A CONVEXITY PROPERTY OF INHOMOGENEOUS INCOMPRESSIBLE ELASTIC CYLINDERS

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Abstract

This note draws attention to an interesting convexity property of incompressible, inhomogeneous isotropic linear elastic cylinders. Considering a right cylinder of general cross-section of this type, whose shear modulus is a smooth function of the cross-sectional coordinates, subject to a class of displacement boundary conditions on its lateral boundary, it is found that a non-negative cross-sectional measure of deformation is a convex function of the axial coordinate provided that the shear modulus is a convex function of the cross-sectional coordinates. Some implications are discussed.

1. Introduction

The purpose of this note is to draw attention to an easily stated, striking property of inhomogeneous incompressible elastic cylinders. Considering a right cylinder of general cross-section consisting of incompressible, inhomogeneous isotropic linear elastic material, whose shear modulus is a smooth function of the cross-sectional coordinates, subject to a class of displacement boundary conditions on its lateral boundary, it is found that a non-negative cross-sectional measure of the deformation is a *convex* function of the axial coordinate provided that the shear modulus is a *convex* function of the cross-sectional coordinates. Some applications of this property are discussed.

Extensions of the fundamental proposition in the *homogeneous* case have been studied previously, both for compressible materials [4] and (in the context of zero displacement on the lateral boundary) for incompressible materials [5]. Other extensions of the proposition, for example to compressible inhomogeneous materials, can also be contemplated, but with a loss of the transparency evident in the fundamental proposition discussed in this note.

2. The convexity property and its implications

Rectangular Cartesian coordinates are denoted by x_i, Latin indices take the values $1, 2, 3$, Greek indices take the values $1, 2$, while the usual comma notation and summation convention are employed.

We consider a right cylinder of elastic material as previously described, whose generators are in the x_3 direction. Its cross-section is denoted by D, and its boundary ∂D is supposed to be sufficiently smooth to allow application of the divergence

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theorem. The shear modulus of the elastic material $\mu(x_1, x_2)$ is supposed to be a positive, C^2 function of x_1, x_2.

We consider elastic fields in the cylinder consistent with imposed displacements (which may be null) on its lateral boundary, and with zero body force. The displacement components are denoted by u_i, the 'pressure' by p, and these are supposed to be C^3 and C^2 functions respectively. They satisfy, in the relevant range of x_3, the equations

$$-p_{,i} + \mu u_{i,jj} + \mu_{,\gamma}\left(u_{i,\gamma} + u_{\gamma,i}\right) = 0, \quad \text{in} \quad D, \tag{2.1}$$

$$u_{i,i} = 0, \quad \text{in} \quad D \tag{2.2}$$

and the boundary conditions on the lateral boundary are supposed to be

$$\left.\begin{aligned} u_\beta &= f_\beta(x_1, x_2), \\ u_3 &= g(x_1, x_2) + x_3 h(x_1, x_2) \end{aligned}\right\} \quad \text{on} \quad \partial D, \tag{2.3}$$

(i.e. $u_{\beta,3} = u_{3,33} = 0$ thereon).

We introduce the non-negative cross-sectional measure of deformation

$$F(x_3) = \int_{D(x_3)} \mu\left[u_{\beta,3}u_{\beta,3} + u_{3,\beta}u_{3,\beta}\right] dA \tag{2.4}$$

where $D(x_3)$ denotes the cross-section at x_3. Differentiating successively,

$$F'(x_3) = 2\int_{D(x_3)} \mu\left[u_{\beta,3}u_{\beta,33} + u_{3,\beta}u_{3,\beta3}\right] dA, \tag{2.5}$$

$$F''(x_3) = 2\int_{D(x_3)} \mu\left[u_{\beta,33}u_{\beta,33} + u_{3,\beta3}u_{3,\beta3} + u_{\beta,3}u_{\beta,333} + u_{3,\beta}u_{3,\beta33}\right] dA. \tag{2.6}$$

Applying (2.1)–(2.3) together with the divergence theorem to the last two terms in (2.6) yields

$$\begin{aligned} F''(x_3) = 2\textstyle\int_{D(x_3)} &\left[\mu\left(u_{\beta,33}u_{\beta,33} + u_{3,\beta3}u_{3,\beta3} + u_{3,33}^2 + u_{\beta,\gamma3}u_{\beta,\gamma3}\right)\right. \\ &\left. + \mu_{,\beta\gamma}u_{\beta,3}u_{\gamma,3}\ \right] dA. \end{aligned} \tag{2.7}$$

In view of the fact that the first four terms in the integrand are non-negative, and the fact that the last term is also provided that $\mu(x_1, x_2)$ is convex (i.e. $\mu_{,11} \geqslant 0, \mu_{,22} \geqslant 0, \mu_{,11}\mu_{,22} - \mu_{,12}^2 \geqslant 0$), it follows that $F''(x_3) \geqslant 0$, giving

Proposition 1. *The cross-sectional measure* (2.4) *is a convex function of the axial coordinate* x_3 *provided that the shear modulus* μ *is a convex function of the cross-sectional coordinates* x_1, x_2.

The convexity of μ is not, of course, a necessary condition for the convexity of F.

Two implications of this result are now mentioned—*assuming henceforward that* μ *is convex*. The first of the two propositions is new even in the homogeneous case.

Proposition 2. *In the context of an infinite cylinder* $-\infty < x_3 < \infty$ *for which* $h \equiv 0$, *in order that* $F(x_3)$ *be everywhere bounded, it is necessary and sufficient that*

$$-p_{,\beta} + (\mu u_{\beta,\gamma})_{,\gamma} + \mu_{,\gamma} u_{\gamma,\beta} = 0, \quad u_{\beta,\beta} = 0, \tag{2.8}$$

and

$$(\mu u_{3,\gamma})_{,\gamma} = 0. \tag{2.9}$$

(In standard terminology, these conditions mean that the deformation is a superposition of plane and anti-plane deformations.)

Using the tangent property of convexity, the necessity implies $F' \equiv 0$. The latter implies that $F'' \equiv 0$, and this, together with an examination of the integrand in (2.7), bearing in mind (2.1), (2.2), gives rise to (2.8), (2.9). Thus the necessity is established, and the sufficiency is obvious.

This latter proposition is reminiscent of previous work concerning the structure and role of the set of all possible elastostatic fields in an infinitely long cylinder in the absence of lateral loading and body force, but in the presence of a restriction on the size of a suitable cross-sectional norm of the associated strain field: see [1]–[3], [6], [4].

The positive square root of F is denoted by $F^{1/2}$ in the following proposition, (a) of which is stronger than Proposition 1.

Proposition 3. (a) $F^{1/2}(x_3)$ *is also convex provided* $F \neq 0$.

(b) *For a finite cylinder* $0 \leqslant x_3 \leqslant L$,

$$F^{1/2}(x_3) \leqslant F^{1/2}(0) + \big(1 - x_3/L\big)\, F^{1/2}(L)$$

where $F(0)$, $F(L)$ *are available in terms of conventional data if* u_3 *and the complementary components of shear stress* $\tau_{3\beta} = \mu(u_{3,\beta} + u_{\beta,3})$ *are specified functions on the plane ends* $x_3 = 0$, $x_3 = L$.

PROOF. Part (a) follows on using

$$\left(F^{1/2}\right)'' = \left(2FF'' - \{F'\}^2\right)\left(4F^{3/2}\right)^{-1}, \quad (F \neq 0),$$

together with (2.5), (2.7) and Schwarz's inequality. Part (b) is obvious.

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