SPECTRA OF TENSOR PRODUCT ELEMENTS III: HOLOMORPHIC PROPERTIES

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ABSTRACT

We show that under fairly general conditions, for $\mathscr{A}$ a commutative unital Banach algebra and X a Banach space, each $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$ admits a compact-open continuous holomorphic functional calculus and, when X is separable and has the bounded approximation property, that this is uniquely implemented. Compactness, non-emptiness and polynomial convexity of the left spectrum, coincidence of left and right spectra, and countably generated uniform Banach algebras are also discussed.

1. Introduction

The functional calculus for single elements of a Banach algebra is now a standard part of every graduate course in functional analysis (see, for instance, Conway [5]), and the theory for the joint spectrum, developed by Šilov [30], Arens and Calderón [1] and Waelbroeck [33] during the 1950s for a finite number of elements of a commutative Banach algebra is also well known (see for instance Mujica [24, theorem 31.7] and Wermer [36]). Waelbroeck [34] initiated an infinite-dimensional theory in the early 1970s by defining the X-spectrum of $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\pi X$ where $\mathscr{A}$ is a commutative unital Banach algebra, X is a Banach space and π is the projective tensor norm. This theory was further developed by Chidami [4], Ortega Aramburu [25], Galé [13] and Matos [20; 21].

Our motivation is to develop further the infinite-dimensional spectral theory along the lines initiated by Waelbroeck so as to include arbitrary tensor norms and non-commutative Banach algebras. In [8] we considered the basic theory of the left spectrum of $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$, where $\mathscr{A}$ is a unital Banach algebra, X is a Banach space and γ is a uniform cross-norm.

We defined $\sigma^{\text{left}}(\mathbf{a})$ to be the left Harte spectrum of $([I_{\mathscr{A}} \otimes x'](\mathbf{a}))_{x' \in X'}$ (see [16]). This object has been widely discussed in the literature for finite indexing sets (we refer to [31] for an interesting survey of results that are particularly relevant to this article), but most

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results for finite indexing sets tacitly use the product topology. The introduction of tensor norms places a much stronger structure on the indexing set, and this in turn leads to a different theory. In this regard we mention the polynomial functional calculus and the one-way spectral mapping theorem given in [9]. In our case polynomials are defined between Banach spaces (by using symmetric n-linear mappings [7]) and are different from the polynomials in non-commutative indeterminates considered in [31] and [16]. In this article we continue our investigations from [8] and [9]. In §2 we show that $\sigma^{\text{left}}(\mathbf{a})$ is a norm compact subset of X and that $\sigma^{\text{left}}(\mathbf{a})=\sigma^{\text{right}}(\mathbf{a})=\{[h\otimes I_X](\mathbf{a}):h\in\mathscr{M}(\mathscr{A})\}$ when $\mathbf{a}$ generates a dense subspace of $\mathscr{A}$. In §3 we show that a commutative unital Banach algebra is a countably generated uniform algebra if and only if it is isometrically isomorphic to the closure of the polynomials in $\mathscr{C}(K)$ where K is a polynomially convex compact subset of a Banach space with the approximation property. In §4 we prove a holomorphic spectral mapping theorem.

We use the terminology of the previous two papers in this series but recall for convenience some of the more frequently used notation. We refer to [7] for background information on polynomials and holomorphic mappings between Banach spaces and to [6; 7] for tensor products of Banach spaces.

All vector spaces are over the complex numbers $\mathbb{C}$. We let $\mathscr{A}$ denote a unital Banach algebra with identity $1_{\mathscr{A}}$; X and Y will denote Banach spaces; $\mathscr{P}(X;Y)$ and $\mathscr{P}(^nX;Y)$ will denote, respectively, the spaces of continuous polynomials and continuous n-homogeneous polynomials from X to Y; and γ will denote a uniform cross-norm. If $\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X$ then

$$\sigma^{\text{left}}(\mathbf{a})=\left\{x''\in X'':1_{\mathscr{A}}\notin\sum_i b_i([I_{\mathscr{A}}\otimes x_i'](\mathbf{a})-x''(x_i')1_{\mathscr{A}})\right\}$$

where $b_i\in\mathscr{A}$, $x_i'\in X'$ and finite sums are taken. By [8, proposition 5], $\sigma^{\text{left}}(\mathbf{a})\subset J_X(X)$ where $J_X:X\to X''$ is the canonical mapping from the Banach space X into its bidual. For this reason we consider the spectrum as lying in X. Since γ is a uniform cross-norm, the canonical action of $\mathscr{A}\otimes X'$ on $\mathscr{A}\otimes X$ defined for elementary tensors by

$$(a\otimes x')\cdot(b\otimes x):=x'(x)ab$$

extends by linearity and continuity to define an action of $\mathscr{A}\otimes X'$ on $\mathscr{A}\hat{\otimes}_\gamma X$.

Since

$$b([I_{\mathscr{A}}\otimes x'](\mathbf{a}-1_{\mathscr{A}}\otimes x))=(b\otimes x')(\mathbf{a}-1_{\mathscr{A}}\otimes x))$$

for $b\in\mathscr{A}$, $x\in X$, $x'\in X'$ and $\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X$, we can describe the spectrum in a manner comparable to that employed in the classical case of a single element of a Banach algebra (see also [10]).

Proposition 1. *If $\mathscr{A}$ is a unital Banach algebra, X is a Banach space, γ is a uniform cross-norm and $\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X$, then*

$$\sigma^{\text{left}}(\mathbf{a})=\{x\in X:\nexists\ \mathbf{b}\in\mathscr{A}\otimes X' \text{ such that } \mathbf{b}\cdot(\mathbf{a}-1_{\mathscr{A}}\otimes x)=1_{\mathscr{A}}\}.$$

Suppose $(a_i)_{i=1}^n\subset\mathscr{A}$. Let X_0 denote an n-dimensional subspace of X with basis $(e_i)_{i=1}^n$ and dual basis $(e_i')_{i=1}^n$. A simple calculation shows that the left Harte spectrum of the n-tuple $(a_1,\ldots,a_n)$ can be identified with the left spectrum of $\sum_{i=1}^n a_i\otimes e_i$ in $\mathscr{A}\hat{\otimes}_\gamma X$ by means of the mapping

$$(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \to \sum_{i=1}^{n} \lambda_i e_i \in \mathscr{A} \otimes X \subset \mathscr{A} \hat{\otimes}_\gamma X.$$

2. Compactness of the spectrum

Proposition 2. *If $\mathscr{A}$ is a unital Banach algebra, X is a Banach space, γ is a uniform cross-norm and $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$, then $\sigma^{left}(\mathbf{a})$ is a norm compact subset of X.*

PROOF. By [8, lemma 3],

$$\sigma^{\mathrm{left}}(\mathbf{a}) \subset \{x : \|x\| \leq \|\mathbf{a}\|\}.$$

Now suppose that $\mathbf{b} := \sum_{i=1}^{n} b_i \otimes x_i' \in \mathscr{A} \otimes X'$ and $\langle \mathbf{b},\ \mathbf{a} - 1_{\mathscr{A}} \otimes x \rangle = 1_{\mathscr{A}}$. If $w \in X$ and $\|w\| < (\sum_{i=1}^{n} \|b_i\| \cdot \|x_i'\|)^{-1}$ then $u := 1_{\mathscr{A}} - \langle \mathbf{b},\ 1_{\mathscr{A}} \otimes w \rangle$ is invertible. Let

$$\tilde{\mathbf{b}} := u^{-1} \cdot \sum_{i=1}^{n} b_i \otimes x_i' = \sum_{i=1}^{n} u^{-1} b_i \otimes x_i' \in \mathscr{A} \otimes X'.$$

Then

$$\begin{aligned} \left\langle \tilde{\mathbf{b}}, \mathbf{a} - 1_{\mathscr{A}} \otimes (x + w) \right\rangle &= u^{-1} \langle \mathbf{b},\ \mathbf{a} - 1_{\mathscr{A}} \otimes x \rangle - u^{-1} \langle \mathbf{b},\ 1_{\mathscr{A}} \otimes w \rangle \\ &= u^{-1}(1_{\mathscr{A}} - \langle \mathbf{b},\ 1_{\mathscr{A}} \otimes w \rangle) \\ &= u^{-1} u \\ &= 1_{\mathscr{A}}. \end{aligned}$$

Hence, if $x \notin \sigma^{\mathrm{left}}(\mathbf{a})$, then $\{y : \|x - y\| < \delta\} \cap \sigma^{\mathrm{left}}(\mathbf{a}) = \emptyset$ for some $\delta > 0$ and $\sigma^{\mathrm{left}}(\mathbf{a})$ is closed.

Now consider the map $T_{\mathbf{a}} : (\mathscr{A}', \sigma(\mathscr{A}', \mathscr{A})) \to (X, \|\cdot\|)$, $T_{\mathbf{a}}(\varphi) = [\varphi \otimes I_X](\mathbf{a})$. Since γ is a uniform cross-norm, $T_{\mathbf{a}}$ is well defined. If $\mathbf{a} \in \mathscr{A} \otimes X$, that is, $\mathbf{a} = \sum_{i=1}^{n} a_i \otimes x_i$, then $T_{\mathbf{a}}(\varphi) = \sum_{i=1}^{n} \varphi(a_i) x_i$ and $T_{\mathbf{a}}$ is continuous.

Let $(\varphi_\alpha)_\alpha$ denote a bounded net in $\mathscr{A}'$, $\|\varphi_\alpha\| \leq M$ all α, and suppose $\varphi_\alpha \to \varphi$ as $\alpha \to \infty$ in the $\sigma(\mathscr{A}', \mathscr{A})$ topology. If $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ and $\varepsilon > 0$ are arbitrary, then we can choose $\mathbf{b} \in \mathscr{A} \otimes X$ such that $\|\mathbf{b} - \mathbf{a}\| \leq \varepsilon$. Hence

$$\begin{aligned} \|T_{\mathbf{a}}(\varphi_\alpha) - T_{\mathbf{a}}(\varphi)\| &= \|[\varphi_\alpha \otimes I_X](\mathbf{a}) - [\varphi \otimes I_X](\mathbf{a})\| \\ &\leq \|[(\varphi_\alpha - \varphi) \otimes I_X](\mathbf{b})\| + \|[(\varphi_\alpha - \varphi) \otimes I_X](\mathbf{a} - \mathbf{b})\| \\ &\leq \|[(\varphi_\alpha - \varphi) \otimes I_X](\mathbf{b})\| + \sup_\alpha \|\varphi_\alpha - \varphi\| \cdot \varepsilon \\ &\leq \varepsilon + 2M\varepsilon \end{aligned}$$

for all α sufficiently large, since $\mathbf{b} \in \mathscr{A} \otimes X$. Hence $T_{\mathbf{a}}$ is continuous on bounded sets for $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ and $T_{\mathbf{a}}(\{\varphi \in \mathscr{A}', \|\varphi\| \leq 1\})$ is a norm compact subset of X.

If $x_0 \in \sigma^{\mathrm{left}}(\mathbf{a})$ let

$$\alpha_{x_0}([I_{\mathscr{A}} \otimes x'](\mathbf{a})) = x'(x_0)$$

for all $x' \in X'$. By the one-way spectral mapping theorem [9, proposition 18], α_{x_0} is well defined. Since

$$\sum_{i=1}^{n} c_i([I_{\mathscr{A}} \otimes x_i'](\mathbf{a})) = \left[I_{\mathscr{A}} \otimes \left(\sum_{i=1}^{n} c_i x_i' \right) \right](\mathbf{a}),$$

α_{x_0} is defined on a subspace of $\mathscr{A}$. By the one-way spectral mapping theorem [9, proposition 18], $x'(x_0) \in \sigma^{\text{left}}([I_{\mathscr{A}} \otimes x'](\mathbf{a}))$. Hence $|x'(x_0)| \leq \|[I_{\mathscr{A}} \otimes x'](\mathbf{a})\|$ and α_{x_0} is a continuous linear mapping of norm ≤ 1. By the Hahn–Banach theorem, α_{x_0} extends to define an element $\tilde{\alpha}_{x_0} \in \mathscr{A}'$ and $\|\tilde{\alpha}_{x_0}\| \leq 1$. If $x' \in X'$ is arbitrary then

$$\begin{aligned} x'(T_{\mathbf{a}}(\tilde{\alpha}_{x_0})) &= x'([\tilde{\alpha}_{x_0} \otimes I_X](\mathbf{a})) \\ &= \alpha_{x_0}([I_{\mathscr{A}} \otimes x'](\mathbf{a})) \\ &= x'(x_0). \end{aligned}$$

By the Hahn–Banach theorem, $x_0 = T_{\mathbf{a}}(\tilde{\alpha}_{x_0})$, and hence

$$\sigma^{\text{left}}(\mathbf{a}) \subset T_{\mathbf{a}}(\{\varphi \in \mathscr{A}', \|\varphi\| \leq 1\}).$$

This shows that $\sigma^{\text{left}}(\mathbf{a})$ is a norm compact subset of X and completes the proof. ∎

If X and Y are Banach spaces and $P \in \mathscr{P}(^nX; Y)$, let $\check{P}$ denote the symmetric n-linear form associated with P, that is, $\check{P}(x, \ldots, x) = P(x)$ for all $x \in X$. If $P := \sum_{j=0}^{n} P_j \in \mathscr{P}(X; Y)$, $P_j \in \mathscr{P}(^jX; Y)$ for $0 \leq j \leq n$, $\mathscr{A}$ is a unital Banach algebra and γ is a uniform cross-norm, then we say that P can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ if there exists for each j a (necessarily unique) $(P_j)_{\mathscr{A}} \in \mathscr{P}(^j\mathscr{A} \hat{\otimes}_\gamma X; \mathscr{A} \hat{\otimes}_\gamma Y)$ such that

$$[(P_j)_{\mathscr{A}}]^{\vee}(a_1 \otimes x_1, \ldots, a_j \otimes x_j) = a_1 \ldots a_j \otimes \check{P}_j(x_1, \ldots, x_j)$$

for all $(a_i)_{i=1}^{n} \subset \mathscr{A}$ and all $(x_i)_{i=1}^{n} \subset X$. Examples of unital Banach algebras and uniform cross-norms γ such that all $P \in \mathscr{P}(X; Y)$, X and Y arbitrary Banach spaces, can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ are given in [9]. We let $P_{\mathscr{A}} = \sum_{j=0}^{n} (P_j)_{\mathscr{A}}$, and if all $P \in \mathscr{P}(X) := \mathscr{P}(X; \mathbb{C})$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ let

$$\mathscr{P}_X[\mathbf{a}] = \{P_{\mathscr{A}}(\mathbf{a}) : P \in \mathscr{P}(X)\}.$$

If $\mathscr{A}$ is commutative then $\mathscr{P}_X[\mathbf{a}]$ is a subalgebra of $\mathscr{A}$ ([9, proposition 4]), and, in addition, if X has the bounded approximation property then the same closed subalgebra of $\mathscr{A}$ is generated by $\{[I_{\mathscr{A}} \otimes x'](\mathbf{a})\}_{x' \in X'}$ as by $\mathscr{P}_X[\mathbf{a}]$ ([9, proposition 19]). In the general case we have

$$2(x'y')_{\mathscr{A}}(\mathbf{a}) = (x')_{\mathscr{A}}(\mathbf{a})\,(y')_{\mathscr{A}}(\mathbf{a}) + (y')_{\mathscr{A}}(\mathbf{a})\,(x')_{\mathscr{A}}(\mathbf{a}),$$

and this suggests that $\mathscr{P}_X[\mathbf{a}]$ may not always be a subalgebra of $\mathscr{A}$.

The following result extends [9, proposition 7].

Proposition 3. *If $\mathscr{A}$ is a unital Banach algebra, X is a Banach space, γ is a uniform cross-norm, all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$, and $\mathscr{P}_X[\mathbf{a}]$ is a dense subspace of $\mathscr{A}$, then*

$$\sigma^{\text{left}}(\mathbf{a}) = \sigma^{\text{right}}(\mathbf{a}) = \{[h \otimes I_X](\mathbf{a}) : h \in \mathscr{M}(\mathscr{A})\}.$$

PROOF. If $h \in \mathscr{M}(\mathscr{A})$ then, by considering finite sums, a density argument and continuity, one sees easily that

$$h(\mathbf{b} \cdot (\mathbf{a} - 1_{\mathscr{A}} \otimes ([h \otimes I_X](\mathbf{a})))) = 0$$

for all $\mathbf{b} \in \mathscr{A} \otimes X'$ and $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$. Hence $[h \otimes I_X](\mathbf{a}) \in \sigma^{\text{left}}(\mathbf{a})$ for all $h \in \mathscr{M}(\mathscr{A})$.

