FREDHOLM THEORY IN AN ALGEBRA WITH RESPECT TO A BANACH SUBALGEBRA

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[Received 19 September 1999. Read 15 May 2000. Published 16 October 2001.]

Abstract

In this paper, Fredholm and perturbation theory for elements of an algebra with respect to a Banach subalgebra is developed. In this development, ideas from the theory of closed operators are combined with general Fredholm theory in a Banach algebra.

1. Introduction

Throughout this paper, A is a complex unital algebra, and $(B, \|\cdot\|_B)$ is a Banach subalgebra of A which contains the unit. An element $t \in A$ is affiliated with B if, for some $\lambda \in \mathbf{C}$, $(\lambda - t)^{-1} \in B$. When t is affiliated with B, then there is a B-spectral theory for t. For the case that A is an LMC-algebra and B is a continuously embedded Banach subalgebra, this spectral theory has been studied and applied in [5] and [7]. In fact, many of the results in these papers hold when A is an arbitrary algebra.

The main purpose of this paper is to introduce and study Fredholm theory in A with respect to B. Fredholm theory in an arbitrary Banach algebra is developed in [6]. Here we extend this development to an even more general setting. The theory presented in this paper is a synthesis of the material in [6] and the theory of closed operators. In fact, many of our results are motivated by the Fredholm and perturbation theory of closed operators. In this regard, we now give a short list of concepts from the theory of closed operators along with the section of this paper in which the corresponding generalisation is considered:

spectrum of a closed operator—Section 2;
T-bounded operator [12, pp 189–94]—Section 2;
Fredholm operator [10, chapter IV]—Section 3;
T-compact operator [10, section V.3]—Section 4;
generalised convergence (convergence in gap) [12, pp 197–208]—Section 5.

2. Some spectral theory

In our first theorem we establish that the B-spectrum of the elements of A (defined below) is related to the operator spectrum of certain closed operators. First, some notation.

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Definition 1. $\mathrm{Inv}_B(A) = \{t \in A : t^{-1} \in B\}$; for $t \in A$:

$$\mathrm{res}_B(t) = \{\lambda \in \mathbf{C} : (\lambda - t) \in \mathrm{Inv}_B(A)\}; \quad \sigma_B(t) = \mathbf{C} \backslash \mathrm{res}_B(t).$$

For $t \in A$, let $\mathrm{dom}\,(t) = \{b \in B : bt \in B\}$. Define the operator $R_t : \mathrm{dom}\,(t) \to B$ by $R_t b = bt, b \in \mathrm{dom}\,(t)$. When t is affiliated with B, then it is easy to see that R_t is a closed operator.

Theorem 2. *Assume that t is affiliated with B. Denote by* $\boldsymbol{B}(B)$ *the algebra of all bounded linear operators on B. The closed operator* R_t *has an inverse* $S \in \boldsymbol{B}(B)$ *if and only if* $t \in \mathrm{Inv}_B(A)$, *in which case* $S = R_{t^{-1}}$.

Proof. Assume that R_t has an inverse $S \in \boldsymbol{B}(B)$, so:
(i) for all $b \in B$, $S(b) \in \mathrm{dom}\,(t)$ and $S(b)t = b$;
(ii) for all $d \in \mathrm{dom}\,(t)$, $S(dt) = d$.
Let $c = S(1) \in B$. Then, by (i), $ct = S(1)t = 1$. Now for all $b \in B$, $S(b) - bc \in B$ and $(S(b) - bc)t = 0$. Thus $S(b) - bc \in \mathrm{dom}\,(t)$, and by (ii), $0 = S((S(b) - bc)t) = S(b) - bc$. It follows that $S = R_c$.

For all $d \in \mathrm{dom}\,(t)$, $dtc = S(dt) = d$. Therefore, for all $d \in \mathrm{dom}\,(t)$, $d(tc - 1) = 0$. Now choose $\lambda \in \mathrm{res}_B(t)$, so $(\lambda - t)^{-1} \in \mathrm{dom}(t)$. Then $(\lambda - t)^{-1}(tc - 1) = 0$, so $tc - 1 = 0$. This proves one implication, and the other is easy. ∎

Theorem 2 implies that $\mathrm{res}_B(t)$ is exactly the usual resolvent set of the closed operator R_t.

Before turning to the main topic of this paper, the Fredholm theory of elements of A with respect to B, we consider a result concerning B-spectral theory.

Definition 3. Fix $t \in A$. Elements of A of the form $tb + c$ or $bt + c$, where $b, c \in B$, are called t-bounded.

The concept of a t-bounded element is motivated by the idea of a T-bounded operator which occurs in the perturbation theory of closed operators; see [12], for example. This concept has proved to be useful in the theory of certain commutative LMC-algebras [5]. Here we prove only an elementary result concerning t-bounded elements.

For $b \in B$, let $\rho_B(b)$ denote the spectral radius of b.

Proposition 4. *Let* $t \in A$. *Let* $s = tb + c$ *where* $b, c \in B$. *If* $\lambda \in \mathrm{res}_B(t)$ *and* $\rho_B((\lambda - t)^{-1}s) < 1$, *then* $(\lambda - t - s)^{-1} \in B$.

Proof. First note that $(\lambda - t)^{-1}s \in B$, since $(\lambda - t)^{-1}t = -1 + \lambda(\lambda - t)^{-1} \in B$. Now $\rho_B((\lambda - t)^{-1}s) < 1$ implies that $1 - (\lambda - t)^{-1}s \in \mathrm{Inv}_B(A)$, so $\lambda - t - s = (\lambda - t)(1 - (\lambda - t)^{-1}s) \in \mathrm{Inv}_B(A)$. ∎

Corollary 5. *Let t and s be as in Proposition 4. If*

$$\inf\{\rho_B((\lambda - t)^{-1}s) : \lambda \in \mathrm{res}_B(t)\} < 1,$$

then $t + s$ is affiliated with B.

