|$ for all $w \in V$, we have $G \cap B_r \subset f(V \cap B_r)$. By taking the inverse image and cutting down to a smaller set, we may assume that $f(V \cap B_r)$ is relatively weak*-open in B_r, and we consider $f : \overline{B_r} \to \overline{B_r}$, as $\|f(z)\| \leq \|z\|$ for $z \in B$.

We first show that, for any non-empty relatively weak*-open subset U of $V \cap B_r$, the image $f(U)$ contains a non-empty relatively weak*-open subset of B_r. We have

$$f(U) \supset f(V \cap B_r) \backslash f(\overline{\overline{V \cap B_r}} \backslash U),$$

and the latter set is relatively weak*-open in B_r since f maps weak*-compact sets to weak*-compact sets. It is also non-empty since f is exact.

Let $z \in (V \cap B_r) \backslash \{0\}$. Let $\mathcal{N} = \{V_\alpha\}$ be a weak*-open neighbourhood base at z which forms a decreasing net. Then each $f(V_\alpha \cap V \cap B_r)$ contains a non-empty relatively weak*-open subset in B_r, and we can choose $\dfrac{rf(z_\alpha)}{\|f(z_\alpha)\|} \in \mathbf{R} f(V_\alpha \cap V \cap B_r) \cap \operatorname{Ext}(\overline{B_r})$ with $z_\alpha \in V_\alpha$, by the density of $\operatorname{Ext}(\overline{B_r})$ in ∂B_r. We have $\dfrac{f(z_\alpha)}{\|f(z_\alpha)\|} \in \operatorname{Ext}(\overline{B})$ and $z = w^*\text{-}\lim_\alpha z_\alpha$.

For each α, we set $\varphi_\alpha(\zeta) = \zeta z_\alpha / \|z_\alpha\|$ for $\zeta \in \Delta$. Then $f \circ \varphi_\alpha$ is a complex geodesic of B because $C_B(f \circ \varphi(0), f \circ \varphi(\|z_\alpha\|)) = C_B(0, f(z_\alpha)) = \tanh^{-1} \|f(z_\alpha)\| = \tanh^{-1} \|z_\alpha\| = C_\Delta(0, \|z_\alpha\|)$.

Since $\dfrac{f(z_\alpha)}{\|f(z_\alpha)\|} \in \operatorname{Ext}(\overline{B})$, by a result of Vesentini [18, lemma 3.5], $\varphi(\zeta) = \dfrac{\zeta f(z_\alpha)}{\|f(z_\alpha)\|}$ is the unique complex geodesic in B passing through 0 and $f(z_\alpha)$. So $f \circ \varphi_\alpha(\zeta) = \zeta w_\alpha / \|w_\alpha\|$ where $w_\alpha = f(z_\alpha)$.

Now let $f(w) = \sum_{k=1}^{\infty} P_k(w)$ be the Taylor expansion of f by k-homogeneous polynomials P_k at 0. By the remarks following Lemma 3.2, each P_k is weak*-continuous. From $\zeta \dfrac{w_\alpha}{\|w_\alpha\|} = f \circ \varphi_\alpha(\zeta) = \sum_{k=1}^{\infty} P_k\left(\dfrac{z_\alpha}{\|z_\alpha\|}\right) \zeta^k$, we obtain $P_k(z_\alpha) = 0$ for $k \geq 2$. Therefore $P_k(z) = \lim_{\alpha \to \infty} P_k(z_\alpha) = 0$ for $k \geq 2$. As $z \in V \cap B_r$ was arbitrary, by analytic continuation we have $f = P_1$, that is, f is linear.