Now suppose $x_0 \in \sigma^{\text{left}}(\mathbf{a})$. Let $\varphi : \mathscr{P}_X[\mathbf{a}] \to \mathbb{C}$ be defined by

$$\varphi(P_{\mathscr{A}}(\mathbf{a})) = P(x_0).$$

If $P_{\mathscr{A}}(\mathbf{a}) = Q_{\mathscr{A}}(\mathbf{a})$ then $(P-Q)_{\mathscr{A}}(\mathbf{a}) = 0$. By the one-way spectral mapping theorem [9, proposition 17],

$$(P-Q)(x_0) \in \sigma^{\text{left}}((P-Q)_{\mathscr{A}}(\mathbf{a})) = \sigma^{\text{left}}(\mathbf{0}) = 0.$$

Hence $P(x_0) = Q(x_0)$ and φ is well defined. A further application of the one-way spectral mapping theorem implies that $P(x_0) \in \sigma^{\text{left}}(P_{\mathscr{A}}(\mathbf{a}))$, and [8, lemma 3] shows that

$$|\varphi(P_{\mathscr{A}}(\mathbf{a}))| = |P(x_0)| \leq \|P_{\mathscr{A}}(\mathbf{a})\|$$

for all $P_{\mathscr{A}}(\mathbf{a}) \in \mathscr{P}_X[\mathbf{a}]$. Hence φ is continuous and, by hypothesis, defined on a dense subspace of $\mathscr{A}$. Let $\tilde{\varphi}$ denote the continuous linear extension of φ to $\mathscr{A}$. If $a \in \mathscr{P}_X[\mathbf{a}]$ then $\varphi(a) \in \sigma^{\text{left}}(a)$.

If $b_n := (P_n)_{\mathscr{A}}(\mathbf{a}) \to b \in \mathscr{A}$ as $n \to \infty$ then $(P_n)_{\mathscr{A}}(\mathbf{a}) - P_n(x_0)1_{\mathscr{A}} \to b - \tilde{\varphi}(b)1_{\mathscr{A}}$ as $n \to \infty$. Since $(P_n)_{\mathscr{A}}(\mathbf{a}) - P_n(x_0)1_{\mathscr{A}}$ is not left-invertible and the left invertibles form an open subset of $\mathscr{A}$, $b - \tilde{\varphi}(b)1_{\mathscr{A}}$ is not left-invertible. Hence $\tilde{\varphi}(b) \in \sigma^{\text{left}}(b) \subset \sigma(b)$ for all $b \in \mathscr{A}$. A result of Gleason [14] and Kahane and Żelazko [17] (see also [3, p. 80] and [28]) implies that $\tilde{\varphi} \in \mathscr{M}(\mathscr{A})$. If $x' \in X'$ then $(x')_{\mathscr{A}}(\mathbf{a}) = [I_{\mathscr{A}} \otimes x'](\mathbf{a})$. Hence

$$x'([\tilde{\varphi} \otimes I_X](\mathbf{a})) = \tilde{\varphi}([I_{\mathscr{A}} \otimes x'](\mathbf{a})) = x'(x_0)$$

for all $x' \in X'$. By the Hahn–Banach theorem, $x_0 = [\tilde{\varphi} \otimes I_X](\mathbf{a})$. We have proved that $\sigma^{\text{left}}(\mathbf{a}) = \{[h \otimes I_X](\mathbf{a}) : h \in \mathscr{M}(\mathscr{A})\}$. The result for $\sigma^{\text{right}}(\mathbf{a})$ is proved in the same way, and this completes the proof. ■

In Proposition 3 we saw that the right and left spectra of certain elements coincided, and it is natural to ask if this ever occurs for all elements in the tensor product. Clearly this will be the case if $\mathscr{A}$ is commutative. If $\mathscr{A}$ is an arbitrary unital Banach algebra, let $\mathscr{R} := \mathscr{R}(\mathscr{A})$ denote the radical of $\mathscr{A}$, that is,

$$\mathscr{R}(\mathscr{A}) := \bigcap \{M : M \text{ is a maximal left ideal in } \mathscr{A}\}.$$

Let $j : \mathscr{A} \to \mathscr{A}/\mathscr{R}$. We claim that $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ is left-invertible if and only if $[j \otimes I_X](\mathbf{a}) \in \mathscr{A}/\mathscr{R} \hat{\otimes}_\gamma X$ is left-invertible. Since j is an algebra homomorphism, $\mathbf{a}$ left-invertible implies that $[j \otimes I_X](\mathbf{a})$ is left-invertible.

Conversely, suppose that $[j \otimes I_X](\mathbf{a})$ is left-invertible. Then there exists $\mathbf{b} \in \mathscr{A} \otimes X'$ such that

$$\langle [j \otimes I_{X'}](\mathbf{b}),\ [j \otimes I_X](\mathbf{a}) \rangle = 1_{\mathscr{A}/\mathscr{R}}.$$

A density argument and the fact that γ is a uniform cross-norm imply that

$$j(\langle \mathbf{b}, \mathbf{a} \rangle) = j(1_{\mathscr{A}}).$$

Hence there exists $\mu \in \mathscr{R}(\mathscr{A})$ such that

$$\langle \mathbf{b}, \mathbf{a} \rangle = 1_{\mathscr{A}} + \mu.$$

By [2, theorem 3.3], $1_{\mathscr{A}} + \mu$ is left-invertible in $\mathscr{A}$. Hence $\mathbf{a}$ is left-invertible in $\mathscr{A} \hat{\otimes}_\gamma X$ and $\sigma^{\text{left}}(\mathbf{a}) = \sigma^{\text{left}}([j \otimes I_X](\mathbf{a}))$. Since a similar result holds for the right spectrum, it follows that $\sigma^{\text{left}}(\mathbf{a}) = \sigma^{\text{right}}(\mathbf{a})$ for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ if $\mathscr{A}/\mathscr{R}$ is commutative. The converse follows from the following result of Fong and Sołtysiak [12]:

$$\sigma^{\text{right}}(a_1, \ldots, a_n) = \sigma^{\text{left}}(a_1, \ldots, a_n)$$

for any finite set $\{a_1, \ldots, a_n\}$ of elements in $\mathscr{A}$ implies that $\mathscr{A}/\mathscr{R}$ $(\mathscr{A})$ is commutative and the fact, noted at the end of §1, that, when X is infinite-dimensional, the set $\{\sigma^{\text{left}}(\mathbf{a}) \mid \mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X\}$ contains the set of all left joint spectra of finite sets of elements in $\mathscr{A}$. We have proved the following result, which extends results in [12].

Proposition 4. *If $\mathscr{A}$ is a unital Banach algebra then $\sigma^{\text{left}}(\mathbf{a}) = \sigma^{\text{right}}(\mathbf{a})$ for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$, where X is an infinite-dimensional Banach space, γ is uniform cross-norm and all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{P}(\mathscr{A} \hat{\otimes}_\gamma X)$ if and only if $\mathscr{A}/\mathscr{R}$ is commutative.*

We remark that the above shows the following: if the left and right spectra agree for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ for one infinite-dimensional Banach space X and one uniform cross-norm γ, then they agree for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_{\gamma'} Y$, Y an arbitrary Banach space and γ' an arbitrary uniform cross-norm, provided that all $P \in \mathscr{P}(Y)$ can be adapted to $\mathscr{P}(\mathscr{A} \hat{\otimes}_{\gamma'} Y)$.

Proposition 5. *If $\mathscr{A}$ is a unital Banach algebra, X is an infinite-dimensional Banach space, γ is a uniform cross-norm and all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{P}(\mathscr{A} \hat{\otimes}_\gamma X)$, then*

$$\sigma^{\text{left}}(\mathbf{a}) \neq \varnothing \text{ for all } \mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$$

if and only if $\mathscr{A}$ admits a non-zero multiplicative functional.

Proof. If $\mathscr{M}(\mathscr{A}) \neq \varnothing$, the proof of Proposition 4 shows that $\sigma^{\text{left}}(\mathbf{a}) \neq \varnothing$ for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$. Conversely, if $\sigma^{\text{left}}(\mathbf{a}) \neq \varnothing$ for all $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$, then the left Harte spectrum of any n-tuple of elements of $\mathscr{A}$ is non-empty. By [11] this implies that $\mathscr{A}$ admits a non-zero multiplicative functional. This completes the proof. ■

Examples of infinite-dimensional Banach spaces X such that $\mathscr{L}(X)$, the (non-commutative) Banach algebra of continuous linear mappings from X into itself, admits non-zero multiplicative functionals can be found in [19; 22; 27; 29; 31].

3. Polynomial convexity and countably generated commutative Banach algebras

The polynomially convex hull $\hat{U}_{\mathscr{P}(X)}$ of a subset U of a Banach space X is defined as follows:

$$\hat{U}_{\mathscr{P}(X)} = \{x \in X : |P(x)| \leq \|P\|_U \text{ for all } P \in \mathscr{P}(X)\}.$$

Clearly $U \subset \hat{U}_{\mathscr{P}(X)}$, and if $U = \hat{U}_{\mathscr{P}(X)}$ we say that U is *polynomially convex*.

If $\mathscr{A}$ is a unital Banach algebra we let $\rho(a)$ denote the spectral radius of $a \in \mathscr{A}$. We have

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \lim_{n\to\infty} \|a^n\|^{1/n},$$

and, if $\mathscr{A}$ is commutative, then

$$\rho(a) = \sup\{|h(a)| : h \in \mathscr{M}(\mathscr{A})\}.$$

Moreover, $\rho(a) \le \|a\|$ for all $a \in \mathscr{A}$ and ρ is a norm on $\mathscr{A}$ if and only if

$$\mathscr{R}(\mathscr{A}) = \bigcap_{h \in \mathscr{M}(\mathscr{A})} h^{-1}(0) = \{0\}.$$

In this case we say that $\mathscr{A}$ is semi-simple.

The following two propositions generalise [9, proposition 22] in different ways.

Proposition 6. *If $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform cross-norm, X is a Banach space and all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$, then*

$$\begin{aligned}\widehat{\sigma(\mathbf{a})}_{\mathscr{P}(X)} &= \{x \in X : |P(x)| \le \rho(P_{\mathscr{A}}(\mathbf{a})) \text{ for all } P \in \mathscr{P}(X)\}\\ &= \{x \in X : |P(x)| \le \|P_{\mathscr{A}}(\mathbf{a})\| \text{ for all } P \in \mathscr{P}(X)\}\end{aligned}$$

for all $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$.

PROOF. By [8, proposition 7],

$$\begin{aligned}\rho(P_{\mathscr{A}}(\mathbf{a})) &= \sup_{h \in \mathscr{M}(\mathscr{A})} |h(P_{\mathscr{A}}(\mathbf{a}))| = \sup_{h \in \mathscr{M}(\mathscr{A})} |P_{\mathscr{A}}([h \otimes I_X](\mathbf{a}))|\\ &= \|P\|_{\sigma(\mathbf{a})}\end{aligned}$$

for all $P \in \mathscr{P}(X)$. Hence

$$\begin{aligned}x \in \widehat{\sigma(\mathbf{a})}_{\mathscr{P}(X)} &\Leftrightarrow |P(x)| \le \|P\|_{\sigma(\mathbf{a})} \text{ for all } P \in \mathscr{P}(X)\\ &\Leftrightarrow |P(x)| \le \rho(P_{\mathscr{A}}(\mathbf{a})) \text{ for all } P \in \mathscr{P}(X).\end{aligned}$$

Since $\rho(P_{\mathscr{A}}(\mathbf{a})) \le \|P_{\mathscr{A}}(\mathbf{a})\|$, this also shows $|P(x)| \le \|P_{\mathscr{A}}(\mathbf{a})\|$ for all $P \in \mathscr{P}(X)$ and all $x \in \widehat{\sigma(\mathbf{a})}_{\mathscr{P}(X)}$. If $|P(x)| \le \|P_{\mathscr{A}}(\mathbf{a})\|$ for all $P \in \mathscr{P}(X)$, then for any positive integer n we may apply [9, proposition 4] to obtain

$$|P^n(x)| \le \|(P^n)_{\mathscr{A}}(\mathbf{a})\| = \|(P_{\mathscr{A}})^n(\mathbf{a})\| = \|(P_{\mathscr{A}}(\mathbf{a}))^n\|$$

for all $P \in \mathscr{P}(X)$. Hence

$$|P(x)| = |P^n(x)|^{1/n} \le \|(P_{\mathscr{A}}(\mathbf{a}))^n\|^{1/n}$$

and

$$|P(x)| \le \lim_{n\to\infty} \|(P_{\mathscr{A}}(\mathbf{a}))^n\|^{1/n} = \rho(P_{\mathscr{A}}(\mathbf{a})).$$

This shows that $x \in \widehat{\sigma(\mathbf{a})}_{\mathscr{P}(X)}$ by the first part of the proof. Hence

$$\widehat{\sigma(\mathbf{a})}_{\mathscr{P}(X)} = \{x \in X : |P(x)| \le \|P_{\mathscr{A}}(\mathbf{a})\| \text{ for all } P \in \mathscr{P}(X)\}.$$

This completes the proof. ■

The commutator ideal $\mathscr{C} := \mathscr{C}_{\mathscr{A}}$ of a Banach algebra $\mathscr{A}$ is the closed ideal generated by $\{ab - ba \mid a, b \in \mathscr{A}\}$. Following [31, proposition 2.3] and the results above, we generalise [9, proposition 22] to the non-commutative setting.

Proposition 7. *If $\mathscr{A}$ is a unital Banach algebra, X is a Banach space, γ is a uniform cross-norm, all $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_{\gamma}X$ and $\mathscr{P}_X[\mathbf{a}]$ is a dense subspace of $\mathscr{A}$, then $\sigma^{left}(\mathbf{a})$ is a polynomially convex compact subset of X.*

Proof. If $\mathscr{M}(\mathscr{A}) = \emptyset$ then, by Proposition 3, $\sigma^{\text{left}}(\mathbf{a}) = \emptyset$ is polynomially convex. If $\mathscr{M}(\mathscr{A})$ is non-empty and $h \in \mathscr{M}(\mathscr{A})$, then $h \equiv 0$ on the commutator ideal $\mathscr{C}$ in $\mathscr{A}$. Hence there exists $\tilde{h} \in \mathscr{M}(\mathscr{A}/\mathscr{C})$ such that $h = \tilde{h} \circ q$ where q denotes the quotient mapping from $\mathscr{A}$ onto $\mathscr{A}/\mathscr{C}$. Since $h \otimes I_X = (\tilde{h} \otimes I_X) \circ (q \otimes I_X)$, our remarks above imply that

$$\sigma^{\text{left}}(\mathbf{a}) = \sigma^{\text{left}}([q \otimes I_X](\mathbf{a})).$$

Since $[q \otimes I_X](\mathbf{a}) \in \mathscr{A}/\mathscr{C}\hat{\otimes}_{\gamma}X$ and $\mathscr{A}/\mathscr{C}$ is a commutative unital Banach algebra, [9, proposition 22] implies that $\sigma^{\text{left}}([q \otimes I_X](\mathbf{a}))$ and hence $\sigma^{\text{left}}(\mathbf{a})$ is a polynomially convex subset of X. An application of Proposition 2 completes the proof. ■

Our next result will be required in §4 and shows that $\sigma(\mathbf{a})$ can be realised as the intersection of projections of a collection of polynomially convex sets. The result is well known in finite dimensions as the 'Arens–Calderón trick' and is, perhaps, implicit in [34], but we include it for the sake of completeness.

Proposition 8. *If $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform cross-norm, X is a Banach space, $\mathbf{a} \in \mathscr{A}\hat{\otimes}_{\gamma}X$ and U is a neighbourhood of $\sigma(\mathbf{a})$, then there exists a finite-dimensional space Y and $\mathbf{b} \in \mathscr{A}\hat{\otimes}_{\gamma}Y$ such that*

$$\sigma\ (\mathbf{a}) \subset \pi_X(\widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \oplus Y)}) \subset U$$

where π_X is the canonical projection from $X \oplus Y$ onto X.