3. Fredholm theory in *A* with respect to *B*

We assume throughout that K_B is a closed inessential ideal of B [6, F.3.12, p. 42], that F_B is an ideal of B with $F_B \leqslant K_B$, and that the quotient algebra K_B/F_B is a radical algebra. Also, *for the remainder of this paper we make the standing assumption that F_B is a left ideal of A*. As one would expect, results analogous to those proved in what follows hold when F_B is a right ideal of A.

Let $\mathrm{rad}\,(B)$ denote the Jacobson radical of B. Let $b \to b'$ be the usual quotient map of $B \to B' = B/\mathrm{rad}\,(B)$; here b' is the residue class of b in the quotient algebra. Since B' is semisimple, the socle of B' exists; denote this by $\mathrm{soc}\,(B')$. Also, set

$$p\mathrm{soc}\,(B) = \{b \in B : b' \in \mathrm{soc}\,(B')\}.$$

Now we prove some technical results involving the relationship between K_B and F_B.

Proposition 6. (1) *If $k \in K_B$, then there exist $h \in K_B$ and $f \in F_B$ such that $(1-k)(1-h) = 1 - f$. Also, there exist $h_1 \in K_B$ and $f_1 \in F_B$ such that $(1 - h_1)(1 - k) = 1 - f_1$.*
(2) $K_B \cap p\mathrm{soc}\,(B) \subseteq F_B + \mathrm{rad}\,(B)$.

PROOF. (1) follows easily from the assumption that K_B/F_B is a radical algebra.

We prove (2). Assume that $g \in K_B \cap p\,\mathrm{soc}\,(B)$, so that $g' \in \mathrm{soc}\,(B')$. By [6, F.1.7, p. 25], there exists $u \in K_B$ such that u' is an idempotent in $\mathrm{soc}\,(B')$ with $u'g' = g'$. Now $\sigma_B(u) = \{0, 1\}$. Let $p(1, u)$ be the spectral idempotent of u corresponding to the isolated point $\{1\}$ in $\sigma_B(u)$. It is straightforward to check that $p(1, u)' = p(1, u') = u'$. Set $p = p(1, u)$, and note that $p \in K_B$. As K_B/F_B is a radical algebra, $p \in F_B$. Now $(g - pg)' = g' - u'g' = 0$. Therefore $g - pg \in \mathrm{rad}\,(B)$, and $pg \in F_B$. Finally,

$$g = pg + (g - pg) \in F_B + \mathrm{rad}\,(B). \quad \blacksquare$$

Definition 7. An element $t \in A$ is in Φ_B if there exist $b, c \in B$ and $k, j \in K_B$ such that

$$tb = 1 - k, \quad \text{and} \quad ct = 1 - j.$$

The element t is in Φ_B° if there exists $f \in F_B$ such that $t + f \in \mathrm{Inv}_B(A)$. Clearly, $\mathrm{Inv}_B(A) \subseteq \Phi_B^\circ \subseteq \Phi_B$.

The Fredholm theory of elements in B relative to the closed inessential ideal K_B is developed in [6]. There, the set Φ_B° is the set of all elements $b \in \Phi_B$ which have index function identically equal to zero; temporarily, denote this set by Γ°. To verify that the definition of Φ_B° above when applied to the elements of B is consistent with the development in [6], and for later application, we prove the following result.

Proposition 8. *If $b \in B$ is in Γ°, then there exists $f \in F_B$ such that $b + \mu f \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$.*

Proof. Since $b \in B$ is in Γ°, by [6, F.3.11, p. 41] there is an element g in $K_B \cap \{\text{presocle of } B\}$ such that $b + \mu g \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$. By Proposition 6 (2), $g = f + r$ where r is in $\mathrm{rad}(B)$ and f is in F_B. It follows that $b + \mu f \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$. ■

Proposition 9. *Let $t \in A$. The following are equivalent:*

(1) *$t \in \Phi_B$;*

(2) *there exist $b_1, c_1 \in B$ and $f, g \in F_B$ such that*

$$tb_1 = 1 - f, \quad \textit{and} \quad c_1 t = 1 - g;$$

(3) *there exist $b \in B$ and $p, q \in F_B$ such that*

$$tb = 1 - p, \quad \textit{and} \quad bt = 1 - q.$$

Proof. Assume that (1) holds. Suppose that $b \in B$ and $k \in K_B$ with $tb = 1 - k$. Using Proposition 6 (2), choose $h \in K_B$ and $f \in F_B$ such that $(1-k)(1-h) = 1 - f$. Setting $b_1 = b(1-h)$, we have $tb_1 = 1 - f$. This, plus a similar argument on the other side, proves that (2) holds.