Now we assume $f(w) = 0$. As before, since V is relatively weak*-open in B and $\operatorname{Ext}(\overline{B})$ is weak*-dense in δB, there exists $z \in V$ such that $z/\|z\| \in \operatorname{Ext}(\overline{B})$. We can find $\delta > 0$ with $\{z + \delta\zeta w : \zeta \in \Delta\} \subset V$. Then $\|z + \delta\zeta w\| = \|f(z + \delta\zeta w)\| = \|f(z)\| = \|z\|$, that is,

$$\left\| \frac{z}{\|z\|} + \zeta \frac{\delta w}{\|z\|} \right\| = 1 \quad (\zeta \in \Delta).$$

It follows that $w = 0$, as $\dfrac{z}{\|z\|}$ is a complex extreme point. So f is injective. ■

Example. Let Z be a spin factor. Then the real extreme points in the closed unit ball Z_1^* of the dual Z^* are weak*-dense in Z_1^* [1]. Therefore the above result applies to the open unit ball of Z^*. However, the extreme points of the closed unit ball in Z are not weakly dense.

We note that a complex extreme point of the closed unit ball of a JB^*-triple is also a real extreme point (cf. [10, Satz 2.22]). Infinite-dimensional domains of the type D_{pq} also have weak*-dense extreme points in the boundary. Indeed, let B be the open unit ball of $\mathcal{B}(H)$. The unitary elements in $\mathcal{B}(H)$ are real extreme points of $\overline{B}$, and, if H is infinite-dimensional, then they are weak*-dense in ∂B (in fact, weak*-dense in $\overline{B}$) (cf. [11]). As in [11, 49], every linear isometry from a Hilbert space H into another one K is a real extreme point in the closed unit ball of $\mathcal{B}(H, K)$. Given that $\dim H \leq \dim K$, we may regard H as a closed subspace of K, in which case the unitary elements in $\mathcal{B}(K)$ restrict to isometries between H and K. So the extreme points in the closed unit ball of $\mathcal{B}(H, K)$ are also weak*-dense in the boundary if $\dim K = \infty$. Therefore we have the following result.

Corollary 3.4. *Let B be the open unit ball of $\mathcal{B}(H, K)$ with* $\dim K = \infty$. *Let $f : B \to B$ be a holomorphic map with $f(0) = 0$. If f is exact and weak*-continuous, if there exists a non-empty relatively weak*-open subset V in B such that $\| f(w)\| = \|w\|$ for all $w \in V$, and if $f(V)$ has non-empty interior in the relative weak* topology on B, then f is linear and injective.*

We remark that a linear bijection on $\mathcal{B}(H)$ need not be an isometry even if it preserves norm on an infinite set. For example, the linear map $f : M_2 \to M_2$ given by

$$f(A) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} A$$

is clearly bijective but not isometric, although it restricts to the identity map on the closed set

$$\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{C} \right\}$$

which intersects the open unit ball. Nevertheless, we can improve the above corollary by removing the assumption on $f(V)$, showing that f is linear and injective under a rank assumption but without assuming that f is exact. This is proved in Theorem 3.5 below.

We note that a linear bijection on $\mathcal{B}(H)$ need not preserve ranks. Consider the linear bijection

$$\begin{pmatrix} a & b & c \\ g & h & l \\ s & x & y \end{pmatrix} \in M_3 \mapsto \begin{pmatrix} h & b & c \\ g & a & l \\ s & x & y \end{pmatrix} \in M_3$$

which maps $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and these two matrices are of different ranks but of the same norm $\sqrt{3}$. In fact, the map is isometric on the closed set

$$\left\{ \begin{pmatrix} a & b & 0 \\ g & h & 0 \\ 0 & 0 & y \end{pmatrix} : a, b, g, h, y \in \mathbf{C} \right\}$$

which intersects the open unit ball.

We also note that a *non-surjective* linear isometry on $\mathcal{B}(H)$ need not preserve ranks. For instance, the linear isometry

$$z \in \mathcal{B}(l_2) \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in \mathcal{B}(l_2 \oplus l_2) \approx \mathcal{B}(l_2)$$

does not preserve ranks.

The following result for domains including those of type D_{pq} generalises the finite-dimensional result of Vigué [19] and Morita [16].