Proof. If Y is a finite-dimensional space and $\mathbf{b} \in \mathscr{A} \otimes Y$, then

$$\sigma(\mathbf{a} \oplus \mathbf{b}) = \{([h \otimes I_X](\mathbf{a}) \oplus [h \otimes I_Y](\mathbf{b})) : h \in \mathscr{M}(\mathscr{A})\}.$$

Hence $\sigma(\mathbf{a}) \subset \pi_X(\widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \otimes Y)})$. If $x_0 \notin \sigma(\mathbf{a})$ then, by Proposition 1, there exists $\mathbf{b} \in \mathscr{A} \otimes X'$ such that $\mathbf{b} \cdot (\mathbf{a} - 1_{\mathscr{A}} \otimes x_0) = 1_{\mathscr{A}}$. If $\mathbf{b} = \sum_{i=1}^{l} b_i \otimes y_i$ we let Y denote the finite-dimensional subspace of X' spanned by $(y_i)_{i=1}^{l}$. Now consider the polynomial of degree two on $X \oplus Y$ defined by

$$P(x \oplus x') = x'(x) - x'(x_0) - 1.$$

If $h \in \mathscr{M}(\mathscr{A})$ then

$$P([h \otimes I_X](\mathbf{a}) \oplus [h \otimes I_Y](\mathbf{b})) = h(\mathbf{b} \cdot (\mathbf{a} - 1_{\mathscr{A}} \otimes x_0) - 1_{\mathscr{A}}) = 0,$$

and the polynomial P vanishes on $\sigma(\mathbf{a} \oplus \mathbf{b})$. Since

$$\|P\|_{\widehat{\sigma(\mathbf{a}\oplus\mathbf{b})}_{\mathscr{P}(X\oplus Y)}} \leq \|P\|_{\sigma(\mathbf{a}\oplus\mathbf{b})} = 0,$$

P vanishes on $\widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \oplus Y)}$.

If $x' \in Y \subset X'$ then $P(x_0 \oplus x') = -1$ and $x_0 \oplus x' \notin \widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \oplus Y)}$ for any $x' \in Y$. Hence $x_0 \notin \pi_X(\widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \oplus Y)})$.

Now consider the collection $\mathscr{F}$ of sets

$$\pi_X\left(\widehat{\sigma(\mathbf{a} \oplus \mathbf{b})}_{\mathscr{P}(X \oplus Y)}\right) \cap (X \backslash U)$$

where Y ranges over all finite-dimensional spaces and $\mathbf{b} \in \mathscr{A} \otimes Y$ is arbitrary. By Proposition 2 this collection consists of compact sets, and by the above their intersection is empty. Hence some finite set in $\mathscr{F}$ consists of sets with empty intersection. Suppose $\mathbf{b}_i \in \mathscr{A} \otimes Y_i$, $i = 1, \ldots, m$ and

$$\bigcap_{i=1}^{m} \pi_X\left(\widehat{\sigma(\mathbf{a} \oplus \mathbf{b}_i)}_{\mathscr{P}(X \oplus Y_i)}\right) \cap (X \backslash U) = \varnothing. \tag{3.1}$$

If $x \in \pi_X\left([\sigma(\mathbf{a} \oplus \mathbf{b}_1 \oplus \cdots \oplus \mathbf{b}_m)]^\wedge_{\mathscr{P}(X \oplus Y_1 \oplus \cdots \oplus Y_m)}\right)$, Proposition 6 implies that there exists $y_i \in Y_i$, $i = 1, \ldots, m$ such that

$$|P(x \oplus y_1 \oplus \cdots \oplus y_m)| \leq \|P\|_{\sigma(\mathbf{a} \oplus \mathbf{b}_1 \oplus \cdots \oplus \mathbf{b}_m)}$$

for all $P \in \mathscr{P}(X \oplus Y_1 \oplus \cdots \oplus Y_m)$.

If $Q \in \mathscr{P}(X \oplus Y_i)$ let $\tilde{Q}(w \oplus z_1 \oplus \cdots \oplus z_m) := Q(w \oplus z_i)$ for $w \in X$ and $z_i \in Y_i$, $i = 1, \ldots, m$. Hence

$$\begin{aligned} |Q(x \oplus y_i)| &= |\tilde{Q}(x \oplus y_1 \oplus \cdots \oplus y_m)| \\ &\leq \|\tilde{Q}\|_{\sigma(\mathbf{a} \oplus \mathbf{b}_1 \oplus \cdots \oplus \mathbf{b}_m)} \\ &= \|Q\|_{\sigma(\mathbf{a} \oplus \mathbf{b}_i)} \end{aligned}$$

and $x \oplus y_i \in \widehat{\sigma(\mathbf{a} \oplus \mathbf{b}_i)}_{\mathscr{P}(X \oplus Y_i)}$, $i = 1, \ldots, m$. This shows that

$$\pi_X\left([\sigma(\mathbf{a} \oplus \mathbf{b}_1 \oplus \cdots \oplus \mathbf{b}_m)]^\wedge_{\mathscr{P}(X \oplus Y_1 \oplus \cdots \oplus Y_m)}\right) \subset \bigcap_{i=1}^{m} \pi_X\left(\sigma(\mathbf{a} \oplus \mathbf{b}_i)\right).$$

If $Y := Y_1 \oplus \cdots \oplus Y_m$ and $\mathbf{c} := \mathbf{b}_1 \oplus \cdots \oplus \mathbf{b}_m$, then (3.1) implies

$$\pi_X(\widehat{\sigma(\mathbf{a} \oplus \mathbf{c})}_{\mathscr{P}(X \oplus Y)}) \subset U.$$

This completes the proof. ■

We also require the following polynomial result in §4.

Proposition 9. *If $\mathscr{A}$ is a commutative unital Banach algebra, X, Y and Z are Banach spaces, γ is a uniform cross-norm, $P \in \mathscr{P}(X; Y)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ and $Q \in \mathscr{P}(Y; Z)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma Y$, then $Q \circ P \in \mathscr{P}(X; Z)$ can be adapted to $\mathscr{A} \hat{\otimes}_\gamma X$ and*

$$(Q \circ P)_{\mathscr{A}} = Q_{\mathscr{A}} \circ P_{\mathscr{A}}.$$

PROOF. We may suppose without loss of generality that P is n-homogeneous and that Q is r-homogeneous. Let $\mathbf{a} = \sum_{i=1}^{k} a_i \otimes x_i \in \mathscr{A} \otimes X$. Then

$$P_{\mathscr{A}}(\mathbf{a}) = \sum_{\substack{m = (m_1, \ldots, m_n) \\ 1 \leq m_i \leq k}} \boldsymbol{a}_{[m]} \otimes \check{P}(x_{[m]})$$

where $\boldsymbol{a}_{[m]} = a_{m_1} \cdots a_{m_n}$ and $x_{[m]} = (x_{m_1}, \ldots, x_{m_n})$. Hence

$$Q_{\mathscr{A}}(P_{\mathscr{A}}(\mathbf{a})) = \sum_{\substack{l=(l_1,\ldots,l_r)\\ l_i=(s_{1_i},\ldots,s_{n_i})\\ 1\le s_{j_i}\le k}} \boldsymbol{a}_{\{1\}} \otimes \check{Q}(\check{P}(x_{[l_1]}), \ldots, \check{P}(x_{[l_r]}))$$

where $\boldsymbol{a}_{\{l\}} = a_{[l_1]} \cdot a_{[l_2]} \cdots a_{[l_r]}$.

Since $\boldsymbol{a}_{\{l\}}$ is invariant under permutation of its entries and the sum is over all permutations, we may replace $\check{Q}(\check{P}(x_{[l_1]}), \ldots, \check{P}(x_{[l_r]}))$ with its symmetrisation with respect to the entries x_i, $i \in l_j$, $1 \le j \le r$. The symmetrisation gives a symmetric nr-linear form that coincides with $\check{Q}(\check{P}(x^n), \ldots, \check{P}(x^n))$ if all entries are equal to x and $x^n = (x, \ldots, x)$.

Since $\check{Q}(\check{P}(x^n), \ldots, \check{P}(x^n)) = \check{Q}(P(x), \ldots, P(x)) = Q \circ P(x)$, we may thus replace $\check{Q}(\check{P}(x_{[l_1]}), \ldots, \check{P}(x_{[l_r]}))$ with $Q \check{\circ} P(x_{[l_1,\ldots,l_r]})$, and we obtain

$$\begin{aligned} Q_{\mathscr{A}} \circ P_{\mathscr{A}}(\mathbf{a}) &= \sum_{\substack{l=(l_1,\ldots,l_{nr})\\ 1\le l_i\le k}} \boldsymbol{a}_{[l]} \otimes [Q \check{\circ} P](x_{[l]}) \\ &= [Q \circ P]_{\mathscr{A}}(\mathbf{a}) \end{aligned}$$

for all $\mathbf{a} \in \mathscr{A} \otimes X$. A density argument shows that

$$Q_{\mathscr{A}} \circ P_{\mathscr{A}} = (Q \circ P)_{\mathscr{A}}. \quad \blacksquare$$

Our final result in this section also features polynomial convexity.

If K is a compact subset of a Banach space X, we let $\mathscr{P}(K)$ denote the closure of $\{P|_K; P \in \mathscr{P}(X)\}$ in $(\mathscr{C}(K), \|\cdot\|_K)$.

Proposition 10. *If $\mathscr{A}$ is a commutative unital Banach algebra, then $\mathscr{A}$ is isometrically and algebraically isomorphic to $\mathscr{P}(K)$ where K is a polynomially convex compact subset of a Banach space with the approximation property if and only if $\mathscr{A}$ is a countably generated uniform Banach algebra.*

Proof. First suppose that $\mathscr{A}$ is a countably generated uniform Banach algebra. By rescaling if necessary, we may suppose that $\mathscr{A}$ has an absolutely convergent sequence of generators $(a_n)_n$. Let X denote an infinite-dimensional Banach space and let γ denote a uniform cross-norm. By a result of Pełczynski (see [18, theorem 1.95]), X contains a basic sequence of unit vectors $(x_n)_{n=1}^{\infty}$. Let $\mathbf{a} = \sum_{n=1}^{\infty} a_n \otimes x_n \in \mathscr{A} \hat{\otimes}_{\gamma} X$. Let $(x'_n)_{n=1}^{\infty}$ denote the coordinate functionals associated with $(x_n)_{n=1}^{\infty}$, extended to X by the Hahn–Banach theorem. Since $[I_{\mathscr{A}} \otimes x'_n](\mathbf{a}) = a_n$, $\mathbf{a}$ is, in the terminology of [9], a dense generator. By [9, proposition 22] and Proposition 2, $\sigma(\mathbf{a})$ is a polynomially convex compact subset of X. If $P \in \mathscr{P}(X)$ then

$$\begin{aligned} \|P_{\mathscr{A}}(\mathbf{a})\| &= \sup_{h \in \mathscr{M}(\mathscr{A})} |h(P_{\mathscr{A}}(\mathbf{a}))| \\ &= \sup_{h \in \mathscr{M}(\mathscr{A})} |P([h \otimes I_X](\mathbf{a}))| \\ &= \|P\|_{\sigma(\mathbf{a})} \end{aligned}$$

and the mapping

$$\left(\mathscr{P}(X), \|\cdot\|_{\sigma(\mathbf{a})}\right) \to \mathscr{A}$$
$$P \mapsto P_{\mathscr{A}}(\mathbf{a})$$

is an isometric isomorphism onto its range. By [9, proposition 4 and example 6], the above mapping is an algebraic homomorphism and, since $\mathbf{a}$ is a dense generator, extends to an isometric isomorphism between $(\mathscr{P}(\sigma(\mathbf{a})), \|\cdot\|_{\sigma(\mathbf{a})})$ and $\mathscr{A}$.

Conversely, suppose that $\mathscr{A} \cong \mathscr{P}(K)$ where K is a polynomially convex compact subset of a Banach space X with the approximation property. Since $\mathscr{P}(K)$ is a uniform algebra, our hypothesis implies that $\mathscr{A}$ is also a uniform algebra. Hence to complete the proof we must show that $\mathscr{A}$ is countably generated.

Let $\alpha = \sup_{x \in K} \|x\| + 1$. Since X has the approximation property, there exists, for any $\varepsilon > 0$, $(y_i)_{i=1}^m \subset X$ and $(y_i')_{i=1}^m \subset X'$ such that

$$\left\| x - \sum_{i=1}^m y_i'(x) y_i \right\| \le \varepsilon$$

for all $x \in K$. Let $P \in \mathscr{P}(^n X)$. By [7, lemma 1.10(c)],

$$\left\| P(x) - P\left(\sum_{i=1}^m y_i'(x) y_i\right) \right\| \le \frac{n^n}{n!} \alpha^n (1+\alpha)^n \|P\| \cdot \varepsilon \tag{3.2}$$

for all $x \in K$. If $l := (l_1, \dots, l_m)$ where each l_i is a non-negative integer and $|l| := \sum_{i=1}^m l_i = n$, let $(y')^l(x) = y_1'(x)^{l_1} \dots y_m'(x)^{l_m}$ for all $x \in X$ and let $\check{P}(y^l) = \check{P}(y_1^{l_1}, \dots, y_m^{l_m})$. Note that $\check{P}(y^l) \in \mathbb{C}$ and $(y')^l \in \mathscr{P}(^n X)$ for all l. By (3.2),

$$\left\| P - \sum_{|l|=n} \binom{n}{l} \check{P}(y^l)(y')^l \right\|_K \le c \cdot \varepsilon$$

where $c = \frac{n^n}{n!} \alpha^n (1+\alpha)^n \|P\|_K$ is independent of ε. Hence the subspace of $\mathscr{P}(K)$ generated by $(y_i')_{i=1}^\infty$ is dense in $\mathscr{P}(K)$ and thus $\mathscr{P}(K)$ is countably generated. This completes the proof. ■

4. Holomorphic functional calculus

If U is an open subset of a Banach space X, let $\mathscr{H}(U)$ denote the set of holomorphic mappings from U into $\mathbb{C}$ endowed with the compact open topology τ_0. If $f \in \mathscr{H}(U)$ and $x_0 \in U$ then there exists a sequence $(P_n)_{n=0}^\infty$ of polynomials, $\mathscr{P}_n \in P(^n X)$ all n, and $\delta > 0$, such that

$$f(x_0 + y) = \sum_{n=0}^\infty P_n(y) \tag{4.1}$$

for all $y \in X$ for which $\|y\| < \delta$. The sequence $(P_n)_{n=1}^\infty$ is uniquely determined by f and x_0, and we use the notation

$$P_n := \frac{\hat{d}^n f(x_0)}{n!}$$

for all n. The expansion (4.1) is called the Taylor series expansion of f about x_0.

Let $\mathscr{H}_b(U)$ denote the subspace of $\mathscr{H}(U)$ consisting of all holomorphic mappings that are bounded on the bounded subsets of X that lie strictly inside U. Endowed with the topology τ_b of uniform convergence on these sets, $\mathscr{H}_b(U)$ is a Fréchet space. If U is an open subset of a finite-dimensional space then $(\mathscr{H}_b(U), \tau_b) = (\mathscr{H}(U), \tau_0)$ is a Fréchet nuclear space.

If X is a Banach space and $r > 0$, we let $B_r(X) = \{x \in X : \|x\| < r\}$, and, if there is no likelihood of confusion, we write B_r in place of $B_r(X)$. If $f : B_r(X) \to \mathbb{C}$, we let $\|f\|_{r,X}$ or $\|f\|_r$ denote sup $\{|f(x)| : x \in B_r(X)\}$.

If U_1 is an open subset of a finite-dimensional Banach space X and U_2 is an open subset of a Banach space Y, then the mapping

$$T : \mathscr{H}(U_1)\hat{\otimes}_\pi \mathscr{H}_b(U_2) \to \mathscr{H}_b(U_1 \oplus U_2)$$

defined on elementary tensors by

$$[T(f \otimes g)]\ (z, w) = f(z)g(w)$$

(for $f \in \mathscr{H}_b(U_1)$, $g \in \mathscr{H}_b(U_2)$, $z \in U_1$ and $w \in U_2$) is a linear and algebraic isomorphism (the product $(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) := f_1 f_2 \otimes g_1 g_2$ extends to define a product on $\mathscr{H}_b(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$).

If n is a positive integer, $A \subset U_1, B \subset U_2$ and $Q_n \in \mathscr{H}(U_1)\hat{\otimes}_\pi \mathscr{P}(^n Y)$, we let

$$\|Q_n\|_{N;A,B} = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| : Q_n = \sum_{i=1}^{\infty} \lambda_i f_i \otimes P_i, \right.$$
$$\left. f_i \in \mathscr{H}(U_1), \|f_i\|_A \le 1,\ P_i \in \mathscr{P}(^n Y), \|P_i\|_B \le 1 \right\}.$$

The finite-dimensional holomorphic functional calculus is due to Arens and Calderón [1], Šilov [30] and Waelbroeck [33]. Uniqueness under progressively weaker assumptions was established by Waelbroeck [33], Zame [37] and Putinar [26]. We require in an essential way the following finite-dimensional result.