Now we show that (2) $\Rightarrow$ (3). Assume that $tb_1 = 1 - f$, and $c_1 t = 1 - g$, where b_1, c_1, f and g are as in (2). Then $c_1 t b_1 = (1-g)b_1 = c_1(1-f)$, so $c_1 = c_1 f + (1-g)b_1$. Therefore $tc_1 = tc_1 f + tb_1 - tgb_1 = 1 + \{\text{something in } F_B\}$. Thus we can take $b = c_1$ in the conclusion of (3). ■

Theorem 10. *Assume that $\gamma \in \mathrm{res}_B(t)$.*

(1) *For $\lambda \neq \gamma$, $(\lambda - t) \in \Phi_B \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \Phi_B$;*

(2) *for $\lambda \neq \gamma$, $(\lambda - t) \in \Phi_B^\circ \Leftrightarrow (\gamma - \lambda)^{-1} - (\gamma - t)^{-1} \in \Phi_B^\circ$;*

(3) *if for some $k \in K_B$, $(\lambda - t - k) \in \mathrm{Inv}_B(A)$, then $(\lambda - t) \in \Phi_B^\circ$.*

Proof. We prove the case where $\gamma = 0$, so $t^{-1} \in B$. To prove (1), first we note an equality which is used repeatedly:

$$(\lambda - t)t^{-1} = -\lambda(\lambda^{-1} - t^{-1}). \tag{*}$$

Now assume that $b \in B$ and $f, g \in K_B$ such that

$$(\lambda - t)b = 1 - f; \quad b(\lambda - t) = 1 - g.$$

Then tb and bt are in B, and using (*) we have

$$(\lambda^{-1} - t^{-1})(-\lambda tb) = 1 - f, \text{ and } (-\lambda bt)(\lambda^{-1} - t^{-1}) = 1 - g.$$

This proves that $(\lambda - t) \in \Phi_B \Rightarrow (\lambda^{-1} - t^{-1}) \in \Phi_B$. The reverse implication is proved in a similar fashion.

Now suppose that $(\lambda - t) \in \Phi_B^\circ$. By definition, there exists $f \in F_B$ such that $(\lambda - t + f) \in \mathrm{Inv}_B(A)$. Then, just as in (*), $(\lambda^{-1} - t^{-1} - \lambda^{-1}ft^{-1})t = -\lambda^{-1}(\lambda - t + f)$. It follows that the right-hand side of this equality has an inverse c in B. Note that this implies that $tc \in B$. Then $(\lambda^{-1} - t^{-1} - \lambda^{-1}ft^{-1})$ has right inverse tc. Also, $c(\lambda^{-1} - t^{-1} - \lambda^{-1}ft^{-1})t = 1$, so $tc(\lambda^{-1} - t^{-1} - \lambda^{-1}ft^{-1}) = 1$. This proves that $(\lambda^{-1} - t^{-1}) \in \Phi_B^\circ$.

Conversely, suppose that $g \in F_B$ is such that $\lambda^{-1} - t^{-1} + g \in \mathrm{Inv}_B(A)$. Now $t^{-1}(\lambda - t - \lambda tg) = -\lambda(\lambda^{-1} - t^{-1} + g)$ has an inverse b in B. Therefore $bt^{-1} \in B$ is an inverse for $(\lambda - t - \lambda tg)$. Note here that $tg \in F_B$. Thus $(\lambda - t) \in \Phi_B^\circ$.

Again, in the proof of (3), we assume that $t^{-1} \in B$. Suppose that $\lambda \neq 0$, and, as in the statement of (3), $(\lambda - t - k) \in \mathrm{Inv}_B(A)$ where $k \in K_B$. Denote by b the inverse of this element. Then $bt \in B$, and bt is the inverse of

$$t^{-1}(\lambda - t - k) = -\lambda(\lambda^{-1} - t^{-1}) - t^{-1}k.$$

It follows by [6, F.3.8, p. 39] that $(\lambda^{-1} - t^{-1}) \in \Phi_B^\circ$. Thus, by (2), $(\lambda - t) \in \Phi_B^\circ$. ■

Definition 11 (two essential spectra). For $t \in A$, define:

$$F\sigma_B(t) = \{\lambda \in \mathbf{C} : (\lambda - t) \notin \Phi_B\};$$

$$W\sigma_B(t) = \cap\{\sigma_B(t + k) : k \in K_B\}.$$

The set $F\sigma_B(t)$ is the Fredholm spectrum of t, and $W\sigma_B(t)$ is the Weyl spectrum of t.

Theorem 12. *Assume that t is affiliated with B.*

(1) *For $k \in K_B, (\lambda - t) \in \Phi_B^\circ \Leftrightarrow (\lambda - t - k) \in \Phi_B^\circ$.*
(2) *If $s \in \Phi_B^\circ$, b is in B and $b \in \Phi_B^\circ$, then sb is in Φ_B°.*
(3) $W\sigma_B(t) = \cap\{\sigma_B(t + f) : f \in F_B\} = \{\lambda \in \mathbf{C} : (\lambda - t) \notin \Phi_B^\circ\}$.
(4) *For $k \in K_B$, $W\sigma_B(t) = W\sigma_B(t + k)$.*

PROOF. First we prove (1). We assume without loss of generality that $t^{-1} \in B$. Fix $k \in K_B$. In the case that $\lambda = 0$, $t \in \mathrm{Inv}_B(A) \subseteq \Phi_B^\circ$ automatically. We prove $t+k \in \Phi_B^\circ$. By Proposition 8 there exists $f \in F_B$ such that $1 + t^{-1}k + f \in \mathrm{Inv}_B(A)$. Therefore $t + k + tf = t(1 + t^{-1}k + f) \in \mathrm{Inv}_B(A)$ so $t + k \in \Phi_B^\circ$.

Now suppose that $\lambda \neq 0$. Assume that $(\lambda - t) \in \Phi_B^\circ$. We have

$$t^{-1}(\lambda - t - k) = -\lambda(\lambda^{-1} - t^{-1}) - t^{-1}k.$$

By Theorem 10, $(\lambda^{-1} - t^{-1}) \in \Phi_B^\circ$, so $-\lambda(\lambda^{-1} - t^{-1}) - t^{-1}k \in \Phi_B^\circ$. Thus there exists $f \in F_B$ such that $t^{-1}(\lambda - t - k) + f \in \mathrm{Inv}_B(A)$. It follows that $(\lambda - t - k) + tf \in \mathrm{Inv}_B(A)$, so $(\lambda - t - k) \in \Phi_B^\circ$. Conversely, assume that $(\lambda - t - k) \in \Phi_B^\circ$. By definition there exists $f \in F_B$ such that $(\lambda - t - k) + f \in \mathrm{Inv}_B(A)$. Applying Theorem 10 (3), we have $(\lambda - t) \in \Phi_B^\circ$.

