A KRASNOSELSKII CONE COMPRESSION THEOREM FOR $\mathcal{U}_c^k$ MAPS

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[Received 31 July 2001. Read 24 January 2002. Published 30 June 2003.]

Abstract

The main purpose of this paper is to prove a generalisation of the Krasnoselskii cone compression theorem for the $\mathcal{U}_c^k$ maps of Park. Our analysis is elementary and relies on the fact that the unit sphere (in a cone) is a retract of the unit ball (in a cone).

1. Introduction

In this paper we establish various Krasnoselskii cone compression theorems for the $\mathcal{U}_c^k$ maps of Park. The result is well known for Kututani maps [1; 9] (and for some other classes). We note that the usual technique (index or essential map theory) cannot be extended to $\mathcal{U}_c^k$ maps. However, using the fact that there exists a retraction from the unit ball (in a cone) to the unit sphere (in a cone), we are able to present a Krasnoselskii cone compression theorem for compact $\mathcal{U}_c^k$ maps. An added bonus is that the proof is elementary. Also in this paper we discuss compression theorems for countably k-set-contractive $\mathcal{U}_c^k(\overline{U}, E)$ maps where U is an open set in an infinite-dimensional normed linear space E.

For the remainder of this section we present some definitions and known results that will be needed. Let C be a non-empty, convex subset of a Hausdorff topological vector space X. Recall that a *polytope* P in C is any convex hull of a non-empty finite subset of C. Of particular importance in this paper will be the class $\mathcal{U}_c^k$ (see [8]). X and Y are Hausdorff topological vector spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F: X \to 2^Y$ (non-empty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

- (i) $\mathcal{U}$ contains the class $\mathcal{C}$ of single-valued continuous functions;
- (ii) each $F \in \mathcal{U}_c$ is upper-semicontinuous and compact-valued; and
- (iii) for any polytope P, $F \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.1. $F \in \mathcal{U}_c^k(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Let (E, d) be a pseudo-metric space. For $S \subseteq E$, let $B(S, \varepsilon) = \{x \in E : d(x, S) \leq \varepsilon\}$, $\varepsilon > 0$, where $d(x, S) = \inf_{y \in S} d(x, y)$. The measure of non-compactness [5] of the set $M \subseteq E$ is

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Mathematical Proceedings of the Royal Irish Academy, **103**A (1), 55–59 (2003)

defined by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\varepsilon > 0 : M \subseteq B(A, \varepsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F: S \to 2^E$; here $S \subseteq E$. The map F is said to be a *countably P-concentrative mapping* [5] if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \leq \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e. X is not precompact in the pseudo-normed space (E, p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(.)$ denotes the measure of non-compactness in the pseudo-normed space (E, p).

In [7] we established the following fixed-point theorem.

Theorem 1.1. *Let Ω be a non-empty, closed, convex subset of a Fréchet space E (P is a defining system of seminorms). Suppose that $F \in \mathcal{U}_c^k(\Omega, \Omega)$ is a countably P-concentrative mapping. Then F has a fixed point in Ω.*

Remark 1.1. Theorem 1.1 was established by Park [8] for compact $\mathcal{U}_c^k$ maps.

Finally for completeness we also give the definition of countably k-set-contractive maps. Let X be a metric space and $P_B(X)$ the bounded subsets of X. The Kuratowskii measure of non-compactness is the map $\alpha : P_B(X) \to [0, \infty)$ defined by

$$\alpha(A) = \inf\left\{ \varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n} X_i \text{ and } \operatorname{diam}(X_i) \leq \varepsilon \right\};$$

here $A \in P_B(X)$. Let S be a non-empty subset of X and let $H: S \to 2^X$. H is called *countably k-set-contractive* ($k \geq 0$) if $H(S)$ is bounded and $\alpha(H(\Omega)) \leq k\alpha(\Omega)$ for all countably bounded sets Ω of S.

2. Fixed-point theory

Let $E = (E, \|.\|)$ be a normed linear space and let $C \subseteq E$ be a cone (i.e. C is a closed, convex invariant under multiplication by non-negative real numbers, and $C \cap (-C) = \{0\}$). For $\rho > 0$ let

$$B_\rho = \{x \in C : \|x\| < \rho\}, \quad \overline{B_\rho} = \{x \in C : \|x\| \leq \rho\}$$

with

$$S_\rho = \{x \in C : \|x\| = \rho\} \quad \text{and} \quad EB_\rho = \{x \in C : \|x\| \geq \rho\}.$$

Our first result is a Krasnoselskii cone compression theorem for $\mathcal{U}_c^k$ maps.