Proposition 11 ([37]). *If $\mathscr{A}$ is a commutative unital Banach algebra, X is a finite-dimensional space, $\mathbf{a} \in \mathscr{A} \otimes X$ and U is an open subset of X containing $\sigma(\mathbf{a})$, then there exists a unique continuous homomorphism*

$$\theta_{\mathbf{a}} : \mathscr{H}(U) \to \mathscr{A}$$

such that

(a) $h(\theta_{\mathbf{a}}(f)) = f([h \otimes I_X](\mathbf{a}))$ *and*

(b) $\theta_{\mathbf{a}}(x') = x'_{\mathscr{A}}(\mathbf{a})$

for all $x' \in X'$.

In the above proposition $\mathscr{H}(U)$ is endowed with the compact open topology τ_0.

The infinite-dimensional holomorphic functional calculus was first studied by Waelbroeck [34; 35] for the projective tensor product and afterwards developed in different directions by Chidami [4], Ortega Aramburu [25], Galé [13] and Matos [20; 21]. In this section we extend this calculus to a rather general setting and obtain an intrinsic uniqueness result for certain Banach spaces including all separable Banach spaces with the bounded approximation property. We first recall certain definitions and results from functional analysis and infinite-dimensional holomorphy.

Definition 12. A Banach space X has the bounded (respectively bounded projection) approximation property if there exists a bounded net $(T_\alpha)_\alpha$, $T_\alpha \in \mathscr{L}(X, X)$ of finite-rank operators (respectively finite-rank projections), that is, $T_\alpha(X)$ is finite-dimensional for all α, such that $T_\alpha \to I_X$ uniformly on compact subsets of X as $\alpha \to \infty$. A Banach space with the bounded projection approximation property is said to have the π-property.

The π-property is strictly stronger than the bounded approximation property, while a separable Banach space has the bounded approximation property if and only if it is isomorphic to a complemented subspace of a Banach space with a Schauder basis.

We require the following result for Fréchet spaces [15; 32].

Proposition 13. *If X and Y are Fréchet spaces, X is nuclear and B is a bounded subset of $X \hat{\otimes}_\pi Y$, then there exist bounded subsets B_1 and B_2 in X and Y respectively such that*

$$B \subset \left\{ \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i : \sum_{i=1}^{\infty} |\lambda_i| \leq 1,\ x_i \in B_1,\ y_i \in B_2 \right\}.$$

(*The pair (X, Y) is said to have the BB-property in this case.*)

Proposition 14. *If X_1 and X_2 are Banach spaces, $\dim(X_1) < \infty$, U_1 is an open subset of X_1 and $U_2 = B_r(X_2)$, then for each $f \in \mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$ there exists a unique sequence $(T_n(f))_{n=0}^\infty$, $T_n(f) \in \mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{P}(^n X_2)$ such that*

$$f = \sum_{n=0}^{\infty} T_n(f) \tag{4.2}$$

in $\mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$.

The topology on $\mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$ is generated by the semi-norms

$$\|f\|_{N;K, B_\rho(X_2)} := \sum_{n=0}^{\infty} \|T_n(f)\|_{N;K, B_\rho(X_2)}$$

where K ranges over the compact subsets of U_1, and ρ over $(0, r)$.

If B is a bounded subset of $\mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$ then

$$\sum_{n=0}^{\infty} \sup_{f \in B} \|T_n(f)\|_{N;K, B_\rho(X_2)} < \infty \tag{4.3}$$

for all K compact in U_1 and all $\rho < r$. The compact open topology when transferred from $\mathscr{H}_b(U_1 \oplus U_2)$ to $\mathscr{H}(U_1) \hat{\otimes}_\pi \mathscr{H}_b(U_2)$ by T^{-1} is generated by the semi-norms

$$\|f\|_{N;K_1, K_2} := \sum_{n=0}^{\infty} \|T_n(f)\|_{N;K_1, K_2} \tag{4.4}$$

where K_i ranges over the compact subsets of U_i, $i = 1, 2$.

Remark 15. (a) If $\{Y_n\}_n$ is an $\mathscr{S}$-absolute decomposition ([7, §3.3]) for the locally convex space Y and X is a locally convex space, then $\{X \hat{\otimes}_\pi Y_n\}_{n \in \mathbb{N}}$ is an $\mathscr{S}$-absolute decomposition for $X \hat{\otimes}_\pi Y$. The decomposition in Proposition 14 is obtained by letting

$\mathscr{H}(U_1)=X$, $Y=(\mathscr{H}_b(U_2), \tau_b)$ and $Y_n=(\mathscr{P}(^nX_2), \|.\|)$ for all n. This implies that $\{\mathscr{H}(U_1)\hat{\otimes}_\pi\mathscr{P}(^nX_2)\}_{n=0}^{\infty}$ is an $\mathscr{S}$-absolute decomposition for $\mathscr{H}(U_1)\hat{\otimes}_\pi\mathscr{H}_b(U_2)$. Note that

$$\mathscr{H}(U_1)\hat{\otimes}_\pi\mathscr{P}(^nX_2)=\mathscr{P}(^nX_2;\mathscr{H}(U_1))$$

and modulo this identification the expansion (4.2) is the $\mathscr{H}(U_1)$-valued Taylor series expansion of f at the origin. For $f\in\mathscr{H}(U_1)\hat{\otimes}_\pi\mathscr{H}_b(U_2)$, $T_n(f)=[I_{\mathscr{H}(U_1)}\otimes\pi_n](f)$ where $\pi_n(g)=\hat{d}^ng(0)/n!$ for all $g\in\mathscr{H}_b(U_2)$. Since

$$\|Q_n\|_{N;K,B_{\rho'}(X_2)}=\left(\frac{\rho'}{\rho}\right)^n\|Q_n\|_{N;K,B_\rho(X_2)}$$

for all K compact in U_1 and all ρ and ρ' positive, (4.3) follows immediately from (4.2).

(b) If $X_1=\{0\}$ then $X_1\oplus X_2=X_2$, $U_1\oplus U_2=B_r(X_2)$ and (4.2) reduces to (4.1) with $x_0=0$. Moreover,

$$\|f\|_{N;\{0\},B_\rho(X_2)}=\sum_{n=0}^{\infty}\left\|\frac{\hat{d}^nf(0)}{n!}\right\|_{B_\rho(X_2)}=:|||f|||_\rho \tag{4.5}$$

generates the τ_b-topology on $\mathscr{H}_b(U_2)$.

If K is a compact subset of the Banach space X, we let $\mathscr{H}(K)$ denote the space of holomorphic germs on K, that is,

$$\mathscr{H}(K)=\bigcup\{\mathscr{H}(U):K\subset U\text{ open}\}/\sim$$

where $f\sim g$ if f and g agree on a neighbourhood of K. We denote by $[f]$ the germ determined by the holomorphic function f and let

$$(\mathscr{H}(K),\tau_0)=\varinjlim_{\substack{K\subset U\\ U\text{ open}}}(\mathscr{H}(U),\tau_0).$$

A collection $\mathscr{F}$ of bounded (respectively compact) subsets of a locally convex space X is fundamental if every bounded (respectively compact) subset of X is a subset of some $F\in\mathscr{F}$.

Proposition 16 ([23; 6, §5.1]). *If K is a compact subset of a Banach space X then*

(a) *the sets $\{f\in\mathscr{H}(K+B_r):\|f\|_{K+B_r}\leq j\}$, $r>0$, $j>0$, form a fundamental system of bounded and compact subsets of $(\mathscr{H}(K),\tau_0)$.*

(b) *$(\mathscr{H}(K),\tau_0)$ is a k-space, that is, mappings from $\mathscr{H}(K)$ into a topological space are continuous if and only if their restrictions to compact sets are continuous.*

We also have $\mathscr{H}(K)=\cup\{\mathscr{H}_b(U):K\subset U\text{ open}\}/\sim$, and we let

$$(\mathscr{H}(K),\tau_b)=\varinjlim_{\substack{K\subset U\\ U\text{ open}}}(\mathscr{H}(U),\tau_b).$$

When X is infinite-dimensional τ_b is strictly stronger than τ_0 on $\mathscr{H}(K)$.

If $\mathscr{A}$ is a commutative unital Banach algebra and γ is a uniform cross-norm, we say that the Banach space X has the $(\mathscr{A},\gamma)$-*extension property* if all $P\in\mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$ and there exists $c>0$ such that

$$\|P_{\mathscr{A}}\|\leq c^n\|P\| \tag{4.6}$$

for all $P\in\mathscr{P}(^nX)$ and all n.

A positive real number c that satisfies (4.6) is called an $(\mathscr{A}, \gamma)$-extension constant for X. All extension constants c satisfy ≥ 1, and [9, example 10] shows that we may require $c > 1$.

Results in [9] show that any Banach space X has the $(\mathscr{A}, \pi)$-extension property for any Banach algebra $\mathscr{A}$ and the $(\mathscr{U}, \varepsilon)$-extension property for any uniform algebra $\mathscr{U}$ (π and ε as usual denote the projective and injective tensor norms). The $(\mathscr{A}, \gamma)$-extension property places uniform bounds on the norms of polynomial extensions and may be rephrased as a holomorphic extension.

Proposition 17. *If $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform cross-norm, X is a Banach space and each $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$, then X has the $(\mathscr{A}, \gamma)$-extension property if and only if for each $f := \sum_{n=0}^{\infty} P_n \in \mathscr{H}_b(X)$ we have $f_{\mathscr{A}} := \sum_{n=0}^{\infty} (P_n)_{\mathscr{A}} \in \mathscr{H}_b(\mathscr{A}\hat{\otimes}_\gamma X)$.*

PROOF. Since $f = \sum_{n=0}^{\infty} P_n \in \mathscr{H}_b(X)$ if and only if $\limsup_{n\to\infty} \|P_n\|^{1/n} = 0$, it is immediate that $f_{\mathscr{A}} \in \mathscr{H}_b(\mathscr{A}\hat{\otimes}_\gamma X)$ when X has the $(\mathscr{A}, \gamma)$-extension property.

Conversely, suppose $f_{\mathscr{A}} \in \mathscr{H}_b(\mathscr{A}\hat{\otimes}_\gamma X)$ whenever $f \in \mathscr{H}_b(X)$. If $\mathbf{a} = \sum_{i=1}^{t} a_i \otimes x_i \in \mathscr{A} \otimes X$ and $P \in \mathscr{P}(^m X)$ then, by [9, remarks after proposition 2],

$$P_{\mathscr{A}}(\mathbf{a}) = \sum_{\substack{|m|=n \\ m \in \mathbb{N}^t}} \frac{n!}{m!} \check{P}(w^m)\boldsymbol{a}^m \tag{4.7}$$

where $\boldsymbol{a}^m = a_1^{m_1} \cdots a_t^{m_t}$ and $w^m = (\underbrace{w_1, \ldots, w_1}_{m_1 \text{ times}}, \underbrace{w_2, \ldots, w_2}_{m_2 \text{ times}}, \ldots, \underbrace{w_t, \ldots, w_t}_{m_t \text{ times}})$ for $m := (m_1, \ldots, m_t) \in \mathbb{N}^t$. Hence the mapping

$$P \in \mathscr{P}(X) \to \|P_{\mathscr{A}}(\mathbf{a})\|$$

defines a continuous semi-norm on $(\mathscr{P}(^n X), \|.\|)$. Since each $P \in \mathscr{P}(^n X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$, $\|P_{\mathscr{A}}\| = \sup_{\substack{\mathbf{a} \in \mathscr{A}\otimes X \\ \|\mathbf{a}\| \leq 1}} \|P_{\mathscr{A}}(\mathbf{a})\| < \infty$, and, as $(\mathscr{P}(^n X), \|.\|)$ is a Banach space, $P \to \|\mathscr{P}_{\mathscr{A}}\|$ defines a continuous semi-norm on $\mathscr{P}(^n X)$. This implies, since $(\mathscr{H}_b(X), \tau_b)$ is a Fréchet space, that $f \to |||f_{\mathscr{A}}|||_\rho$ defines a continuous semi-norm on $\mathscr{H}_b(X)$. Hence there exists $M > 0$ and $c > 0$ such that

$$|||f_{\mathscr{A}}||| \leq M|||f|||_c$$

for all $f \in \mathscr{H}_b(X)$. If $f = P \in \mathscr{P}(^n X)$, this implies that

$$\|P_{\mathscr{A}}\| \leq M\|P\|_{B_c(X)} = Mc^n\|P\|$$

and X satisfies the $(\mathscr{A}, \gamma)$-extension property. This completes the proof. ■

We now state our main result in this section.

Theorem 18. *If $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform cross-norm and X is a Banach space with the $(\mathscr{A}, \gamma)$-extension property, then for each $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$ there exists a continuous homomorphism*

$$\theta_{\mathbf{a}} : (\mathscr{H}(\sigma(\mathbf{a})), \tau_0) \to \mathscr{A}$$

such that for all $h \in \mathscr{M}(\mathscr{A})$, $f \in \mathscr{H}(\sigma(\mathbf{a}))$, $P \in \mathscr{P}(X)$ *and* $x' \in X'$

$$h(\theta_{\mathbf{a}}(f)) = f([h \otimes I_X](\mathbf{a})), \tag{4.8}$$

$$\theta_{\mathbf{a}}(P) = P_{\mathscr{A}}(\mathbf{a}) \tag{4.9}$$

and

$$\theta_{\mathbf{a}}(x') = [1_{\mathscr{A}} \otimes x'](\mathbf{a}) = x'_{\mathscr{A}}(\mathbf{a}). \tag{4.10}$$

Moreover, if X is a complemented subspace of a Banach space with the π*-property* (*in particular if X is separable and has the bounded approximation property*) *and the* $(\mathscr{A}, \gamma)$*-extension property, then*

(a) $\theta_{\mathbf{a}}$ *is the unique* τ_0*-continuous homomorphism from* $\mathscr{H}(\sigma(\mathbf{a}))$ *into* $\mathscr{A}$ *satisfying* (4.8) *and* (4.10);

(b) $\theta_{\mathbf{a}}$ *is the unique* τ_b*-continuous homomorphism from* $\mathscr{H}(\sigma(\mathbf{a}))$ *into* $\mathscr{A}$ *satisfying* (4.8) *and* (4.9).

Since $\tau_b \gneq \tau_0$ when X is infinite-dimensional, the existence result in Theorem 18 is strictly stronger than those obtained by Waelbroeck [34] and Matos [20; 21]. Uniqueness results for $\gamma = \pi$ and τ_b in place of τ_0 have also been obtained by Waelbroeck [34] and Galé [13]. The results in [34] involve the functional calculus on a collection of Banach spaces including X, while in [13] uniqueness does not involve the full space of holomorphic germs. These results are not subsumed by Theorem 18, but the methods we develop can easily be adapted to prove analogous results, and we give these without proof in Proposition 23. By using τ_0 on the space of germs and the deep results of J. Mujica quoted in Proposition 16, we obtain uniqueness for the full space of holomorphic germs.

Our approach to proving Theorem 18 follows the general scheme outlined by Waelbroeck in [34] but contains significant modifications, particularly in Lemmas 20 and 21. The final part of the proof of Lemma 20 is similar to a proof by Galé in [13].

Before proceeding to prove Theorem 18 we discuss the relationship between the given conditions and examine how necessary our hypotheses are for the existence of a functional calculus. This analysis helps clarify the roles played by the various conditions and hypotheses in the actual proof. For the sake of convenience we say that an algebra homomorphism $T : \mathscr{B} \to \mathscr{A}$, $\mathscr{B}$ a subalgebra of $\mathscr{H}(\sigma(\mathbf{a}))$, satisfies (4.8), (4.9), (4.10), (a) or (b) if these conditions are satisfied with $\theta_{\mathbf{a}}$ replaced by T. If T satisfies (4.8) we say that $\mathbf{a}$ admits a $\mathscr{B}$-functional calculus and that T implements a $\mathscr{B}$-functional calculus. If $\mathscr{B} = \mathscr{H}(\sigma(\mathbf{a}))$ we use the term *holomorphic functional calculus* in place of $\mathscr{B}$-functional calculus. By Proposition 17 and Theorem 18, if X has the $(\mathscr{A}, \gamma)$-extension property then $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ admits a holomorphic functional calculus if and only if it admits a $\mathscr{H}_b(X)$-functional calculus. We also note that $P_{\mathscr{A}}(\mathbf{a})$ is well defined in the following two cases: (a) $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ and P a finite-rank polynomial; (b) $\mathbf{a} \in \mathscr{A} \otimes X$ and P arbitrary—use (4.7). In case (b) Proposition 11 shows that $\mathbf{a}$ admits a holomorphic functional calculus that is uniquely implemented if (4.10) is satisfied.