Now assume that s and b are as in the statement of (2). Choose $g, f \in F_B$ such that $s + g$ and $b + f$ are in $\mathrm{Inv}_B(A)$. Then $sb + (gb + sf + gf) \in \mathrm{Inv}_B(A)$. Since the element in parentheses is in F_B, $sb \in \Phi_B^\circ$.

To prove (3), denote the set on the right by Γ. If $\lambda \notin \Gamma$, then $(\lambda - t) \in \Phi_B^\circ$. Then, for some $f \in F_B$, $(\lambda - t - f)^{-1} \in B$, and so $\lambda \notin \sigma_B(t + f)$. Therefore,

$$W\sigma_B(t) \subseteq \cap\{\sigma_B(t+f) : f \in F_B\} \subseteq \Gamma.$$

Now suppose that $\lambda \notin W\sigma_B(t)$, so, for some $k \in K_B$, $(\lambda - t - k) \in \mathrm{Inv}_B(A) \subseteq \Phi_B^\circ$. Therefore, by (1), $(\lambda - t) \in \Phi_B^\circ$, and thus $\lambda \notin \Gamma$. This proves the equality of the sets in (3).

Part (4) follows directly from the definition of $W\sigma_B(t)$. ∎

Theorem 13. *Assume that t is affiliated with B. If $(\lambda - t) \in \Phi_B^\circ$, then there exists $f \in F_B$ such that $\lambda - t + \mu f \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$.*

Proof. We assume without loss of generality that $t^{-1} \in B$. When $\lambda = 0$, we can choose $f = 0$ in the statement of the theorem. Suppose that $\lambda \neq 0$. Then $t^{-1}(\lambda - t) = -\lambda(\lambda^{-1} - t^{-1})$ and, by Theorem 10, $\lambda^{-1} - t^{-1} \in \Phi_B^\circ$. By Proposition 8 there exists $f \in F_B$ such that $-\lambda(\lambda^{-1} - t^{-1}) + \mu f \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$. Therefore $(\lambda - t + \mu t f) = t(-\lambda(\lambda^{-1} - t^{-1}) + \mu f) \in \mathrm{Inv}_B(A)$ for all $\mu \neq 0$. Since $tf \in F_B$, the result is proved. ∎

4. Perturbation results

As before, we assume that F_B is a left ideal of A. Let $t \in A$. An element $a \in A$ is t-inessential if a is of the form $a = tk + j$, where $k, j \in K_B$ (when F_B is a right ideal of A, t-inessential elements have the form $kt + j$, where $k, j \in K_B$).

Note 14. Assume that a is t-inessential, so $a = tk + j$ where $k, j \in K_B$. Then $-a$ is $(t + a)$-inessential.

Proof. Now $t + a = t(1 + k) + j$. By Proposition 6 (1) we can choose $h \in K_B$ and $f \in F_B$ such that $(1+k)(1+h) = 1 + f$. Therefore $(t+a)(1+h) = t(1+f) + j(1+h)$, so $(t+a) + (t+a)h = t + d$ where $d \in K_B$. This implies that $a = -(t+a)(h) + d$, and thus $-a$ is $(t + a)$-inessential. ∎

Theorem 15. *Let $a = tk + j$ where $k, j \in K_B$. Then*

$$(\lambda - t) \in \Phi_B \Leftrightarrow (\lambda - (t + a)) \in \Phi_B.$$

Thus $F\sigma_B(t) = F\sigma_B(t + a)$.

Proof. Let $b \in B$, and $f, g \in K_B$ have the properties that

$$(\lambda - t)b = 1 - f; \quad b(\lambda - t) = 1 - g.$$

Note that $bt \in B$. Now

$$b(\lambda - t - a) = 1 - g - [btk + bj];$$

and note that $[btk + bj] \in K_B$.

By Proposition 6 (1) we can choose $h \in K_B$ such that $(1+k)(1+h) = 1+p$ where $p \in F_B$. Then

$$\begin{aligned}(\lambda - t - a)(1+h)b &= (\lambda - t(1+k) - j)(1+h)b \\ &= \lambda(1+h)b - t(1+p)b - j(1+h)b \\ &= (\lambda - t)b - tpb + m\end{aligned}$$

where $m \in K_B$. Also, since F_B is a left ideal of A, $tpb \in F_B$. Thus $(\lambda - t - a)(1+h)b = 1 - f +$ something in K_B. This proves $(\lambda - t) \in \Phi_B \Rightarrow (\lambda - t - a) \in \Phi_B$.

To prove the reverse implication, assume that $(\lambda - t - a) \in \Phi_B$. By Note 14, $-a$ is $(t+a)$-inessential. Then, applying the previous argument, we have

$$(\lambda - t - a) \in \Phi_B \Rightarrow (\lambda - t) = (\lambda - t - a + a) \in \Phi_B. \quad \blacksquare$$

Theorem 16. *Assume that t is affiliated with B. Let $a = tk + j$ where $k, j \in K_B$;*

$$(\lambda - t) \in \Phi_B^\circ \Leftrightarrow (\lambda - (t+a)) \in \Phi_B^\circ.$$

Thus $W\sigma_B(t) = W\sigma_B(t+a)$.

