ITERATION OF TOTAL NEGATION IN CONSTRAINED ENVIRONMENTS

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Abstract

Reiterative application of Bankston's total negation operator 'anti-' upon an arbitrary topological invariant is known to lead rapidly to repetition in one of just seven patterns. The authors have recently shown that a great deal of the total negation procedure can be constrained to take place within a fixed class of topological spaces (the 'constraint' for the discussion) without impairing much of the theory. The present article explores iterative behaviour within a constraint. We show that, provided the constraint is hereditary, at most eight patterns of repetition are possible. An example reveals that in non-hereditary constraints the (unending) sequence of invariants generated may consist entirely of distinct terms, without ever entering a cycle of repetition.

1. Introduction

In [1] Paul Bankston demonstrated a method for producing a new topological property anti($\mathscr{P}$) that is, in a well-defined sense, the 'opposite' of a given property $\mathscr{P}$. This process is known as total negation and, frequently, the term 'anti-operator' is used to describe the transition from a property $\mathscr{P}$ to its total negation anti($\mathscr{P}$). It is known [2] that repeatedly applying this process to a given property $\mathscr{P}$ will result in the generated sequence of properties becoming repetitive in one of only seven 'iteration patterns', and that no more than four distinct properties can appear in this sequence.

We observe the following notational conventions throughout this article. We shall use script capitals such as $\mathscr{C}$, $\mathscr{P}$, $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ to represent topological properties and, specifically, we employ $\mathscr{U}$ to denote the universal property satisfied by all topological spaces. We shall, in any given problem, use $\mathscr{C}$ to represent the property known as the *constraint* for the context, which is further discussed below, and we may assume that the property $\mathscr{P}$ is always contained in $\mathscr{C}$ (for, were it not so, we could simply replace $\mathscr{P}$ with $\mathscr{C} \cap \mathscr{P}$ before proceeding). Italic capitals such as X, Y and Z will represent individual topological spaces.

The following definition is derived from [1] but presented as in [2]. The three

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succeeding results were established by Matier and McMaster in [2] and, in particular, Theorem 4 is referred to as the 'classical iteration theorem'.

Definition 1. For a given topological property $\mathscr{P}$ we define
spec($\mathscr{P}$) = {λ: λ is a non-zero cardinal and all spaces on λ-many points are $\mathscr{P}$},
ind($\mathscr{P}$) = {λ: λ is a non-zero cardinal and there is a $\mathscr{P}$ space and a not-$\mathscr{P}$ space on λ-many points},
proh($\mathscr{P}$) = {λ: λ is a non-zero cardinal and there is no $\mathscr{P}$ space on λ-many points},
anti($\mathscr{P}$) = {X: X has no $\mathscr{P}$ subspaces, excepting those subspaces Y for which the cardinality $|Y| \in spec(\mathscr{P})$}.
We may further define $anti^0(\mathscr{P}) = \mathscr{P}$, $anti^1(\mathscr{P}) = anti(\mathscr{P})$ and $anti^{n+1}(\mathscr{P}) = anti(anti^n(\mathscr{P}))$ for $n \geq 1$.

Lemma 2. *For a given topological property $\mathscr{P}$ the sequence*

$\mathscr{P}$, anti($\mathscr{P}$), anti^2($\mathscr{P}$),...

hereafter called the Bankston iteration sequence, contains no new terms after the fourth term. More precisely, either $\text{anti}^n(\mathscr{P}) = \mathscr{U}$ *for all* $n \geq 3$, *or else both* $\text{anti}^{2n}(\mathscr{P}) = \text{anti}^2(\mathscr{P})$ *and* $\text{anti}^{2n+1}(\mathscr{P}) = \text{anti}^3(\mathscr{P})$ *for all* $n \geq 1$.

Lemma 3. *For a given topological property $\mathscr{P}$:*
(i) $\mathscr{P} = \text{anti}(\mathscr{P}) \Rightarrow \mathscr{P} = \mathscr{U}$,
(ii) $\mathscr{P} = \text{anti}^3(\mathscr{P}) \Rightarrow \mathscr{P} = \mathscr{U}$.

Theorem 4. *The pattern of the Bankston iteration sequence beginning with an arbitrary topological property $\mathscr{P}$ is one of the following:*

(1)	$(\mathscr{U})'$	(5)	$(\mathscr{P}, \mathscr{Q})'$
(2)	$\mathscr{P}, (\mathscr{U})'$	(6)	$\mathscr{P}, (\mathscr{Q}, \mathscr{R})'$
(3)	$\mathscr{P}, \mathscr{Q}, (\mathscr{U})'$	(7)	$\mathscr{P}, \mathscr{Q}, (\mathscr{R}, \mathscr{S})'$
(4)	$\mathscr{P}, \mathscr{Q}, \mathscr{R}, (\mathscr{U})'$		

where we use $(*)'$ as a short form for an infinite repetition of the sequence segment $*$ inside parentheses. Furthermore, (4) *and* (7) *cannot occur when $\mathscr{P}$ is hereditary.*

A question frequently posed has been what effect the decision to work exclusively within a given separation axiom might have on the process of total negation. To investigate this, we consider restricting ourselves to work inside a collection of spaces $\mathscr{C}$, rather than $\mathscr{U}$. The natural restructuring of the classical definitions then assumes the following form.