Condition (4.9) implies (4.10), since $X' \subset \mathscr{P}(X)$. In our next example we consider the implications (4.8) $\Rightarrow$ (4.9) and (4.10) $\Rightarrow$(4.9). By combining Theorem 18 and Example 19(b) we see that, when X has the bounded approximation property, there exists a holomorphic functional calculus satisfying (4.10) for each $\mathbf{a} \in \mathscr{A} \hat{\otimes}_\gamma X$ if and only if X has the $(\mathscr{A}, \gamma)$-extension property.

Example 19. In this example we assume only that $\mathscr{A}$ is a commutative unital Banach algebra, γ is a uniform cross-norm, X is a Banach space and each $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$ admits a holomorphic functional calculus implemented by the homomorphism $R_\mathbf{a}$. Hence $R_\mathbf{a}$ satisfies (4.8). In particular we do not assume any continuity property on $R_\mathbf{a}$, that each $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$ or that X satisfies the $(\mathscr{A}, \gamma)$-extension property.

(a) If $h \in \mathscr{M}(\mathscr{A})$ and $x' \in X'$ then, by (4.8),

$$h(R_\mathbf{a}(x')) = x'([h \otimes I_X](\mathbf{a})) = h([1_\mathscr{A} \otimes x'](\mathbf{a})).$$

Hence, if $\mathscr{A}$ is semi-simple, $R_\mathbf{a}$ satisfies (4.10).

The converse is also true. Let $a \in \mathscr{R}(\mathscr{A})$, $a \neq 0$. Then $h(a) = 0$ for all $h \in \mathscr{M}(\mathscr{A})$ and $\sigma(a) = \{0\}$.

If $f \in \mathscr{H}(\{z \in \mathbb{C} : |z| < \varepsilon\}), f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, let $R_a(f) = f(0)1_\mathscr{A}$. Then $R_a : \mathscr{H}(\sigma(a)) \to \mathscr{A}$ is easily seen to be a homomorphism. Since

$$h(R_a(f)) = f(0) = f(h(a)),$$

R_a implements the holomorphic functional calculus. In this simple case we take $X = \mathbb{C}$. If $f(z) = z$ for all $z \in \mathbb{C}$ then $R_a(f) = f(0)1_\mathscr{A} = 0$ and $f_\mathscr{A}(a) = a$. Since f is linear, this shows that R_a does not satisfy (4.10).

The homomorphism $R'_a(f) := \sum_{n=0}^{\infty} \alpha_n a^n$ also implements the holomorphic functional calculus for a. This also shows that the holomorphic functional calculus is not uniquely implemented if $\mathscr{A}$ is not semi-simple. This example can be lifted to $\mathscr{A}\hat{\otimes}_\gamma X$ for any γ and any X.

(b) Now suppose that X is a Banach space with the bounded approximation property and that $R_\mathbf{a}$ is τ_0-continuous for all $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$. If $T : X \to X$ is a finite-rank operator and $P \in \mathscr{P}(X)$, then $P \circ T$ lies in the algebra generated by all $x' \in X'$ (see the proof of Proposition 9). If $R_\mathbf{a}$ satisfies (4.10) then $R_\mathbf{a}(P \circ T) = (P \circ T)_\mathscr{A}(\mathbf{a}) = P_\mathscr{A}(T_\mathscr{A}(\mathbf{a}))$ for all $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$. Let $(T_\alpha)_\alpha$ denote a bounded net of finite-rank operators from X to X that converges to the identity on compact subsets of X. Let U denote a bounded neighbourhood of $\sigma(\mathbf{a})$. Since $R_\mathbf{a}$ is τ_0-continuous, there exists a compact subset K of U and $C > 0$ such that $\|R_\mathbf{a}(P)\| \leq C\|P\|_K$ for all $P \in \mathscr{P}(X)$.

Hence

$$\|P_\mathscr{A}(T_\mathscr{A}(\mathbf{a})) - R_\mathbf{a}(P)\| \leq C\|P \circ T_\alpha - P\|_K \to 0$$

as $\alpha \to \infty$. Hence the mapping $P^\mathscr{A} : \mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X \to R_\mathbf{a}(P)$ is the pointwise limit of a bounded net of polynomials of bounded degree on the Banach space $\mathscr{A}\hat{\otimes}_\gamma X$ and $P^\mathscr{A} \in \mathscr{P}(\mathscr{A}\hat{\otimes}_\gamma X)$.

If $P \in \mathscr{P}(^n X)$ and $\mathbf{a} = \sum_{i=1}^{l} a_i \otimes x_i \in \mathscr{A} \otimes X$ then (4.7) implies

$$P_\mathscr{A}((T_\alpha)_\mathscr{A}(\mathbf{a})) = \sum_{\substack{|m|=n \\ m \in \mathbb{N}^l}}^{n} \frac{n!}{m!} \check{P}(T_\alpha(x)^m)\mathbf{a}^m \to P_\mathscr{A}(\mathbf{a})$$

as $\alpha \to \infty$. Hence $P^\mathscr{A}(\mathbf{a}) = P_\mathscr{A}(\mathbf{a})$ for all $\mathbf{a} \in \mathscr{A} \otimes X$. Hence each $P \in \mathscr{P}(X)$ can be adapted to $\mathscr{A}\hat{\otimes}_\gamma X$ and

$$P^\mathscr{A}(\mathbf{a}) = P_\mathscr{A}(\mathbf{a}) = R_\mathbf{a}(P).$$

This shows that $R_\mathbf{a}$ satisfies (4.9).

We now show that X has the $(\mathscr{A}, \gamma)$-extension property. If $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$ then $\sigma(\mathbf{a}) \subset \{x : \|x\| \leq \|\mathbf{a}\|\}$. Since $R_\mathbf{a}$ is τ_0-continuous, our analysis so far shows that for all $\mathbf{a} \in \mathscr{A}\hat{\otimes}_\gamma X$

there exists $c(\mathbf{a})>0$ such that for every polynomial $P\in\mathscr{P}(X)$

$$\|P_{\mathscr{A}}(\mathbf{a})\|\leq c(\mathbf{a})\|P\|_{B_{\|\mathbf{a}\|+1}(X)}. \tag{4.11}$$

Hence for all $\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X$, $\|\mathbf{a}\|=1$, all n and all $P\in\mathscr{P}(^nX)$

$$\|P_{\mathscr{A}}(\mathbf{a})\|\leq c(\mathbf{a})2^n\|P\|. \tag{4.12}$$

For $c>0$ let

$$V_c=\{\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X : \|P_{\mathscr{A}}(\mathbf{a})\|\leq c^n\|P\|\|\mathbf{a}\|^n \text{ for all } n \text{ and all } P\in\mathscr{P}(^nX)\}.$$

By (4.12), $\bigcup_{m=1}^{\infty} V_m=\mathscr{A}\hat{\otimes}_\gamma X$. If $\mathbf{a}_j\in V_m$ for all j and $\mathbf{a}_j\to\mathbf{a}$ as $j\to\infty$ then for each $P\in\mathscr{P}(^nX)$

$$\|P_{\mathscr{A}}(\mathbf{a})\|=\lim_{j\to\infty}\|P_{\mathscr{A}}(\mathbf{a}_j)\|\leq\lim_{j\to\infty}m^n\|P\|\|\mathbf{a}_j\|^n=m^n\|P\|\|\mathbf{a}\|^n$$

and $\mathbf{a}\in V_m$. Hence each V_m is closed. By the Baire category theorem there exists m_0 such that V_{m_0} has non-empty interior. If $\mathbf{a}_0+B_\delta(\mathscr{A}\hat{\otimes}_\gamma X)\subset V_{m_0}$ then, since V_{m_0} is balanced, [7, lemma 1.10(a)] implies $B_\delta(\mathscr{A}\hat{\otimes}_\gamma X)\subset V_{m_0}$. If $\|\mathbf{a}\|\leq\delta$ then

$$\|P_{\mathscr{A}}(\mathbf{a})\|\leq m_0^n\|P\|\|\mathbf{a}\|^n$$

and $\|P_{\mathscr{A}}\|\leq m_0^n\|P\|$ for all n and all $P\in\mathscr{P}(^nX)$. Hence X has the $(\mathscr{A},\gamma)$-extension property.

In the following two lemmas we assume that $\mathscr{A}$, γ and X satisfy the hypotheses of Theorem 18 and that c is the $(\mathscr{A},\gamma)$-extension constant for X. Since the mapping $\theta_{\mathbf{a}}$ that we eventually construct in Theorem 18 is an extension of the mappings constructed in Lemmas 20 and 21, we use the same notation.

Lemma 20. *If $U=B_r$ and $r>c\|\mathbf{a}\|$ then there exists a continuous homomorphism*

$$\theta_{\mathbf{a}}:\mathscr{H}_b(B_r)\to\mathscr{A}$$

such that

(1) *$h(\theta_{\mathbf{a}}(f))=f([h\otimes I_X](\mathbf{a}))$ for all $h\in\mathscr{M}(\mathscr{A})$ and $f\in\mathscr{H}_b(B_r)$,*
(2) *$\theta_{\mathbf{a}}$ is τ_0-continuous on bounded subsets of $\mathscr{H}_b(B_r)$,*
(3) *$\theta_{\mathbf{a}}(P)=P_{\mathscr{A}}(\mathbf{a})$ for all $P\in\mathscr{P}(X)$,*
(4) *$\theta_{\mathbf{a}}(x')=[1_{\mathscr{A}}\otimes x'](\mathbf{a})=x'_{\mathscr{A}}(\mathbf{a})$ for all $x'\in X'$.*

The mapping $\theta_{\mathbf{a}}$ is the unique continuous homomorphism satisfying (1) *and* (3), *and if X has the bounded approximation property then $\theta_{\mathbf{a}}$ is the unique continuous homomorphism satisfying* (1) *and* (2) *and* (4).

Proof. By the $(\mathscr{A},\gamma)$-extension property, $\|P_{\mathscr{A}}\|\leq c^n\|P\|$ for all $P\in\mathscr{P}(^nX)$ and all n. If $f=\sum_{n=0}^{\infty}P_n\in\mathscr{H}_b(B_r)$ then, by Remark 15(b),

$$\sum_{n=0}^{\infty}\|(P_n)_{\mathscr{A}}(\mathbf{a})\|\leq\sum_{n=0}^{\infty}\|(P_n)_{\mathscr{A}}\|\ \|\mathbf{a}\|^n\leq\sum_{n=0}^{\infty}(\|c\mathbf{a}\|)^n\|P_n\|$$
$$=\sum_{n=0}^{\infty}\|P_n\|_{c\|\mathbf{a}\|}=|||f|||_{c\|\mathbf{a}\|}<\infty.$$

Hence

$$f_{\mathscr{A}}(\mathbf{a}) := \sum_{n=0}^{\infty} (P_n)_{\mathscr{A}}(\mathbf{a}) \in \mathscr{A},$$

and the mapping $\theta_{\mathbf{a}} : f \in \mathscr{H}_b(B_r) \to f_{\mathscr{A}}(\mathbf{a}) \in \mathscr{A}$ is a well-defined τ_b-continuous mapping that satisfies (3) and (4) and, by [9, proposition 4(a)], is an algebra homomorphism. By [9, proposition 16], if $h \in \mathscr{M}(\mathscr{A})$ and $f \in \mathscr{H}_b(B_r)$, then

$$\begin{aligned} f([h \otimes I_X](\mathbf{a})) &= \sum_{n=0}^{\infty} (P_n)_{\mathscr{A}}([h \otimes I_X](\mathbf{a})) \\ &= \sum_{n=0}^{\infty} h((P_n)_{\mathscr{A}}(\mathbf{a})) \\ &= h\left(\sum_{n=0}^{\infty} (P_n)_{\mathscr{A}}(\mathbf{a})\right) = h(\theta_{\mathbf{a}}(f)). \end{aligned}$$

Hence $\theta_{\mathbf{a}}$ satisfies (1).

To prove that $\theta_{\mathbf{a}}$ satisfies (2) it suffices, since $\theta_{\mathbf{a}}$ is linear, to show the following: if $(f_\alpha)_\alpha$ is a bounded net in $(\mathscr{H}_b(B_r), \tau_b)$ and $f_\alpha \to 0$ uniformly on compact subsets of B_r as $\alpha \to \infty$, then $\theta_{\mathbf{a}}(f_\alpha) \to 0$ as $\alpha \to \infty$.

Let $\varepsilon > 0$ be arbitrary. By Remark 15(b), $\sup_\alpha |||f_\alpha|||_{c\|\mathbf{a}\|} < \infty$. Hence

$$\sum_{n=0}^{\infty} \sup_\alpha \left\| \theta_{\mathbf{a}} \left(\frac{\hat{d}^n f_\alpha(0)}{n!} \right) \right\| < \infty,$$

and we can choose a positive integer n_0 such that

$$\sum_{n=n_0}^{\infty} \sup_\alpha \left\| \theta_{\mathbf{a}} \left(\frac{\hat{d}^n f_\alpha(0)}{n!} \right) \right\| < \varepsilon. \tag{4.13}$$

We now fix a positive integer n and let $P_\alpha := \hat{d}^n f_\alpha(0)/n!$ for all α. By Remark 15(a) and (b), $(P_\alpha)_\alpha$ is a bounded subset of $(\mathscr{P}(^nX), \|.\|)$ and $P_\alpha \to 0$ uniformly on compact subsets of X as $\alpha \to \infty$.

Lemma 1.10(c) in [7] says that

$$|P(x) - P(y)| \leq \frac{n^n}{n!}(1+\varepsilon)^n \|x-y\| \, \|P\|$$

whenever $P \in \mathscr{P}(^nX)$, $\|y\| < \varepsilon$ and $\|x-y\| < 1$. Since $\sup_\alpha \|P_\alpha\| < \infty$, this result and the $(\mathscr{A}, \gamma)$-extension property allow us to choose $\mathbf{b} = \sum_{i=1}^{t} b_i \otimes w_i \in \mathscr{A} \otimes X$ such that for all α

$$\sup_\alpha \|(P_\alpha)_{\mathscr{A}}(\mathbf{a}) - (P_\alpha)_{\mathscr{A}}(\mathbf{b})\| < \varepsilon. \tag{4.14}$$

By (4.7),

$$(P_\alpha)_{\mathscr{A}}(\mathbf{b}) = \sum_{\substack{|m|=n \\ m \in \mathbb{N}^t}} \frac{n!}{m!} \check{P}_\alpha(w^m) b^m. \tag{4.15}$$

Let K denote the closed convex hull of the set $A := \{w_1, \ldots, w_t\}$. Since A is finite, K is a compact subset of X. By the Polarization Formula,

$$|\check{P}(w^m)| \leq \frac{n^n}{n!} \|P_\alpha\|_K$$

for all α and m. Hence (4.15) implies $(P_\alpha)_{\mathscr{A}}(\mathbf{b}) \to 0$ as $\alpha \to \infty$, and, combining this with (4.14), we see that $\theta_{\mathbf{a}}(P_\alpha) = (P_\alpha)_{\mathscr{A}}(\mathbf{a}) \to 0$ as $\alpha \to \infty$. Hence, if we use this result and (4.13), $\theta_{\mathbf{a}}$ satisfies (2).

By Remark 15(b), $\mathscr{P}(X)$ is dense in $(\mathscr{H}_b(B_r), \tau_b)$. Hence $\theta_{\mathbf{a}}$ is the unique homomorphism satisfying (1) and (3). Now suppose that ψ is a continuous homomorphism satisfying (1), (2) and (4) and that X has the bounded approximation property. By the uniqueness established above, it suffices, since $\tau_b \geq \tau_0$, to show that $\psi(P) = P_{\mathscr{A}}(\mathbf{a})$ for all $P \in \mathscr{P}(X)$. By the bounded approximation property there exists a bounded net of finite-rank operators, $(T_\alpha)_\alpha$, $T_\alpha \in \mathscr{L}(X, X)$, such that $T_\alpha \to I_X$ uniformly on compact subsets of X as $\alpha \to \infty$. By the remarks preceding proposition 19 in [9] (see also Proposition 9),

$$(P \circ T_\alpha)_{\mathscr{A}}(\mathbf{a}) = P_{\mathscr{A}} \circ [I_{\mathscr{A}} \otimes T_\alpha](\mathbf{a}) \to P_{\mathscr{A}}(\mathbf{a}) \text{ as } \alpha \to \infty.$$

By (2), $\psi(P \circ T_\alpha) \to \psi(P)$ as $\alpha \to \infty$, and (by (4)) $\psi(P \circ T_\alpha) \to P_{\mathscr{A}} \circ [I_{\mathscr{A}} \otimes T_\alpha](\mathbf{a})$ for all α (see also Example 19(b)).