PROOF. We assume without loss of generality that $t^{-1} \in B$. Suppose that $(\lambda - t) \in \Phi_B^\circ$. First assume that $\lambda \neq 0$. Then

$$t^{-1}(\lambda - (t+a)) = -\lambda(\lambda^{-1} - t^{-1}) - k - t^{-1}j.$$

By Theorem 10, $(\lambda^{-1} - t^{-1}) \in \Phi_B^\circ$, so the expression on the right above is in Φ_B°. Call this element d. By Proposition 8 there exists $f \in F_B$ such that $d + f \in \mathrm{Inv}_B(A)$. Therefore $(\lambda - (t+a)) + tf = t(d+f) \in \mathrm{Inv}_B(A)$. This proves $(\lambda - (t+a)) \in \Phi_B^\circ$.

Next we prove $t + a \in \Phi_B^\circ$. Now $t^{-1}(t+a) = 1 + k + t^{-1}j \in \Phi_B^\circ$. Call the element on the right b. Again by Proposition 8 there exists $f \in F_B$ such that $b + f \in \mathrm{Inv}_B(A)$. Thus $t + a + tf = t(b+f) \in \mathrm{Inv}_B(A)$.

Now assume that $(\lambda - (t+a)) \in \Phi_B^\circ$. By Proposition 6 (1) we can choose $h \in K_B$ such that $(1+k)(1+h) = 1+f$ where $f \in F_B$. By Theorem 12 (2), $(\lambda - t(1+k) - j)(1+h) \in \Phi_B^\circ$, and this element is equal to $(\lambda - t) + \lambda h - tf - j(1+h)$. Since $\lambda h - tf - j(1+h) \in K_B$, Theorem 12 (1) implies that $(\lambda - t) \in \Phi_B^\circ$. $\blacksquare$

Definition 17. A number λ is a Riesz point of t if either
(i) $\lambda \in \mathrm{res}_B(t)$, or
(ii) λ is an isolated point of $\sigma_B(t)$ with $\lambda - t \in \Phi_B$.

Theorem 18. *Assume that $\lambda_0 \in \mathrm{bndy}(\sigma_B(t))$ and $\lambda_0 - t \in \Phi_B$. Then λ_0 is a Riesz point of t.*

PROOF. Note that t is affiliated with B. Therefore we may assume that t^{-1} is in B, and that $\lambda_0 \neq 0$. Now $\lambda_0^{-1} \in \mathrm{bndy}(\sigma_B(t^{-1}))$ and, by Theorem 10, $\lambda_0^{-1} - t^{-1} \in \Phi_B$. Applying [6, R.2.4, p. 57], we have that λ_0^{-1} is a Riesz point of t^{-1}. Thus λ_0^{-1} is an isolated point of $\sigma_B(t^{-1})$, and it follows that λ_0 is an isolated point of $\sigma_B(t)$. Therefore, by Definition 16, λ_0 is a Riesz point of t. ■

Assume that $d \in B$, and that η is an isolated point of $\sigma_B(d)$. Let p be the spectral idempotent (defined by the holomorphic functional calculus) corresponding to the set $\{\eta\}$. It is a useful fact that $\eta - d + p$ is invertible. To verify this, choose disjoint open sets U and V such that $\eta \in U$, $|\eta - \lambda| < 1$ for all $\lambda \in U$, and $\sigma_B(d)\backslash\{\eta\} \subseteq V$. Define $g(\lambda)$ by

$$g(\lambda) = \eta - \lambda \text{ for } \lambda \in V; \quad g(\lambda) = \eta - \lambda + 1 \quad \text{for } \lambda \in U.$$

Then $g(d) = \eta - d + p$ and, by the Spectral Mapping Theorem, $\sigma_B(\eta - d + p) = g(\sigma_B(d))$. Since $g(\lambda) \neq 0$ for all $\lambda \in U \cup V$, it follows that $\eta - d + p$ is invertible.

Theorem 19. *Let λ_0 be a Riesz point of t. Then there exists $f \in F_B$ such that $\lambda_0 - t + f \in \mathrm{Inv}_B(A)$. In particular, $\lambda_0 - t \in \Phi_B^\circ$.*

Proof. We assume that λ_0 is in $\sigma_B(t)$. As in the proof of Theorem 18, we may assume that t^{-1} is in B, and that $\lambda_0 \neq 0$. Also, by that proof, λ_0^{-1} is a Riesz point of t^{-1}. Let p_0 be the spectral idempotent of t^{-1} corresponding to the set $\{\lambda_0^{-1}\}$. As noted previously, we have $(\lambda_0^{-1} - t^{-1} + p_0)$ is invertible in B. Denote by b the inverse of this element. Then

$$(\lambda_0 - t - \lambda_0 t p_0)b = -\lambda_0 t(\lambda_0^{-1} - t^{-1} + p_0)b = -\lambda_0 t.$$

By [6, R.2.3, p. 56] p_0 is in K_B and, since K_B/F_B is radical, $p_0 \in F_B$. Therefore $f = -\lambda_0 t p_0 \in F_B$, and $\lambda_0 - t + f$ has right inverse $b(-\lambda_0^{-1})t^{-1}$ in B. The same argument works to show that $\lambda_0 - t + f$ has a left inverse (in this regard, note that $tp_0 = p_0 t$). ■

5. GS-convergence relative to B

Definition 20. A sequence $\{t_k\} \subseteq A$ converges to $t \in A$ *in the generalised sense* (GS-convergence) if there exists a positive integer N such that there is a number $\lambda \in \mathrm{res}_B(t_k) \cap \mathrm{res}_B(t)$ for all $k \geq N$, with the property that

$$\|(\lambda - t)^{-1} - (\lambda - t_k)^{-1}\|_B \to 0 \quad \text{as} \quad N \leq k \to \infty.$$

GS-convergence is extensively studied and applied in [4]. The idea of GS-convergence is derived from the concept of convergence in gap for a sequence of closed operators; see [12, pp 197–208]. Although the results in [4] are for the case where A is an LMC-algebra, it is easy to see that they hold for any algebra.