Theorem 2.1. *Let E and C be as above and let r, R be constants with $0 < r < R$. Suppose that $F \in \mathcal{U}_c^k(\overline{B_R}, C)$ is compact, with*

$$F(S_r) \subseteq EB_r \quad \text{and} \quad F(S_R) \subseteq \overline{B_R}. \tag{2.1}$$

Then F has a fixed point in $B_{r,R} = \{x \in C : r \leq \|x\| \leq R\}$.

PROOF. We know that there exists a continuous retraction $r_0 : \overline{B_r} \to S_r$ (indeed if we fix $x_0 \in S_r$ then

$$r_0(x) = \frac{r\{(r - \|x\|)x_0 + x\}}{\|(r - \|x\|)x_0 + x\|}$$

will do; note that $(r - \|x\|)x_0 + x \neq 0$ since $C \cap (-C) = \{0\}$). Now let

$$g(x) = \begin{cases} r_0(x) \text{ if } x \in \overline{B_r} \\ x \text{ if } x \in B_{r,R} \\ R\dfrac{x}{\|x\|} \text{ if } x \in EB_R. \end{cases}$$

Clearly $g : C \to \overline{B_R}$ is continuous, and since $\mathscr{U}_c^k$ is closed under compositions we have that $G = F \circ g \in \mathscr{U}_c^k(C, C)$ is a compact map. Now Theorem 1.1 guarantees that there exists $x \in C$ with $x \in G(x)$. If $\|x\| < r$ then

$$x \in Fr_0(x) \subseteq F(S_r) \subseteq EB_r,$$

a contradiction. If $\|x\| > R$ then

$$x \in F\left(R\frac{x}{\|x\|}\right) \subseteq F(S_R) \subseteq \overline{B_R},$$

a contradiction. Thus $x \in B_{r,R}$ and $x \in G(x) = F(x)$. ■

Remark 2.1. Let $\mathscr{A}$ be a subclass of the $\mathscr{B}^k$ maps of Park [2; 8] such that, for any Hausdorff topological spaces X_1, X_2 and X_3, if $F \in \mathscr{A}(X_1, X_3)$ and $g \in \mathscr{C}(X_2, X_1)$ then $F \circ g \in \mathscr{B}^k(X_2, X_3)$. Then the result in Theorem 2.1 is again true if $F \in \mathscr{U}_c^k(\overline{B_R}, C)$ is replaced by $F \in \mathscr{A}(\overline{B_R}, C)$.

Remark 2.2. In (2.1) it is clear that $F(S_R) \subseteq \overline{B_R}$ could be replaced by

$$x \notin \lambda Fx \quad \text{for } x \in S_R \quad \text{and } \lambda \in (0, 1). \tag{2.2}$$

To see this, note as in Theorem 2.1 that $x \in G(x)$ for $x \in C$. If $\|x\| > R$ then $x \in F(R\frac{x}{\|x\|})$ and so

$$y \in \lambda F(y) \quad \text{with } y = R\frac{x}{\|x\|} \quad \text{and } \lambda = \frac{R}{\|x\|}.$$

Next let $E = (E, \|.\|)$ be an infinite-dimensional normed linear space. For this case with $\rho > 0$ let

$$B_\rho = \{x \in E : \|x\| < \rho\}, \quad \overline{B_\rho} = \{x \in E : \|x\| \leq \rho\}$$

with

$$S_\rho = \{x \in E : \|x\| = \rho\} \quad \text{and} \quad EB_\rho = \{x \in E : \|x\| \geq \rho\}.$$

We know [4] that there exists a continuous retraction $r_0 : \overline{B_r} \to S_r$. Essentially the same reasoning as in Theorem 2.1 establishes our next result.

Theorem 2.2. *Let* $E=(E, \|.\|)$ *be an infinite-dimensional normed linear space and let* r, R *be constants with* $0<r<R$. *Suppose that* $F\in\mathscr{U}_c^k(\overline{B_R}, E)$ *is compact with* (2.1) *holding. Then* F *has a fixed point in* $B_{r,R}=\{x\in E: r\leq \|x\| \leq R\}$.

Remark 2.3. In (2.1) we could replace $F(S_R)\subseteq\overline{B_R}$ with (2.2).