This completes the proof. ■

Lemma 21. *Let $X = X_1 \oplus X_2$, $\dim(X_1) < \infty$, and suppose $\mathbf{a} = \mathbf{a}_1 \oplus \mathbf{a}_2$ where $\mathbf{a}_1 \in \mathscr{A}\hat{\otimes}_\gamma X_1$ and $\mathbf{a}_2 \in \mathscr{A}\hat{\otimes}_\gamma X_2$. If U_1 is an open neighbourhood of $\sigma(\mathbf{a}_1)$ in X_1 and $U_2 = B_r(X_2)$ where $r > c\|\mathbf{a}_2\|$, then there exists a continuous homomorphism*

$$\theta_{\mathbf{a}} : \mathscr{H}_b(U_1 \oplus U_2) \to \mathscr{A}$$

such that

(1) *$h(\theta_{\mathbf{a}}(f)) = f([h \otimes I_X](\mathbf{a}))$ for all $h \in \mathscr{M}(\mathscr{A})$ and $f \in \mathscr{H}_b(U_1 \oplus U_2)$,*
(2) *$\theta_{\mathbf{a}}$ is continuous on each bounded subset of $\mathscr{H}_b(U_1 \oplus U_2)$ endowed with the compact open topology,*
(3) *$\theta_{\mathbf{a}}(P) = P_{\mathscr{A}}(\mathbf{a})$ for all $P \in \mathscr{P}(X)$,*
(4) *$\theta_{\mathbf{a}}(x') = [1_{\mathscr{A}} \otimes x'](\mathbf{a}) = x'_{\mathscr{A}}(\mathbf{a})$ for all $x' \in X'$.*

The mapping $\theta_{\mathbf{a}}$ is the unique continuous homomorphism satisfying (1) *and* (3), *and if X has the bounded approximation property then $\theta_{\mathbf{a}}$ is the unique homomorphism satisfying* (1), (2) *and* (4).

Proof. As previously noted, the mapping

$$T : \mathscr{H}(U_1)\hat{\otimes}_\pi \mathscr{H}_b(U_2) \to \mathscr{H}_b(U_1 \oplus U_2)$$

is an algebra isomorphism. Let $\theta_{\mathbf{a}_1}$ denote the continuous non-zero homomorphism $: \mathscr{H}(U_1) \to \mathscr{A}$ given by Proposition 11 and let $\theta_{\mathbf{a}_2}$ denote the continuous homomorphism from $\mathscr{H}_b(U_2)$ into $\mathscr{A}$ constructed in Lemma 20. We let

$$\theta_{\mathbf{a}} = [\theta_{\mathbf{a}_1} \otimes \theta_{\mathbf{a}_2}] \circ T^{-1} : \mathscr{H}_b(U_1 \oplus U_2) \to \mathscr{A}.$$

Since $\theta_{\mathbf{a}_1}$, $\theta_{\mathbf{a}_2}$ and T^{-1} are continuous homomorphisms, $\theta_{\mathbf{a}}$ is also a τ_b-continuous homomorphism.

If $f \in \mathscr{H}_b(U_1 \oplus U_2)$, then $T(\sum_i \lambda_i f_i \otimes g_i) = f$ where $\sum_{i=1}^{\infty} |\lambda_i| < \infty$, $(f_i)_{i=1}^{\infty}$ is a bounded sequence in $\mathscr{H}(U_1)$ and $(g_i)_{i=1}^{\infty}$ is a bounded sequence in $\mathscr{H}_b(U_2)$. If $h \in \mathscr{M}(\mathscr{A})$ then

$$\begin{aligned} h(\theta_{\mathbf{a}}(f)) &= h\left([\theta_{\mathbf{a}_1} \otimes \theta_{\mathbf{a}_2}]\left(\sum_i \lambda_i f_i \otimes g_i\right)\right) \\ &= \sum_i \lambda_i h(\theta_{\mathbf{a}_1}(f_i)) h(\theta_{\mathbf{a}_2}(g_i)) \\ &= \sum_i \lambda_i f_i([h \otimes I_{X_1}](\mathbf{a}_1)) g_i([h \otimes I_{X_2}](\mathbf{a}_2)) \\ &= \sum_i \lambda_i T(f_i \otimes g_i)([h \otimes I_{X_1}](\mathbf{a}_1) \oplus [h \otimes I_{X_2}](\mathbf{a}_2)) \\ &= \left[T\left(\sum_i \lambda_i f_i \otimes g_i\right)\right]([h \otimes I_X](\mathbf{a}_1 \oplus \mathbf{a}_2)) \\ &= f([h \otimes I_X](\mathbf{a})). \end{aligned}$$

Hence $\theta_{\mathbf{a}}$ satisfies condition (1).

To show that $\theta_{\mathbf{a}}$ satisfies condition (2) we use the truncation method employed in Lemma 20 and the expansion given in Proposition 14. This shows that it suffices to prove the following: if $(P_\alpha)_\alpha$ is a bounded net in $\mathscr{H}(U_1)\hat{\otimes}_\pi \mathscr{P}(^n X_2)$ and $\|P_\alpha\|_{N;K_1,K_2} \to 0$ as $\alpha \to \infty$ for every compact subset K_1 in U_1 and every compact subset K_2 in U_2, then $\theta_{\mathbf{a}}(P_\alpha) \to 0$.

Let $\varepsilon > 0$ be arbitrary. Since $\theta_{\mathbf{a}_1}$ and $\theta_{\mathbf{a}_2}$ are continuous, there exist K_1 compact in U_1, $0 < \rho < r$ and $C > 0$ such that

$$\|\theta_{\mathbf{a}_1}(f)\| \le C\|f\|_{K_1} \quad \text{for all } f \in \mathscr{H}(U_1)$$

and

$$\|\theta_{\mathbf{a}_2}(P)\| \le C\|P\|_\rho \quad \text{for all } P \in \mathscr{P}(^n X_2).$$

This implies

$$\|\theta_{\mathbf{a}}(f)\| \le C^2 \|f\|_{N;K_1,\rho}$$

for all $f \in \mathscr{H}(U_1)\hat{\otimes}_\pi \mathscr{P}(^n X_2)$. Since $(P_\alpha)_\alpha$ is bounded, Proposition 13 implies that there exist bounded sets B_1 in $\mathscr{H}(U_1)$ and B_2 in $\mathscr{P}(^n X_2)$ such that for all α we have representations

$$P_\alpha = \sum_{i=1}^{\infty} \lambda_i^\alpha f_i^\alpha \otimes P_i^\alpha$$

where $\sum_{i=1}^{\infty} |\lambda_i^\alpha| \le 1$, $f_i^\alpha \in B_1$ and $P_i^\alpha \in B_2$. Let $M = \sup_{i,\alpha} \|f_i^\alpha\|_{K_1}$. As in the proof of Lemma 20 (see [9]) we can find $\mathbf{b} \in \mathscr{A}\hat{\oplus}_\gamma X_2$ such that

$$\|\theta_{\mathbf{a}_2}(P_i^\alpha) - \theta_{\mathbf{b}}(P_i^\alpha)\|_\rho \le \frac{\varepsilon}{CM}$$

for all i and α. Let $\mathbf{b} = \sum_{i=1}^{t} b_i \otimes w_1$ where $b_i \in \mathscr{A}$ and $w_i \in X_2$. Hence

$$\begin{aligned} \|\theta_{\mathbf{a}}(P_\alpha) - [\theta_{\mathbf{a}_1} \otimes \theta_{\mathbf{b}}](P_\alpha)\| &= \|[\theta_{\mathbf{a}_1} \otimes \theta_{\mathbf{a}_2 - \mathbf{b}}](P_\alpha)\| \\ &\le \left\|\sum_{i=1}^{\infty} \lambda_i^\alpha \theta_{\mathbf{a}_1}(f_i^\alpha)\theta_{\mathbf{a}_2 - \mathbf{b}}(P_i^\alpha)\right\| \\ &\le \sum_{i=1}^{\infty} |\lambda_i^\alpha| \cdot C\|f_i^\alpha\|_{K_1} \cdot \frac{\varepsilon}{CM} \\ &\le \varepsilon \end{aligned}$$

for all α. Let K_2 denote the closed convex hull of the set $A:=\{w_1,\ldots,w_t\}$. Since A is finite, K_2 is a compact subset of X_2. Using the Polarization Formula, as in the proof of Lemma 20, we can find $C'>0$ such that

$$\|\theta_{\mathbf{b}}(P)\|\le C'\|P\|_{K_2}$$

for all $P\in\mathscr{P}({}^nX_2)$. This implies

$$\begin{aligned}\|[\theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{b}}](P_\alpha)\| &\le \sum_{i=1}^{\infty}|\lambda_i^\alpha|\cdot\|\theta_{\mathbf{a}_1}(f_i^\alpha)\|\cdot\|\theta_{\mathbf{b}}(P_i^\alpha)\|\\ &\le C'\sum_{i=1}^{\infty}|\lambda_i^\alpha|\cdot C\|f_i^\alpha\|_{K_1}\cdot\|P_i^\alpha\|_{K_2}.\end{aligned}$$

Since this holds for any representation, we have

$$\|[\theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{b}}](P_\alpha)\|\le CC'\|P_\alpha\|_{N;K_1,K_2}$$

for all α. Hence $\|[\theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{b}}](P_\alpha)\|\to 0$ as $\alpha\to\infty$, and our estimate above shows that

$$\|\theta_{\mathbf{a}}(P_\alpha)\|\to 0$$

as $\alpha\to\infty$. Hence $\theta_{\mathbf{a}}$ satisfies condition (2).

If $(e_j)_{j=1}^k$ is a basis for X_1 let $\varphi_j\left(\sum_{j=1}^k\lambda_je_j\oplus x\right)=\lambda_j$ for each j where $(\lambda_j)_j\in\mathbb{C}^k$ and $x\in X_2$. For $m=(m_1,\ldots,m_2)\in\mathbb{N}^k$ let $\varphi^m=\varphi_1^{m_1}\cdots\varphi_k^{m_k}$ and $e^m=(\underbrace{e_1,\ldots,e_1}_{m_1},\ldots,\underbrace{e_k,\ldots,e_k}_{m_k})$. For l a positive integer let $x^l=\underbrace{(x,\ldots,x)}_{l\text{ times}}$. We have

$$P\left(\sum_{j=1}^k\lambda_je_j\oplus x\right)=\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}}\binom{n}{m}\lambda^m\check{P}(e^m,x^{n-|m|}).$$

Let $P_m:X\to\mathbb{C}$ be defined by

$$P_m\left(\sum_{j=1}^k\lambda_je_j\oplus x\right)=\binom{n}{m}\check{P}(e^m,x^{n-|m|})$$

for $m\in\mathbb{N}^k$. Then $P_m\in\mathscr{P}({}^{n-|m|}X)$ and

$$P=\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}}\varphi^m\cdot P_m.$$

Since

$$P(z\oplus w)=\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}}\varphi^m(z)\cdot P_m(w)$$

for $z\in X_1$ and $w\in X_2$,

$$P=\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}}T(\varphi^m|_{X_1}\otimes P_m|_{X_2})$$

and

$$\theta_{\mathbf{a}}(P) = \sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}} \theta_{\mathbf{a}_1}(\varphi^m|_{X_1})\cdot\theta_{\mathbf{a}_2}(P_m|_{X_2})$$
$$= \sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}} (\varphi^m|_{X_1})_{\mathscr{A}}(\mathbf{a}_1)\cdot(P_m|_{X_2})_{\mathscr{A}}(\mathbf{a}_2).$$

Since $(\varphi^m|_{X_1})_{\mathscr{A}} = (\varphi^m)_{\mathscr{A}}|_{\mathscr{A}\otimes X_1}$, $(\varphi^m)_{\mathscr{A}}(\mathbf{a}_1\oplus\mathbf{a}_2) = (\varphi^m)_{\mathscr{A}}(\mathbf{a}_1)$, and, as $(P_m|_{X_2})_{\mathscr{A}} = (P_m)_{\mathscr{A}}|_{\mathscr{A}\hat{\otimes}_\gamma X_2}$, $(P_m)_{\mathscr{A}}(\mathbf{a}_1\oplus\mathbf{a}_2) = (P_m)_{\mathscr{A}}(\mathbf{a}_2)$. Hence, using [9, proposition 4], we have

$$\theta_{\mathbf{a}}(P) = \sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}} (\varphi^m)_{\mathscr{A}}(\mathbf{a}_1)\cdot(P_m)_{\mathscr{A}}(\mathbf{a}_2)$$
$$= \left[\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}} (\varphi^m)_{\mathscr{A}}\cdot(P_m)_{\mathscr{A}}\right](\mathbf{a}_1\oplus\mathbf{a}_2)$$
$$= \left(\sum_{\substack{|m|\le n\\ m\in\mathbb{N}^k}} \varphi^m P_m\right)_{\mathscr{A}}(\mathbf{a})$$
$$= P_{\mathscr{A}}(\mathbf{a}).$$

Hence $\theta_{\mathbf{a}}$ satisfies conditions (3) and (4).

Let $\psi:\mathscr{H}_b(U_1\oplus U_2)\to\mathscr{A}$ denote a continuous algebra homomorphism that satisfies either (1) and (2) or, if X has the bounded approximation property, (1), (3) and (4). Let $1_{U_1}(x) = 1_{B_r(X_2)}(y) = 1$ for $x\in U_1$ and $y\in B_r(X_2)$. By Proposition 11,

$$\psi\circ T|_{\mathscr{H}(U_1)\otimes 1_{B_r(X_2)}} = \theta_{\mathbf{a}_1}.$$

By Lemma 20,

$$\psi\circ T|_{1_{U_1}\otimes\mathscr{H}(B_r(X_2))} = \theta_{\mathbf{a}_2}.$$

If $f\in\mathscr{H}(U_1)$ and $g\in\mathscr{H}_b(B_r(X_2))$ then

$$\psi\circ T(f\otimes g) = \psi\circ T(f\otimes 1_{B_r(X_2)}\cdot 1_{U_1}\otimes g)$$
$$= \psi\circ T(f\otimes 1_{B_r(X_2)})\cdot\psi\circ T(1_{U_1}\otimes g)$$
$$= \theta_{\mathbf{a}_1}(f)\cdot\theta_{\mathbf{a}_2}(g)$$
$$= [\theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{a}_2}](f\otimes g).$$

By density, $\psi\circ T = \theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{a}_2}$ and $\psi = (\theta_{\mathbf{a}_1}\otimes\theta_{\mathbf{a}_2})\circ T^{-1} = \theta_{\mathbf{a}}$, and we have established uniqueness. This completes the proof. ■

PROOF OF THEOREM 18 WHEN X IS A COMPLEMENTED SUBSPACE OF A BANACH SPACE WITH THE π-PROPERTY

We discuss this case first as the proof is shorter than that for the general case and gives an alternative proof of existence for this collection of Banach spaces.