The following properties of generalised convergence can be found in [4].

Assume that $t_k \to t$ (GS).

(i) When $\lambda \in \mathrm{res}_B(t)$, then $\lambda \in \mathrm{res}_B(t_k)$ for all k sufficiently large, and $\|(\lambda - t_k)^{-1} - (\lambda - t)^{-1}\|_B \to 0$ as $k \to \infty$ [4, proposition 3].

(ii) When K is a compact subset of $\mathrm{res}_B(t)$, then there exists N such that for $k \geq N$, $K \subseteq \mathrm{res}_B(t_k)$ [4, theorem 6].

(iii) Fix $b \in B$. There exists $\varepsilon > 0$ such that for all $\delta \in \mathbf{C}$ with $|\delta| < \varepsilon$, $t_k + \delta b \to t + \delta b$ (GS) [4, theorem 5].

Lemma 21. *Let $t \in A$, and assume that D is an open disk such that $\lambda - t \in \Phi_B$ for all $\lambda \in D$. Also, assume that there exist $\gamma \in \mathrm{res}_B(t) \cap (D^C)$ and $\eta \in \mathrm{res}_B(t) \cap D$. Then every point in D is a Riesz point of t.*

PROOF. Let $g(\lambda) = (\gamma - \lambda)^{-1}$. Set $\Omega = g(D)$. Then Ω is a connected set which by Theorem 10 is in $[F\sigma_B((\gamma - t)^{-1})]^C$. Also, $(\gamma - \eta)^{-1}$ is in both $\mathrm{res}_B((\gamma - t)^{-1})$ and Ω. It follows from [6, theorem R.2.7, p. 60] that Ω consists entirely of Riesz points of $(\gamma - t)^{-1}$. It is not difficult to see that this implies that every point in D is a Riesz point of t. ■

Theorem 22. (1) *Assume that $t \in \Phi_B^{\circ}$, and $t_k \to t$* (GS). *Then $t_k \in \Phi_B^{\circ}$ for all k sufficiently large.*

(2) *Assume that λ_0 is a Riesz point of t, and $t_k \to t$* (GS). *There exists $\varepsilon > 0$ such that whenever $|\lambda - \lambda_0| < \varepsilon$, then λ is a Riesz point of t_k for all k sufficiently large.*

PROOF. Assume that $t \in \Phi_B^{\circ}$, and $t_k \to t$ (GS). By Theorem 13 there exists $f \in F_B$ such that $t + \lambda f \in \mathrm{Inv}_B(A)$ whenever $\lambda \neq 0$. By (iii) above, we can choose $\delta \neq 0$ so that $t_k + \delta f \to t + \delta f$ (GS). Then, by (i), $t_k + \delta f \in \mathrm{Inv}_B(A)$ for all k sufficiently large. This proves (1).

Now assume that λ_0 is a Riesz point of t, and $t_k \to t$ (GS). We assume that λ_0 is in $\sigma_B(t)$. Thus, by definition, λ_0 is an isolated point of $\sigma_B(t)$ and $\lambda_0 - t \in \Phi_B$. By Theorem 19, $\lambda_0 - t \in \Phi_B^{\circ}$. Now (i) implies that $(\lambda_0 - t_k) \to (\lambda_0 - t)$ (GS). As argued in the proof of (1), we can choose $f \in F_B$ such that $\lambda_0 - t + f \in \mathrm{Inv}_B(A)$ and $(\lambda_0 - t_k + f) \to (\lambda_0 - t + f)$ (GS). Choose $\varepsilon > 0$ such that

$$D = \{\mu \in \mathbf{C} : |\mu - \lambda_0| \leq \varepsilon\} \subseteq \mathrm{res}_B(t - f).$$

By (ii), there exists N_1 such that, for all $k \geq N_1$ and for all $\mu \in D$, $(\mu - t_k + f)^{-1} \in B$. It follows that for $k \geq N_1$, $D \subseteq W\sigma_B(t_k)^C$. Now fix $\lambda_2 \in \mathrm{res}_B(t)$ with $0 < |\lambda_2 - \lambda_0| < \varepsilon$. By (i), there exists N_2 such that $(\lambda_2 - t_k)^{-1} \in B$ whenever $k \geq N_2$. Applying Lemma 21, we have that for all $k \geq N = \max(N_1, N_2)$ and for all μ such that $|\mu - \lambda_0| < \varepsilon$, μ is a Riesz point of t. ■

6. Examples

There are many examples of algebras in analysis to which the Fredholm theory developed here applies. We present briefly two examples, both of which are of some interest in operator theory.

Example I. Let H be a Hilbert space, and assume that $\{H_k\}$, $k \geq 1$, is a sequence of closed subspaces of H with $H_k \perp H_j$ when $k \neq j$, having the property that $H = \oplus \sum_{k=1}^{\infty} H_k$. Let A be the algebra of all sequences $T = \{T_k\}_{k \geq 1}$ where $T_k \in \boldsymbol{B}(H_k)$ for

$k \geq 1$. The algebraic operations in A are done entrywise (e.g. $\{T_k\}+\{S_k\} = \{T_k+S_k\}$). The algebra A has a natural complete LMC-topology defined by the semi-norms:

$$\text{for } T = \{T_k\}_{k\geq 1}, \|T\|_m = \max\{\|T_k\| : 1 \leq k \leq m\}.$$

Also, A has an involution: if $T = \{T_k\}_{k\geq 1}$, then $T^* = \{T_k^*\}_{k\geq 1}$. All the semi-norms have the C*-property, so A is an example of a generalised B*-algebra [8, 9], a class of algebras which has been widely studied.