Theorem 3.5. *Let H and K be Hilbert spaces and let B be the open unit ball of $\mathscr{B}(H, K)$. Let $f : B \to B$ be holomorphic and $f(0) = 0$. Suppose that there exists an open subset V in B, intersecting $\mathscr{K}(H, K)$, such that*

(i) $\|f(z)\| = \|z\|$ *for* $z \in V$;

(ii) $\sup\{\operatorname{rank} f(w) : w \in B, \operatorname{rank} w \leq n\} < \infty$ *for* $n \in \mathbf{N}$.

Then $f|_{B \cap \mathscr{K}(H,K)}$ is linear and isometric. Further, f is linear if it is weak-continuous, and is injective if, in addition, V is relatively weak*-open in B.*

PROOF. We may assume that H and K are infinite-dimensional since the other cases can be proved similarly without condition (ii). By assumption, V contains a finite-rank operator. We can therefore, using condition (ii), find finite-dimensional subspaces $H_1 \subset H$ and $K_1 \subset K$ with $\dim H_1 = n$ and

$$m = \dim K_1 \geq \sup\{\operatorname{rank} f(w) : \operatorname{rank} w \leq n\}$$

such that $V \cap \mathscr{B}(H_1, K_1) \neq \emptyset$ where $\mathscr{B}(H_1, K_1)$ is naturally identified as a subspace of $\mathscr{B}(H, K)$.

For every $w \in B \cap \mathscr{B}(H_1, K_1)$ we have $\operatorname{rank} f(w) \leq m$ and there is a linear isometric embedding $U_w : f(w)(H) \to K_1$ such that $U_w^* U_w f(w) = f(w)$ and $U_w f(w) f(w)^* U_w^* \in \mathscr{B}(K_1)$.

Following Morita [16, p. 491], if we regard an element $w \in \mathscr{B}(H_1, K_1)$ as a complex matrix $w = (w_{\alpha\beta})$ with $w_{\alpha\beta} = x_{\alpha\beta} + i y_{\alpha\beta}$ and $x_{\alpha\beta}, y_{\alpha\beta} \in \mathbf{R}$, then the determinant $\det(\lambda I - ww^*)$ can be regarded as a polynomial with coefficients in the polynomial ring

$$\mathbf{C}[x_{11}, x_{21}, \ldots, x_{mn}, y_{11}, y_{21}, \ldots, y_{mn}] := \mathbf{C}[x, y]$$

over $\mathbf{C}$, where $x_{11}, x_{21}, \ldots, x_{mn}, y_{11}, y_{21}, \ldots, y_{mn}$ are independent indeterminates. We note that $\det(\lambda I - ww^*)$ does not depend on the matrix representation of w since $\det(\lambda I - uww^*u^*) = \det(\lambda I - ww^*)$ for any unitary u.

It has been shown in [16, p. 491] that the polynomial $\varphi(\lambda, x, y) := \det(\lambda I - ww^*) = \lambda^m + a_1(x, y)\lambda^{m-1} + \cdots + a_m(x, y)$ is irreducible in the ring $\mathbf{C}[x, y][\lambda]$. Thus the equation $\varphi(\lambda, x, y) = 0$ defines a multi-valued algebraic function $\lambda = \Phi(x, y)$ of complex variables $x_{11}, \ldots, y_{11}, \ldots, y_{mn}$. Now, for each $w \in V \cap \mathscr{B}(H_1, K_1)$, the characteristic polynomials $\varphi(\lambda, x, y)$ and $\psi(\lambda, x, y) := \det(\lambda I - U_w f(w) f(w)^* U_w^*) = \det(\lambda I - f(w) f(w)^*) = \lambda^m + b_1(x, y)\lambda^{m-1} + \cdots + b_m(x, y) \in \mathbf{C}[\lambda]$ have a common root in λ since $\|ww^*\| = \|f(w)f(w)^*\|$ is the maximum eigenvalue. Hence a suitable branch of $\Phi(x, y)$ satisfies $\psi(\lambda, x, y) = 0$ on $V \cap \mathscr{B}(H_1, K_1)$ and it follows that, as in [16, p. 492], $\varphi(\lambda, x, y) = \psi(\lambda, x, y)$ on $B \cap \mathscr{B}(H_1, K_1)$ by analytic continuation. Therefore ww^* and $f(w)f(w)^*$ have the same eigenvalues for all $w \in B \cap \mathscr{B}(H_1, K_1)$, and, in particular, we have $\|w\| = \|ww^*\|^{\frac{1}{2}} = \|f(w)f(w)^*\|^{\frac{1}{2}} = \|f(w)\|$. We also have