We first suppose that X has the π-property with projections $(T_\alpha)_\alpha$, $\|T_\alpha\| \leq d$ for all α. Let B denote the open unit ball in X and let $T^\alpha = I_X - T_\alpha$. For each α, $T_\alpha(X)$ is finite-dimensional and $X = T_\alpha(X) \oplus T^\alpha(X)$ where both $T_\alpha(X)$ and $T^\alpha(X)$ are given the induced norms from X. Since X has the $(\mathscr{A}, \gamma)$-extension property, it is easily verified that each $T^\alpha(X)$ has the extension property, and the same extension constant C can be chosen for all α. Let $\varepsilon > 0$ be arbitrary and let $3(1+d)\varepsilon' = \varepsilon$. Choose $\mathbf{b} := \sum_{i=1}^t b_i \otimes x_i \in \mathscr{A} \otimes X$ such that $\|\mathbf{a} - \mathbf{b}\| < \varepsilon'$. Since γ is a uniform cross-norm, $\|1_\mathscr{A} \otimes T^\alpha(\mathbf{a} - \mathbf{b})\| \leq (1+d)\varepsilon' < \varepsilon$ for all α. Next choose α_0 such that $\|T^{\alpha_0}(x_i)\| = \|x_i - T_{\alpha_0}(x_i)\| < \varepsilon'\left(1 + \sum_{i=1}^t \|b_i\|\right)^{-1}$ for $i = 1, \ldots, t$. This implies

$$\begin{aligned}
\|[1_\mathscr{A} \otimes T^{\alpha_0}](\mathbf{a})\| &\leq \|[1_\mathscr{A} \otimes T^{\alpha_0}](\mathbf{b})\| + (1+d)\varepsilon' \\
&\leq \sum_{i=1}^t \|b_i\| \cdot \|x_i - T_{\alpha_0}(x_i)\| + (1+d)\varepsilon' \\
&\leq 2(1+d)\varepsilon' \\
&< \varepsilon.
\end{aligned}$$

Let $\mathbf{a}_1 = [1_\mathscr{A} \otimes T_{\alpha_0}(\mathbf{a})]$ and $\mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = [1_\mathscr{A} \otimes T^{\alpha_0}](\mathbf{a})$. By [9, proposition 16], $\sigma(\mathbf{a}_1) \subset T_{\alpha_0}(\sigma(\mathbf{a})) \subset T_{\alpha_0}(X)$ and $\sigma(\mathbf{a}_2) \subset T^{\alpha_0}(X)$.

Let $U_1(\varepsilon) = \sigma(\mathbf{a}_1) + 2\varepsilon d(B \cap T_{\alpha_0}(X))$ and $U_2(\varepsilon) = 2\varepsilon(1+d)C(B \cap T^{\alpha_0}(X))$. By our construction,

$$\begin{aligned}
\sigma(\mathbf{a}) + \varepsilon B &\subset \sigma(\mathbf{a}_1) + 2\varepsilon B \\
&\subset \sigma(\mathbf{a}_1) + 2\varepsilon d(B \cap T_{\alpha_0}(X)) + 2\varepsilon(1+d)C(B \cap T^{\alpha_0}(X)) \\
&= U_1(\varepsilon) \oplus U_2(\varepsilon) \\
&\subset \sigma(\mathbf{a}) + \varepsilon(1 + 2d + 2(1+d)C)B,
\end{aligned}$$

and we obtain the inclusion mappings

$$\mathscr{H}_b(\sigma(\mathbf{a}) + \varepsilon\beta B) \hookrightarrow \mathscr{H}_b(U_1(\varepsilon) \oplus U_2(\varepsilon)) \hookrightarrow \mathscr{H}_b(\sigma(\mathbf{a}) + \varepsilon B) \tag{4.16}$$

where $\beta := 1 + 2d + 2(1+d)C$ is independent of ε. Hence we can choose a strictly decreasing null sequence of positive numbers $(\varepsilon_n)_n$ such that $U_1(\varepsilon_n) \oplus U_2(\varepsilon_n) \subset U_1(\varepsilon_m) \oplus U_2(\varepsilon_m)$ for $n > m$. This implies

$$(\mathscr{H}(\sigma(\mathbf{a})), \tau_b) = \varinjlim_n \mathscr{H}_b(U_1(\varepsilon_n) \oplus U_2(\varepsilon_n)).$$

Since $\|\mathbf{a}_2\| < 2\varepsilon(1+d)$, Lemma 21 implies that there exists for each n a continuous homomorphism

$$\theta_n : \mathscr{H}_b(U_1(\varepsilon_n) \oplus U_2(\varepsilon_n)) \rightarrow \mathscr{A}$$

satisfying conditions (1), (2) and (3) of that lemma.

For $n > m$ we have the diagram

$$\begin{array}{ccc} \mathscr{H}_b\left(U_1(\varepsilon_m) \oplus U_2(\varepsilon_m)\right) & & \\ \Big\downarrow R_{n,m} & \overset{\theta_m}{\searrow} & \\ & & \mathscr{A} \\ & \underset{\theta_n}{\nearrow} & \\ \mathscr{H}_b\left(U_1(\varepsilon_n) \oplus U_2(\varepsilon_n)\right) & & \end{array}$$

where $R_{n,m}$ is the continuous restriction mapping. The mappings $\theta_n \circ R_{n,m}$ and θ_m satisfy (1) and (3) of Lemma 21. By uniqueness, $\theta_m = \theta_n \circ R_{n,m}$ and the above diagram commutes. By the definition of inductive limit there exists a continuous homomorphism $\theta_{\mathbf{a}} : (\mathscr{H}(\sigma(\mathbf{a})), \tau_b) \to \mathscr{A}$ satisfying (4.8) and (4.9). Moreover, by Lemma 21, $\theta_{\mathbf{a}}$ is τ_0-continuous on the compact subsets of $(\mathscr{H}(\sigma(\mathbf{a})), \tau_0)$. By Proposition 16, $\theta_{\mathbf{a}} : (\mathscr{H}(\sigma(\mathbf{a})), \tau_0) \to \mathscr{A}$ is continuous.

Now suppose that $\psi : (\mathscr{H}(\sigma(\mathbf{a})), \tau_b) \to \mathscr{A}$ is a continuous mapping satisfying (4.8) and which either is τ_0-continuous and satisfies (4.10) or satisfies (4.9). By Lemma 21, ψ and $\theta_{\mathbf{a}}$ agree on $\mathscr{H}_b(U_1(\varepsilon_n) \oplus U_2(\varepsilon_n))$ for all n and hence $\psi = \theta_{\mathbf{a}}$. We have completed the proof of Theorem 18 when X has the π-property. ■

Now suppose that X is a complemented subspace of a Banach space with the π-property and the $(\mathscr{A}, \gamma)$-extension property. Then there exist Banach spaces Y and Z such that $X \oplus Y = Z$ and Z has the π-property and the $(\mathscr{A}, \gamma)$-extension property. Let $\imath : X \to Z$ denote the canonical inclusion mapping and let π denote the projection from Z onto X. Let $\mathbf{a}' := [1_{\mathscr{A}} \otimes \imath](\mathbf{a}) = \imath_{\mathscr{A}}(\mathbf{a}) \in \mathscr{A} \hat{\otimes}_\gamma Z$. By [9, proposition 16], $\sigma(\mathbf{a}') = \imath(\sigma(\mathbf{a}))$.

For any mapping $\phi : A \to B$ we let ${}^t\phi$ denote the transpose, that is, ${}^t\phi f = f \circ \phi$ for any function f defined on B. If $\sigma(\mathbf{a}) \subset U \subset X$ then $\sigma(\mathbf{a}') \subset \imath(U) \oplus Y$ and $\theta_{\mathbf{a}'} \circ {}^t\pi : (\mathscr{H}(\sigma(\mathbf{a})), \tau_0) \to \mathscr{A}$ is a continuous homomorphism. If $h \in \mathscr{M}(\mathscr{A})$ then $[h \otimes I_Z](\mathbf{a}') = [h \otimes I_Z] \circ [1_{\mathscr{A}} \otimes \imath](\mathbf{a}) = \imath([h \otimes I_X](\mathbf{a}))$. Since $\theta_{\mathbf{a}'}$ satisfies (4.8),

$$\begin{aligned} h(\theta_{\mathbf{a}'} \circ {}^t\pi(f)) &= {}^t\pi(f)([h \otimes I_Z](\mathbf{a}')) \\ &= f(\pi(\imath([h \otimes I_X](\mathbf{a})))) \\ &= f([h \otimes I_X](\mathbf{a})) \end{aligned}$$

and $\theta_{\mathbf{a}'} \circ {}^t\pi$ satisfies (4.8).

If $P \in \mathscr{P}(X)$ then ${}^t\pi(P) \in \mathscr{P}(Z)$. Since $\theta_{\mathbf{a}'}$ satisfies (4.9), we have

$$\theta_{\mathbf{a}'}({}^t\pi(P)) = (P \circ \pi)_{\mathscr{A}}(\mathbf{a}') = P_{\mathscr{A}} \circ \pi_{\mathscr{A}} \circ \imath_{\mathscr{A}}(\mathbf{a}) = P_{\mathscr{A}}((\imath \circ \pi)_{\mathscr{A}}(\mathbf{a})) = P_{\mathscr{A}}(\mathbf{a}),$$

since $\imath \circ \pi = I_X$. Hence $\theta_{\mathbf{a}'} \circ {}^t\pi$ satisfies (4.9). This proves existence.

Let $\psi : (\mathscr{H}(\sigma(\mathbf{a})), \tau_b) \to \mathscr{A}$ denote a continuous homomorphism satisfying (4.8) and either (a) or (b) of Theorem 18. Since $\imath(\sigma(\mathbf{a})) = \sigma(\mathbf{a}')$ the mapping ${}^t\imath$ from $\mathscr{H}(\sigma(\mathbf{a}'))$ into $\mathscr{H}(\sigma(\mathbf{a}))$ is well defined and a continuous homomorphism when both spaces are endowed with the τ_0 or τ_b topologies. Hence $\psi \circ {}^t\imath : (\mathscr{H}(\sigma(\mathbf{a}')), \tau_b) : \to \mathscr{A}$ is a continuous homomorphism that is τ_0-continuous when ψ is τ_0-continuous. Since

$$\begin{aligned} h(\psi \circ {}^t\imath(f)) &= {}^t\imath(f)([h \otimes I_X](\mathbf{a})) \\ &= f(\imath([h \otimes I_X](\mathbf{a}))) \\ &= f([h \otimes I_Z](\mathbf{a})) \end{aligned}$$

when $f \in \mathscr{H}(\sigma(\mathbf{a}'))$ and $h \in \mathscr{M}(\mathscr{A})$, $\psi \circ {}^t\imath$ satisfies (4.8). If ψ satisfies (b) then, by the uniqueness established for Banach spaces with the π-property, $\psi \circ {}^t\imath = \theta_{\mathbf{a}'}$. If ψ satisfies (4.9) then for all $P \in \mathscr{P}(Z)$

$$\psi \circ {}^t\imath(P) = \psi(P \circ \imath) = (P \circ \imath)_{\mathscr{A}}(\mathbf{a}) = P_{\mathscr{A}}(\mathbf{a}')$$

and ψ satisfies (4.9). Hence, by uniqueness, $\psi \circ {}^t\imath = \theta_{\mathbf{a}'}$ in all cases.

Since $\pi \circ \imath = I_X$, ${}^t\imath \circ {}^t\pi = I_X$ and

$$\psi = \psi \circ {}^t\imath \circ {}^t\pi = \theta_{\mathbf{a}'} \circ {}^t\pi.$$

This shows that $\theta_{\mathbf{a}'} \circ {}^t\pi$ does not depend on how we embed X as a complemented subspace of a Banach space with the π-property and the $(\mathscr{A}, \gamma)$-extension property. Hence ψ is uniquely determined.

Proof of Theorem 18 when X is an arbitrary Banach space

This existence result is modelled on [34], but, since [34] is scarce on details and the extension along similar lines in [4] has not been published, we decided to include full details here.

We need to construct a τ_0-continuous homomorphism from $\mathscr{H}(\sigma(\mathbf{a}))$ into $\mathscr{A}$ satisfying (4.8) and (4.9). Since the construction and proof are rather involved, we begin by indicating the main steps in the proof. Let $U = \sigma(\mathbf{a}) + \varepsilon B$ where $\varepsilon > 0$ and B is the open unit ball in X. We first find a finite-dimensional polynomially convex open set V_1, W a ball in X, and construct continuous homomorphisms

$$\mathscr{H}_b(U) \xrightarrow{R_1^*} \mathscr{H}_b(V_1 \oplus W) \xrightarrow{\theta_1} \mathscr{A}.$$

The homomorphism θ_1 is constructed by using Lemma 21. In order to proceed we need to show that this construction does not depend on our choice of R_1^* and θ_1, and afterwards we need to show coherence in passing from $\mathscr{H}(U)$ to $\mathscr{H}(V)$ where $\sigma(\mathbf{a}) \subset V \subset U$. We achieve both of these by constructing further continuous homomorphisms and obtaining the commutative diagram

$$\begin{array}{ccccc} \mathscr{H}_b(U) & \xrightarrow{R_1^*} & \mathscr{H}_b(V_1 \oplus W) & \xrightarrow{\theta_1} & \\ \Big\downarrow S & & \Big\downarrow T_1^* & & \\ & & \mathscr{H}_b(V_3 \oplus W'') & \xrightarrow{\theta_3} & \mathscr{A} \\ & & \Big\uparrow T_2^* & & \\ \mathscr{H}_b(U') & \xrightarrow{R_2^*} & \mathscr{H}_b(V_2 \oplus W') & \xrightarrow{\theta_2} & \end{array} \tag{4.17}$$

where $U' = \sigma(\mathbf{a}) + \varepsilon' B$; $\varepsilon' \leq \varepsilon$, W' and W'' are balls in X; V_2 and V_3 are polynomially convex open subsets of finite-dimensional spaces; S is the restriction mapping; and θ_2 and θ_3 are constructed by using Lemma 21. We describe T_1^* and T_2^* later.

Let $\eta := \frac{\varepsilon}{4+c}$ and $W = (\varepsilon - 2\eta)B$. Now choose $\mathbf{a}_1 \in \mathscr{A} \otimes X$ such that $\|\mathbf{a} - \mathbf{a}_1\| < \eta$ and next choose a finite-dimensional subspace X_1 of X such that $\mathbf{a}_1 \in \mathscr{A} \otimes X_1$. Let $\sigma(\mathbf{a}_1)$ denote the spectrum of $\mathbf{a}_1$ in X_1. By Proposition 8 or [34, proposition 3, p. 97] there exists a finite-dimensional space Y_1, a continuous linear surjection $\pi_1 : Y_1 \to X_1$, $\mathbf{b}_1 \in \mathscr{A} \otimes Y_1$, and V_1 a polynomially convex neighbourhood of $\sigma(\mathbf{b}_1) \subset Y_1$ such that

$$(\pi_1)_{\mathscr{A}}(\mathbf{b}_1) = [I_{\mathscr{A}} \otimes \pi_1](\mathbf{b}_1) = \mathbf{a}_1 \tag{4.18}$$

and

$$\pi_1(V_1) \subset U_1 := \sigma(\mathbf{a}_1) + \eta(B \cap X_1). \tag{4.19}$$

Since $\|\mathbf{a} - \mathbf{a}_1\| < \eta$, we have $\|[h \otimes I_X](\mathbf{a} - \mathbf{a}_1)\| \leq \|\mathbf{a} - \mathbf{a}_1\| =: \eta_1 < \eta$ for all $h \in \mathscr{M}(\mathscr{A})$ and $\sigma(\mathbf{a}_1) \subset \sigma(\mathbf{a}) + \eta_1 B$.

If $z_1 \in V_1$ and $\xi \in W$ then, by (4.19),

$$\begin{aligned}\pi_1(z_1) + \xi \in U_1 + W &\subset \sigma(\mathbf{a}_1) + \eta B + (\varepsilon - 2\eta)B \\ &\subset \sigma(\mathbf{a}) + \eta_1 B + \eta B + (\varepsilon - 2\eta)B \\ &\subset \sigma(\mathbf{a}) + (\varepsilon + \eta_1 - \eta)B.\end{aligned}$$

Hence $\pi_1(z_1) + W$ is a bounded subset of X and, since $\eta_1 < \eta$, it lies strictly inside U.

Let $R_1 : Y_1 \oplus X \to X$ be given by $R_1(z_1 \oplus \xi) = \pi_1(z_1) + \xi$. The mapping R_1 is linear, and, since $R_1(V_1 \oplus W) \subset U$, we may define the continuous homomorphism $R_1^* : \mathscr{H}_b(U) \to \mathscr{H}_b(V_1 \oplus W)$ by letting $R_1^*(f) = f \circ R_1$.

Since $\|\mathbf{a} - \mathbf{a}_1\| < \eta$, we have $c\|\mathbf{a} - \mathbf{a}_1\| < c\eta < \varepsilon - 2\eta$, and, as $\sigma(\mathbf{b}_1) \subset V_1$, there exists, by Lemma 21, a unique continuous homomorphism $\theta_1 := \theta_{\mathbf{b}_1 \oplus \mathbf{a} - \mathbf{a}_1} : \mathscr{H}_b(V_1 \oplus W) \to \mathscr{A}$ satisfying conditions (1) and (3) of Lemma 21. The composition $\theta_1 \circ R_1^*$ defines a continuous homomorphism from $\mathscr{H}_b(U)$ into $\mathscr{A}$. We will show that this homomorphism does not depend on our choice of $\mathbf{a}_1$, $\mathbf{b}_1$, π_1, V_1, X_1 or Y_1.