Let $B = \{T = \{T_k\}_{k\geq 1} \in A$: the sequence of norms, $\{\|T_k\|\}$, is bounded$\}$; let $\|T\|_B = \sup\{\|T_k\| : 1 \leq k\}$. Then $(B, \|\cdot\|_B)$ is a C*- algebra. Set $K_B = \{\{T_k\}_{k\geq 1} \in B$: each T_k is a compact operator, and $\lim_{k\to 0} \|T_k\| = 0\}$. Finally, set $F_B = \{\{T_k\}_{k\geq 1} \in A$: each T_k has finite-dimensional range, and $T_k = 0$ for all but at most a finite number of indices $k\}$. In this case, F_B is a two-sided ideal of A, and since the closure of F_B in the B-norm is K_B, K_B/F_B is a radical algebra.

Every element of the algebra A can be represented in a natural way as a closed operator on the Hilbert space H. We define this closed operator next.

Definition 23. Assume that $T = \{T_k\}_{k\geq 1} \in A$. Define $\mathrm{dom}(\bar{T})$ to be the set of all $x \in H$, $x = \sum x_k$ ($x_k \in H_k$ for all k), such that $\sum_1^\infty \|T_k x_k\|^2$ is finite.

$$\text{For } x = \sum x_k \in \mathrm{dom}(\bar{T}), \text{ let } \bar{T}x = \sum_1^\infty T_k x_k.$$

It is straightforward to verify that $(\mathrm{dom}(\bar{T}), \bar{T})$ is a closed operator.

When $T = \{T_k\}_{k\geq 1} \in B$, so $\exists M > 0$ such that $\|T_k\| \leq M$ for all k, then for every $x = \sum x_k \in H$, $\sum_1^\infty \|T_k x_k\|^2 \leq \sum_1^\infty M\|x_k\|^2 = M\|x\|^2$. Therefore, in this case $\mathrm{dom}(\bar{T}) = H$, and $\bar{T} \in \boldsymbol{B}(H)$. In fact it is easily checked that for $T = \{T_k\}_{k\geq 1} \in A$:

$$T \in B \Leftrightarrow \bar{T} \in \boldsymbol{B}(H); T \in K_B \Leftrightarrow \bar{T} \text{ is a compact operator on } H;$$
$$T \in F_B \Leftrightarrow \bar{T} \text{ is a bounded operator on } H \text{ with finite-dimensional range.}$$

In the next result we summarise the basic spectral and Fredholm properties of elements T in the algebra A, and how these properties relate to $\bar{T}$. We do not include a proof as the verification of the properties is fairly routine.

Theorem 24. *Assume that $T = \{T_k\}_{k\geq 1} \in A$.*

(1) $T \in \mathrm{Inv}_B(A) \Leftrightarrow$ *each T_k is invertible in $\boldsymbol{B}(H_k)$ and $\{\|T_k^{-1}\|\}$ is a bounded sequence $\Leftrightarrow \bar{T}$ has an inverse in $\boldsymbol{B}(H)$.*

(2) $T \in \Phi_B \Leftrightarrow$ *each T_k is a Fredholm operator on H_k, and $\exists N$ such that T_k is invertible in $\boldsymbol{B}(H_k)$ for $k \geq N$, and $\{\|T_k^{-1}\|_{k\geq N}\}$ is a bounded sequence $\Leftrightarrow \bar{T}$ is a Fredholm operator on H.*

(3) $T \in \Phi_B^\circ \Leftrightarrow T \in \Phi_B$, *and each T_k has index zero on $H_k \Leftrightarrow \bar{T}$ is a Fredholm operator on H, and there exists a bounded finite-rank operator F on H with the properties that $F(H_k) \subseteq H_k$ for all k and $\bar{T} + F$ has an inverse in $\boldsymbol{B}(H)$.*

(4) [*From* (1)] *$\sigma_B(T)$ is the spectrum of the closed operator $\bar{T}$.*

(5) *If $T = T^*$, then $\bar{T}$ is a self-adjoint operator.*

We remark that algebras such as A in this example have been used in operator theory to study unbounded self-adjoint operators (although the algebra structure of A is only implicitly exploited). Let $(\mathrm{dom}\,(S), S)$ be an unbounded self-adjoint operator on H. Then there exists a decomposition of H as above, $H = \oplus \sum_{k=1}^{\infty} H_k$, such that for some T in the corresponding algebra A, $\bar{T} = S$. The construction of this decomposition of H can be found in [13, sections 118 and 120]. In section 120, for $T = T^* = \{T_k\} \in A$, the spectral resolution of the identity for $\bar{T}$ is constructed using the bounded case applied to the bounded self-adjoint operators T_k.

Example II. Let X and Y be Banach spaces which form a dual pair, so (by definition) there is a bounded non-degenerate bilinear form on $X \times Y$, which we denote by $\langle x, y\rangle$. Let A be the algebra $\boldsymbol{B}(X)$. Let B be the Jorgens algebra of the dual pair [1], that is, B is the algebra of all operators $T \in \boldsymbol{B}(X)$ which have an adjoint $T^t \in \boldsymbol{B}(Y)$ with respect to the dual pair:

$$\langle Tx, y\rangle = \langle x, T^t y\rangle \quad \text{for all } x \in X, y \in Y.$$

The Banach algebra norm on B is: $\|T\|_B = \max\{\|T\|, \|T^t\|\}$. Let $K_B = \{T \in B : T$ and T^t are compact operators$\}$. For $x \in X$, $y \in Y$, define the operator $y \otimes x$ on X by

$$(y \otimes x)(w) = \langle w, y\rangle x; \quad w \in X.$$

Then let $F_B = \mathrm{span}\,\{y \otimes x : \text{all } x \in X, y \in Y\}$. Note that F_B is a left ideal of $A = \boldsymbol{B}(X)$.