$$\operatorname{trace}(ww^*) = \operatorname{trace}(f(w)f(w)^*),$$

that is, $\|w\|_2 = \|f(w)\|_2$ for $w \in B \cap \mathscr{B}(H_1, K_1)$, where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm.

For finite-dimensional subspaces $H_2 \subset H$ and $K_2 \subset K$ containing H_1 and K_1 respectively, we have $\mathscr{B}(H_2, K_2) \cap V \supset \mathscr{B}(H_1, K_1) \cap V \neq \emptyset$, and likewise we have $\|f(w)\| = \|w\|$ and $\|f(w)\|_2 = \|w\|_2$ for $w \in B \cap \mathscr{B}(H_2, K_2)$. Let B_2 be the open unit ball of $\mathscr{S}(H, K)$, the space of Hilbert–Schmidt operators. Since the finite-rank operators are $\|\cdot\|_2$-dense in the Hilbert space $\mathscr{S}(H, K)$, we have therefore shown that $f(B_2) \subset B_2$. By [9], $f|_{B\cap\mathscr{S}(H,K)}$ is linear and $\|\cdot\|_2$-isometric on $\mathscr{S}(H, K)$. As $\mathscr{S}(H, K)$ is norm-dense in $\mathscr{K}(H, K)$, we have $f|_{B\cap\mathscr{K}(H,K)} = f'(0)|_{B\cap\mathscr{K}(H,K)}$. Also, f is $\|\cdot\|$-isometric on $\mathscr{K}(H, K)$ since the finite-rank operators are $\|\cdot\|$-dense in $\mathscr{K}(H, K)$.

Let f be weak*-continuous. Then the weak*-density of $\mathscr{K}(H,K)$ in $\mathscr{B}(H,K)$ and the Kaplansky density theorem imply that f is linear.

Finally, one can show that f is injective, as in the last paragraph of the proof of Proposition 3.3, using the weak*-density of the extreme points in the boundary. ■

Remarks. 1. If the domain B above is of type D_{pq}, then $\dim H < \infty$, $\mathscr{B}(H, K) = \mathscr{K}(H, K)$ and condition (ii) is redundant.

2. By a result of Fan [4, corollary 2], if B is the open unit ball in $\mathscr{B}(H)$ and if $f : B \to B$ is of the form $f(z) = \Psi(z)$ for some holomorphic function $\Psi : \Delta \to \Delta$, where $\Psi(z)$ is defined by the Riesz functional calculus, then rank $f(w) \leq$ rank w for any finite-rank operator $w \in B$, in which case condition (ii) above is satisfied.

3. Since $\mathscr{K}(H, K)$ is weak*-dense in $\mathscr{B}(H, K)$, the map f in Theorem 3.5 has a weak*-closed range in $\mathscr{B}(H, K)$, as f is isometric on $\mathscr{K}(H, K)$. If one can show that f preserves the triple product in $\mathscr{K}(H, K)$, then f would be a triple monomorphism on $\mathscr{B}(H, K)$ and therefore an isometry. However, we note that a *non-surjective* linear isometry between *JB**-triples need not preserve triple product, as the following example shows.

Example. Let $\mathbf{c}$ be the *JB**-triple of convergent sequences. Define $f : \mathbf{c} \to \mathbf{c}$ by

$$f(z_1, z_2, z_3, \ldots) = \left(\tfrac{1}{2}(z_1 + z_2), z_1, z_2, z_3, \ldots\right).$$

Then f is a linear isometry but does not preserve the triple product.

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