Let $U' = \sigma(\mathbf{a}) + \eta' B$, where $0 < \varepsilon' \leq \varepsilon$, and let $\eta' = \frac{\varepsilon'}{4+c}$ and $W' = (\varepsilon' - 2\eta')B$. We note that $\eta' \leq \eta$ and $\varepsilon' - 4\eta' \leq \varepsilon - 4\eta$. We apply the above procedure to obtain $\mathbf{a}_2$, $\mathbf{b}_2$, π_2, V_2, X_2 and Y_2 as above. These satisfy similar conditions to the above and allow us to define a linear mapping R_2 and continuous homomorphisms R_2^* and $\theta_2 := \theta_{\mathbf{b}_2 \oplus \mathbf{a} - \mathbf{a}_2}$. It is important to note that $\varepsilon = \varepsilon'$ implies only $\eta' = \eta$ and $W' = W$: the remaining items constructed may be different. We define $\gamma_i : Y_1 \oplus Y_2 \to Y_i$, $i = 1, 2$ and $\pi : Y_1 \oplus Y_2 \to X$ by letting $\gamma_i(z_1 \oplus z_2) = z_i$ for $i = 1, 2$ and $\pi = \pi_1 \circ \gamma_1 - \pi_2 \circ \gamma_2$. We have $\mathbf{b} := \mathbf{b}_1 \oplus \mathbf{b}_2 \in (\mathscr{A} \oplus Y_1) \oplus (\mathscr{A} \oplus Y_2) \cong \mathscr{A} \otimes (Y_1 \oplus Y_2)$, $[I_{\mathscr{A}} \otimes \gamma_i](\mathbf{b}) = \mathbf{b}_i$, $i = 1, 2$, and

$$[I_{\mathscr{A}} \otimes \pi](\mathbf{b}) = [I_{\mathscr{A}} \otimes \pi_1](\mathbf{b}_1) - [I_{\mathscr{A}} \otimes \pi_2](\mathbf{b}_2) = \mathbf{a}_1 - \mathbf{a}_2. \tag{4.20}$$

By the polynomial spectral mapping theorem [9, proposition 16],

$$\sigma(\mathbf{a}_1 - \mathbf{a}_2) = \sigma([I_{\mathscr{A}} \otimes \pi](\mathbf{b})) = \pi(\sigma(\mathbf{b})) \tag{4.21}$$

and

$$\sigma(\mathbf{b}_i) = \sigma([I_A \otimes \gamma_i](\mathbf{b})) = \gamma_i(\sigma(\mathbf{b})) \tag{4.22}$$

for $i = 1, 2$. Since $\|\mathbf{a}_1 - \mathbf{a}_2\| \leq \|\mathbf{a}_1 - \mathbf{a}\| + \|\mathbf{a}_2 - \mathbf{a}\| < \eta + \eta'$, the ball $(\eta + \eta')B$ is a neighbourhood of the compact set $\sigma(\mathbf{a}_1 - \mathbf{a}_2)$. Equations (4.21) and (4.22) imply that there exists a neighbourhood V_3 of $\sigma(\mathbf{b})$ such that $\pi(V_3) \subset (\eta + \eta')B$ and $\gamma_i(V_3) \subset V_i$ for $i = 1, 2$. Since

$(\eta+\eta')B$ and V_i, $i=1$, 2, are polynomially convex, we may suppose, if necessary on replacing V_3 with its polynomially convex hull, that V_3 is polynomially convex. Let $W''=(\varepsilon'-4\eta')B$. Let $T_1 : Y_1 \oplus Y_2 \oplus X \to Y_1 \oplus X$ be defined by $T_1(z_1 \oplus z_2 \oplus \xi)= z_1 \oplus (\xi - \pi(z_1 \oplus z_2))$. If $z \in V_3$ and $\xi \in W''$ then $\|\xi - \pi(z)\| < \eta + \eta' + \varepsilon' - 4\eta' = \varepsilon' - 3\eta' + \eta \leq \varepsilon - 2\eta$ and $\xi - \pi(z) \in W$. Hence $T_1^*(f) := f \circ T_1$ defines a continuous homomorphism from $\mathscr{H}_b(V_1 \oplus W)$ into $\mathscr{H}_b(V_3 \oplus W'')$.

Let $T_2 : Y_1 \oplus Y_2 \oplus X \to Y_2 \oplus X$ be defined by $T_2(z_1 \oplus z_2 \oplus \xi) = z_2 \oplus \xi$. Since $T_2(V_3 \oplus W'') \subset V_2 \oplus W'$, $T_2^*(f) := f \circ T_2$ is a continuous homomorphism from $\mathscr{H}_b(V_2 \oplus W')$ into $\mathscr{H}_b(V_3 \oplus W'')$. If $f \in \mathscr{H}_b(U)$ and $z \oplus \xi \in V_3 \oplus W''$ then

$$\begin{aligned}[][T_1^*(R_1^*)](f)](z \oplus \xi) &= [R_1^*(f)](\gamma_1(z) \oplus (\xi - \pi(z))) \\ &= f(\pi_1(\gamma_1(z)) + \xi - \pi(z)) \\ &= f(\pi_2(\gamma_2(z)) + \xi),\end{aligned}$$

and, if $g \in \mathscr{H}_b(U')$, then

$$\begin{aligned}[][T_2^*(R_2^*)](g)](z \oplus \xi) &= [R_2^*(g)](\gamma_2(z) \oplus \xi) \\ &= g(\pi_2(\gamma_2(z)) + \xi).\end{aligned}$$

Let $S : \mathscr{H}_b(U) \to \mathscr{H}_b(U')$ be the natural restriction mapping, that is, $Sf = f|_{U'}$. By the above, $T_1^*(R_1^*) = T_2^*(R_2^*) \circ S$ on $\mathscr{H}_b(U)$.

Since $c\|\mathbf{a} - \mathbf{a}_2\| < c\eta' = \varepsilon' - 4\eta'$ and $\sigma(\mathbf{b}) \subset V_3$, there exists a unique continuous homomorphism

$$\theta_3 := \theta_{\mathbf{b} \oplus \mathbf{a} - \mathbf{a}_2} : \mathscr{H}_b(V_3 \oplus W'') \to \mathscr{A}$$

satisfying conditions (1) and (3) in Lemma 21. If $P \in \mathscr{P}(Y_1 \oplus X)$ then, by Proposition 9,

$$\begin{aligned}[\theta_3 \circ T_1^*](P) &= \theta_3(P \circ T_1) \\ &= (P \circ T_1)_{\mathscr{A}}(\mathbf{b} \oplus \mathbf{a} - \mathbf{a}_2) \\ &= P_{\mathscr{A}}([T_1]_{\mathscr{A}}(\mathbf{b} \oplus \mathbf{a} - \mathbf{a}_2)).\end{aligned}$$

By (4.20),

$$\begin{aligned}(T_1)_{\mathscr{A}}(\mathbf{b} \oplus \mathbf{a} - \mathbf{a}_2) &= (I_{\mathscr{A}} \otimes T_1](\mathbf{b} \oplus \mathbf{a} - \mathbf{a}_2) \\ &= (I_{\mathscr{A}} \otimes \gamma_1](\mathbf{b}) \oplus ([I_{\mathscr{A}} \otimes I_X](\mathbf{a} - \mathbf{a}_2) - [I_{\mathscr{A}} \otimes \pi](\mathbf{b})) \\ &= \mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_2 - (\mathbf{a}_1 - \mathbf{a}_2)) \\ &= \mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_1).\end{aligned}$$

This implies

$$[\theta_3 \circ T_1^*](P) = P_{\mathscr{A}}(\mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_1)) = \theta_1(P). \tag{4.23}$$

In the same way we obtain $[\theta_3 \circ T_2^*](P) = \theta_2(P)$.

If $h \in \mathscr{M}(\mathscr{A})$ and $f \in \mathscr{H}_b(V_1 \oplus W)$, Lemma 21 implies

$$h([\theta_3 \circ T_1^*](f)) = h(\theta_3(f \circ T_1)) = f \circ T_1([h \otimes I_{Y_1 \oplus Y_2 \oplus X}](\mathbf{b} \oplus (\mathbf{a} - \mathbf{a}_2))).$$

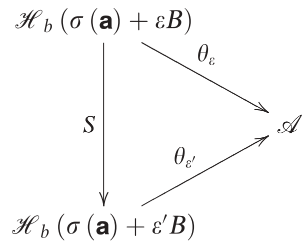
By (4.20),

$$
\begin{aligned}
T_1([h \otimes I_{(Y_1 \oplus Y_2)\oplus X}](\mathbf{b} \oplus (\mathbf{a} - \mathbf{a}_2)))
&= [h \otimes I_{Y_1}](\mathbf{b}_1) \oplus ([h \otimes I_X](\mathbf{a} - \mathbf{a}_2) - [h \otimes \pi](\mathbf{b}))\\
&= [h \otimes I_{Y_1}](\mathbf{b}_1) \oplus ([h \otimes I_X](\mathbf{a} - \mathbf{a}_2) - [h \otimes I_X](\mathbf{a}_1 - \mathbf{a}_2))\\
&= [h \otimes I_{Y_1}](\mathbf{b}_1) \oplus [h \otimes I_X](\mathbf{a} - \mathbf{a}_1)\\
&= [h \otimes I_{Y_1 \oplus X}](\mathbf{b}_1 \oplus \mathbf{a} - \mathbf{a}_1)
\end{aligned}
$$

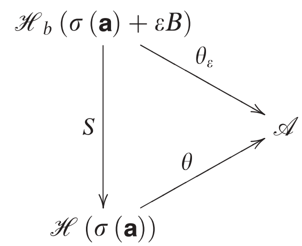
and hence

$$h([\theta_3 \circ T_1^*](f)) = f([h \otimes I_{Y_1 \oplus X}](\mathbf{b}_1 \oplus \mathbf{a} - \mathbf{a}_1)). \tag{4.24}$$

By the uniqueness condition in Lemma 21 and (4.23) and (4.24), we conclude that $\theta_1 = \theta_3 \circ T_1^*$. Similarly $\theta_2 = \theta_3 \circ T_2^*$. We have now defined all terms in diagram (4.17) and shown that the diagram is commutative. By taking $\varepsilon = \varepsilon'$ we see that $\theta_1 \circ R_1^*$ is independent of our method of construction and depends only on ε, and hence we write $\theta_\varepsilon := \theta_1 \circ R_1^*$. Again the commutativity of the diagram shows that for $\varepsilon' < \varepsilon$ we have the following commutative diagram of continuous homomorphisms



where S is the restriction mapping. By the definition of inductive limit there exists a continuous homomorphism $\theta : (\mathscr{H}(\sigma(\mathbf{a})), \tau_b) \to \mathscr{A}$ such that the following diagram commutes



for every $\varepsilon > 0$.

If $P \in \mathscr{P}(X)$ then $\theta_\varepsilon(P) = [\theta_1 \circ R_1^*](P) = \theta_1(P \circ R_1) = [P \circ R_1]_{\mathscr{A}}(\mathbf{b}_1 \oplus \mathbf{a} - \mathbf{a}_1) = P_{\mathscr{A}}([R_1]_{\mathscr{A}}(\mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_1)))$ by Proposition 9. Since $R_1(z_1 \oplus \xi) = \pi_1(z_1) + \xi$ for $z_1 \in Y_1$ and $\xi \in X$,

$$
\begin{aligned}
[R_1]_{\mathscr{A}}(\mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_1)) &= [I_{\mathscr{A}} \otimes R_1](\mathbf{b}_1 \oplus (\mathbf{a} - \mathbf{a}_1))\\
&= ([I_{\mathscr{A}} \otimes \pi_1](\mathbf{b}_1)) + \mathbf{a} - \mathbf{a}_1\\
&= \mathbf{a}_1 + \mathbf{a} - \mathbf{a}_1\\
&= \mathbf{a}
\end{aligned}
$$

and $\theta(P)=\theta_\varepsilon(P)=P_{\mathscr{A}}(\mathbf{a})$ for all $P\in\mathscr{P}(X)$. If $f\in\mathscr{H}(\sigma(\mathbf{a}))$ then f is the germ of a holomorphic function, say f, on $\sigma(\mathbf{a})+\varepsilon B$ for some $\varepsilon>0$. If $h\in\mathscr{M}(\mathscr{A})$ then

$$\begin{aligned} h(\theta(f)) &= h(\theta_1\circ R_1^*(f)) \\ &= h(\theta_1(f\circ R_1)) \\ &= f\circ R_1([h\otimes I_{Y_1\oplus X}](\mathbf{b}_1\oplus \mathbf{a}-\mathbf{a}_1)) \\ &= f([h\otimes I_{Y_1}](\mathbf{a}_1)+[h\otimes I_X](\mathbf{a}-\mathbf{a}_1)) \\ &= f([h\otimes I_X](\mathbf{a})) \end{aligned}$$

and θ satisfies (4.8) in Theorem 18.

By Lemma 21, θ_1 is τ_0-continuous on the set of bounded subsets of $\mathscr{H}_b(V_1\oplus W)$. Since $R_1: V_1\oplus W\to U$ is continuous, R_1^* maps bounded subsets of $\mathscr{H}_b(U)$ into bounded subsets of $\mathscr{H}_b(V_1\oplus W)$ and is continuous when these bounded subsets are endowed with the compact open topology. Hence

$$\theta:\{f:\|f\|_{\sigma(\mathbf{a})+\varepsilon B}\le M\}\subset\mathscr{H}(\sigma(\mathbf{a}))\to\mathscr{A}$$

is continuous when $\{f:\|f\|_{\sigma(\mathbf{a})+\varepsilon B}\le M\}$ is endowed with the compact open topology. By Proposition 16,

$$\theta:(\mathscr{H}(\sigma(\mathbf{a})),\ \tau_0)\to\mathscr{A}$$

is continuous.

This completes the proof of Theorem 18. ■

Remark 22. Equation (4.8) in Theorem 18 implies that

$$\sigma(\theta_{\mathbf{a}}(f))=f(\sigma(\mathbf{a}))$$

for all $f\in\mathscr{H}(\sigma(\mathbf{a}))$.

The following uniqueness results extend those of Waelbroeck [34] and Galé [13] and can be proved using the methods we have developed. We denote by $\mathscr{H}_{\text{wub}}(U)$ the subspace of $\mathscr{H}_b(U)$, U open in a Banach space X, consisting of functions that are weakly uniformly continuous on the bounded subsets of X that lie strictly inside U, and let

$$\mathscr{H}_{\text{wub}}(K)=\{f\in\mathscr{H}(K); f=[g], g\in\mathscr{H}_{\text{wub}}(U) \text{ for some } U \text{ open}, K\subset U\}.$$

In certain cases, for example if X is the original Tsirelson space (see [7, §2.4]), X' has the bounded approximation property and

$$\mathscr{H}_{\text{wub}}(\sigma(\mathbf{a}))=\mathscr{H}(\sigma(\mathbf{a})).$$

Proposition 23. *Let $\mathscr{A}$ denote a unital commutative Banach algebra, γ a uniform cross-norm and X a Banach space with the $(\mathscr{A},\gamma)$-extension property. Let $\mathbf{a}\in\mathscr{A}\hat{\otimes}_\gamma X$ and let $\theta_{\mathbf{a}}$ denote the homomorphism constructed in Theorem* 18. *Then*

(i) (*see* [34]) $\theta_{\mathbf{a}}$ *is the unique τ_b-continuous homomorphism satisfying*

(a) $\theta_{\mathbf{a}}(P)=P_{\mathscr{A}}(\mathbf{a})$ *for all* $P\in\mathscr{P}(X)$,

(b) $\theta_{\mathbf{a}}(f\circ T)=\theta_{T_{\mathscr{A}}(\mathbf{a})}(f)$ *for all* $T:X\to Y$ *Fredholm and all* $f\in\mathscr{H}(\sigma(T_{\mathscr{A}}(\mathbf{a})))$;

(ii) (*see* [13]) $\theta_{\mathbf{a}}|_{\mathscr{H}_{\text{wub}}(\sigma(\mathbf{a}))}$ *is the unique τ_b-continuous homomorphism satisfying*

$$h(\theta_{\mathbf{a}}(f))=f([h\otimes I_X](\mathbf{a}))$$

for all $h \in \mathscr{M}(\mathscr{A})$ *and* $f \in \mathscr{H}_{\text{wub}}(\sigma(\mathbf{a}))$ *whenever* X' *has the bounded approximation property.*

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