Let T be any operator in $A = \boldsymbol{B}(X)$. Define $\mathrm{dom}\,(T^t)$ to be the set of all $y \in Y$ for which there exists $z_y \in Y$ with the property that $\langle Tx, y\rangle = \langle x, z_y\rangle$ for all $x \in X$. Then set $T^t y = z_y$ when $y \in \mathrm{dom}\,(T^t)$. Thus, by definition,

$$\langle Tx, y\rangle = \langle x, T^t y\rangle \text{ for all } x \in X, y \in \mathrm{dom}\,(T^t).$$

It is straightforward to verify that $\mathrm{dom}\,(T^t)$ is a subspace of Y and that T^t is a closed operator.

The following result is a direct consequence of [2, theorem 14 and corollary 15]. We use the following notation:

$$\text{for } E \subseteq X, E^{\perp} = \{y \in Y : \langle x, y\rangle = 0 \text{ for all } x \in E\};$$

$$\text{for } F \subseteq Y, {}^{\perp}F = \{x \in X : \langle x, y\rangle = 0 \text{ for all } y \in F\}.$$

For T an operator, $\mathbf{N}(T)$ denotes the null space of T, and $\mathbf{R}(T)$ denotes the range of T.

Theorem 25. *Assume that $T \in \boldsymbol{B}(X)$ is affiliated with B.*

(1) $\lambda - T \in \Phi_B \Leftrightarrow \lambda - T$ *is Fredholm on X, $\lambda - T^t$ is Fredholm on Y (in the extended sense in* [2]*), and* $\mathrm{ind}(\lambda - T) = -\mathrm{ind}(\lambda - T^t)$.

(2) *When $\lambda - T \in \Phi_B$ then all of the following hold:*

$$\text{(a)}\quad \mathbf{N}(\lambda - T) = {}^{\perp}\mathbf{R}(\lambda - T^t); \quad \text{(b)}\quad \mathbf{N}(\lambda - T)^{\perp} = \mathbf{R}(\lambda - T^t);$$

(c) $\mathbf{N}(\lambda - T^t) = \mathbf{R}(\lambda - T)^{\perp}$; (d) ${}^{\perp}\mathbf{N}(\lambda - T^t) = \mathbf{R}(\lambda - T)$.

Now let Ω be a locally compact Hausdorff topological space, and assume that μ is a positive σ-finite regular Borel measure on Ω with the property that, for any non-empty open set U, $\mu(U) > 0$. Let X be the Banach space of all bounded continuous functions on Ω, and let $Y = L^1(\Omega, \mu) \cap X$. Y has a natural Banach space norm, and X and Y are a dual pair with bounded bilinear form given by $\langle f, g \rangle = \int_\Omega f(x)g(x)\, dx$ (here, and in what follows, we write dx for $d\mu(x)$). This dual pair is widely used in the study of linear integral operators on X; see [11], for example.

Let $K(x,t)$ be a kernel on $\Omega \times \Omega$ with the property that

$$\sup_{x\in\Omega} \int_\Omega |K(x,t)|\, dt \text{ is finite.}$$

For $f \in X$, let

$$T_K(f)(x) = \int_\Omega K(x,t)f(t)\, dt,$$

and assume that, for all $f \in X$, $T_K(f) \in X$. Then $T_K \in \boldsymbol{B}(X)$. Let T_K^t be the transpose of T_K with respect to the dual pair X and Y. Let S_K be the operator defined on functions $g \in Y$ by

$$S_K(g)(t) = \int_\Omega K(x,t)g(x)\, dx.$$

In general $S_K(g)$ need not be in Y; nevertheless, by Fubini's Theorem, $\langle T_K(f), g\rangle = \langle f, S_K(g)\rangle$ for all $f \in X$, $g \in Y$ (where $\langle , \rangle$ is defined just as above). It follows from this equality that for all $g \in \text{dom}(T_K^t)$, $T_K^t(g) = S_K(g)\ \mu -$ a.e.

We can now prove an interesting result concerning the integral operator T_K.

Theorem 26. *Assume all of the notation above. Assume that T_K is affiliated with B and that $\lambda - T_K \in \Phi_B$. Let $N = \{g \in Y : \int_\Omega K(x,t)g(x)\, dx = \lambda g(t)\ \mu - \text{a.e.}\}$. Then N is finite-dimensional, and $\mathbf{R}(\lambda - T_K) = {}^{\perp}N$.*

Proof. Assume that $g \in N$. For any $h \in X$, $\langle(\lambda - T_K)h, g\rangle = \langle h, \lambda g - S_K(g)\rangle = 0 = \langle h, 0\rangle$. Therefore, by definition, $g \in \text{dom}(T_K^t)$ and $(\lambda - T_K)^t g = 0$. Thus $g \in \mathbf{N}(\lambda - T_K^t)$. This proves that $N \subseteq \mathbf{N}(\lambda - T_K^t)$, and the proof of the opposite inclusion is similar. Then, since $N = \mathbf{N}(\lambda - T_K^t)$, we have by Theorem 25 that ${}^{\perp}N = \mathbf{R}(\lambda - T_K)$. ■

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