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# THE COMPLETION OF A METRISABLE BTB SPACE IS ITS STRONG BIDUAL

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[Received 18 June 1997. Read 15 May 2000. Published 16 October 2001.]

Abstract

We show that the completion of the metrisable BTB space $(E, T)$ is $(E'', \beta(E'', E'))$.

It is well known that every strong bidual of a metrisable locally convex space $(E, T)$ contains the completion of $(E, T)$ [2, § 29, 2. (3)]. It is the purpose of this short note to show that if $(E, T)$ is in addition a BTB space then the strong bidual, $(E'', \beta(E'', E'))$, is actually the completion.

By a locally convex space $(E, T)$ we shall mean a separated locally convex space with dual $E'$. $\sigma(E', E)$ and $\beta(E', E)$ denote the weak and strong topology respectively of the separated dual pair $\langle E', E\rangle$. The dual of the strong dual, $(E', \beta(E', E))$, called the *bidual*, is denoted by $E''$.

The locally convex space $(E, T)$ is called a BTB space [3, definition 6-4-10] if every bounded set is precompact. By $(\hat{E}, \hat{T})$ we denote the completion.

*Examples of BTB spaces*

(1) Every semi-Montel space [3, p. 90] is a BTB space. More precisely, a locally convex space is a semi-Montel space if and only if it is quasi-complete [2, § 18.4; 3, p. 73] and a BTB space. Recall that a quasi-barrelled semi-Montel space is called a Montel space.

(2) Every Schwartz space is a BTB space. More precisely, a locally convex space is a Schwartz space if and only if it is quasi-normable and a BTB space [1, prop. 17].

We now state our theorem.

 

**Theorem.** *The completion of the metrisable BTB space* $(E, T)$ *is* $(E'', \beta(E'', E'))$.

PROOF. By [2, § 29, 2. (3)], $(\hat{E}, \hat{T})$ is a subspace of $(E'', \beta(E'', E'))$, and it only remains to be seen that $\hat{E} \supset E''$. Any $u \in E''$ is bounded on some $\beta(E', E)$-neighbourhood of 0, and thus there is a bounded set $B$ in $(E, T)$ such that $|\langle u, y\rangle| \leq 1$ for each $y \in B^0$; i.e., $u$ is a member of the polar $B^{00}$ in $E''$ of the polar $B^0$ in $E'$ of $B$. The bipolar theorem implies (i) that $B^{00}$ is the balanced convex $\sigma(E'', E')$-closed hull of $B$, and (ii) that $C = B^{00} \cap \hat{E}$ is $\sigma(\hat{E}, E')$-closed, hence $\hat{T}$-closed. Since $(E, T)$ is a BTB space, $C$ is $\hat{T}$-compact, hence $\sigma(\hat{E}, E')$-compact, hence $\sigma(E'', E')$-compact, hence $\sigma(E'', E')$-closed. Now $B \subset C$ implies $B^{00} \subset C$, which ensures that $u \in C \subset \hat{E}$. ■

An elementary argument along similar lines characterises those spaces $(E, T)$ for which completions and strong biduals coincide. Metrisable BTB spaces obviously satisfy (a)–(c) below, where $\ddot{E}$ denotes the union of all closures in $(\hat{E}, \hat{T})$ of bounded sets in $(E, T)$.

**Observation.** *The completion of a locally convex space* $E$ *is its strong bidual if and only if*
(a) $E$ *is quasi-barrelled,*
(b) $\hat{E} = \ddot{E}$ *and*
(c) *closures in* $(\hat{E}, T)$ *of balanced convex bounded sets of* $(E, T)$ *are* $\sigma(\hat{E}, E')$*-compact.*

## ACKNOWLEDGEMENT

I hereby acknowledge the assistance of the referee and I thank him for his pointers to the literature and suggestions.

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