In fact we could obtain a more general version of Theorem 2.2, as we now show.

Theorem 2.3. *Let* E *be an infinite-dimensional normed linear space, and let* U_1 *and* U_2 *be open convex subsets of* E *with* $0\in U_1$ *and with* $\overline{U_1}\subset U_2$ *(proper). Suppose that* $F\in\mathscr{U}_c^k(\overline{U_2}, E)$ *is compact with*

$$F(\partial U_1)\subseteq E\backslash U_1 \quad \text{and} \quad F(\partial U_2)\subseteq\overline{U_2}. \tag{2.3}$$

Then F *has a fixed point in* $\overline{U_2}\backslash U_1$.

Proof. It is easy to see [3] that there exists a continuous retraction $r_1:\overline{U_1}\to\partial U_1$. Now let

$$g(x)=\begin{cases} r_1(x) \text{ if } x\in\overline{U_1} \\ x \text{ if } x\in\overline{U_2}\backslash U_1 \\ \dfrac{x}{\mu(x)} \text{ if } x\in E\backslash U_2 \end{cases}$$

where μ is the Minkowski functional on $\overline{U_2}$. Notice that $g:E\to\overline{U_2}$ is continuous, so $G=F\circ g\in\mathscr{U}_c^k(E,E)$ is a compact map. Now Theorem 1.1 guarantees that there exists $x\in E$ with $x\in G(x)$. If $x\in U_1$ then

$$x\in Fr_1(x)\subseteq F(\partial U_1)\subseteq E\backslash U_1,$$

a contradiction. If $E\backslash\overline{U_2}$ then

$$x\in F\left(\frac{x}{\mu(x)}\right)\subseteq F(\partial U_2)\subseteq\overline{U_2},$$

a contradiction. ■

Remark 2.4. In (2.3) it is clear that $F(\partial U_2)\subseteq\overline{U_2}$ could be replaced by

$$x\notin\lambda Fx \quad \text{for } x\in\partial U_2 \quad \text{and } \lambda\in(0,1). \tag{2.4}$$

In fact in [4] the authors showed that if E is infinite-dimensional then there exists a Lipschitzian retraction $r_0:\overline{B_r}\to S_r$ with Lipschitz constant, say, k_0. We refer the reader to [6, chapter 21] for a discussion of upper and lower bounds for k_0; note in particular that $k_0>1$. We can now improve Theorem 2.2.

Theorem 2.4. *Let* $E=(E, \|.\|)$ *be an infinite-dimensional normed linear space and let* r, R *be constants with* $0<r<R$. *Suppose that* $F\in\mathscr{U}_c^k(\overline{B_R}, E)$ *is a countably k-set-contractive map,* $0\leq k<\frac{1}{k_0}$, *with* (2.1) *holding. Then* F *has a fixed point in* $B_{r,R}=\{x\in E: r\leq \|x\| \leq R\}$.

PROOF. Let $r_0:\overline{B_r}\to S_r$ be the retraction with Lipschitz constant k_0 (note that $k_0>1$), and let

$$g(x)=\begin{cases} r_0(x) \text{ if } x\in\overline{B_r} \\ x \text{ if } x\in B_{r,R} \\ R\dfrac{x}{\|x\|} \text{ if } x\in EB_R. \end{cases}$$

Notice that g is k_0-set-contractive; if Ω is a bounded subset of E then $\Omega=\Omega_1\cup\Omega_2\cup\Omega_3$ where $\Omega_1=\Omega\cap B_r, \Omega_2=\Omega\cap B_{r,R}$, $\Omega_3=\Omega\cap AB_R$ (here $AB_R=\{x\in E:\|x\|>R\}$) and

$$\begin{aligned}\alpha(g(\Omega))&\le\max\{\alpha(g(\Omega_1)),\alpha(g(\Omega_2)),\alpha(g(\Omega_3))\}\\ &\le\max\{k_0\alpha(\Omega_1),\alpha(\Omega_2),\alpha(\Omega_3)\}\le k_0\alpha(\Omega),\end{aligned}$$

since $g(\Omega_3)\subseteq co(\Omega_3\cup\{0\})$. Thus $G=F\circ g\in\mathcal{U}_c^k(E,E)$ is a countably kk_0-contractive map. Now Theorem 1.1 guarantees that there exists $x\in C$ with $x\in G(x)$. Also (2.1) implies $x\in B_{r,R}$. ■

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