Definition 5. Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties; then
$\mathscr{C}$-spec($\mathscr{P}$) = {λ: $\lambda \notin$ proh($\mathscr{C}$) and all the $\mathscr{C}$ spaces on λ-many points are $\mathscr{P}$},
$\mathscr{C}$-ind($\mathscr{P}$) = {λ: $\lambda \notin$ proh($\mathscr{C}$) and there is a $\mathscr{C}$ and $\mathscr{P}$ space, and a $\mathscr{C}$ and not-$\mathscr{P}$ space on λ-many points},
$\mathscr{C}$-proh($\mathscr{P}$) = {λ: $\lambda \notin$ proh($\mathscr{C}$) and no $\mathscr{C}$ space on λ-many points is $\mathscr{P}$},

$\mathscr{C}$-anti($\mathscr{P}$) $= \{X: X$ is $\mathscr{C}$, and has no $\mathscr{C}$ and $\mathscr{P}$ subspaces, except for those subspaces Y for which $|Y| \in \mathscr{C}$-spec($\mathscr{P}$)$\}$.
It is easily seen that these definitions collapse to their classical counterparts (in Definition 1) when $\mathscr{C}$ is taken to be $\mathscr{U}$.

It is sometimes useful to characterise the *non-*$\mathscr{C}$-anti($\mathscr{P}$) spaces using the following lemma, which follows directly from the definitions.

Lemma 6. *Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties; then the non-*$\mathscr{C}$-anti($\mathscr{P}$) *spaces are those which are not $\mathscr{C}$, or which contain a $\mathscr{C}$ and $\mathscr{P}$ subspace Y such that $|Y| \in$* $\mathscr{C}$-ind($\mathscr{P}$).

We shall further extend the notion of a hereditary property.

Definition 7. Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties; then we shall call $\mathscr{P}$ *$\mathscr{C}$-hereditary* if and only if, when X is a $\mathscr{P}$ (and $\mathscr{C}$) space, then every $\mathscr{C}$ subspace of X is $\mathscr{P}$.

The following can easily be shown.

Proposition 8. *Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties. Then*
 (i) $\mathscr{C}$-anti($\mathscr{P}$) *is $\mathscr{C}$-hereditary* (*but need not be hereditary*);
 (ii) *if $\mathscr{C}$ is hereditary then* $\mathscr{C}$-anti($\mathscr{P}$) *is hereditary.*

We also use the following lemma, which is an extension of a result found in classical total negation theory [1].

Lemma 9. *Let $\mathscr{C}$ be a hereditary property and let $\mathscr{P}$ be a $\mathscr{C}$-hereditary property. Then* $\mathscr{C} \cap \mathscr{P} \Rightarrow \mathscr{C}$-anti^2($\mathscr{P}$).

Proof. Suppose that X is $\mathscr{C}$ and $\mathscr{P}$ but not $\mathscr{C}$-anti^2($\mathscr{P}$). Then by Lemma 6 there exists a subspace Y of X which is $\mathscr{C}$-anti($\mathscr{P}$), and such that $|Y| \in \mathscr{C}$-ind($\mathscr{C}$-anti($\mathscr{P}$)). Therefore there exists a $\mathscr{C}$ space Z such that $|Y| = |Z|$ but Z is not $\mathscr{C}$-anti($\mathscr{P}$). Hence, in turn, there exists a subspace W of Z such that W is $\mathscr{C}$ and $\mathscr{P}$ but $|W| \in \mathscr{C}$-ind($\mathscr{P}$). We can select a $\mathscr{C}$ subspace $\mathscr{U}$ of X (and of Y) such that $|U| = |W|$.

The space U is $\mathscr{C}$ and $\mathscr{P}$ and $\mathscr{C}$-anti($\mathscr{P}$). Therefore $|U| \in \mathscr{C}$-spec($\mathscr{P}$). This contradicts our choice of W and hence of X. ■

We make a final observation before embarking on the iteration pathway.

Lemma 10. *Let $\mathscr{C}$ be a hereditary property and let $\mathscr{P}$ be a $\mathscr{C}$-hereditary property. Then* $\mathscr{C}$-proh($\mathscr{P}$) *is an increasing subclass of* (spec($\mathscr{C}$) $\cup$ ind($\mathscr{C}$)).

We now establish a collection of lemmas which form the backbone of the proof of the *constrained* iteration theorem.

Lemma 11. *Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties. Then*
(i) $\mathscr{C}$-anti($\mathscr{P}$) $= \mathscr{C} \Leftrightarrow \mathscr{C}$-ind($\mathscr{P}$) $= \emptyset$,
(ii) $\mathscr{C}$-anti($\mathscr{C}$) $= \mathscr{C}$.

Proof. (i) If $\mathscr{C}$-ind($\mathscr{P}$) $= \emptyset$, Lemma 6 shows that all $\mathscr{C}$ spaces will be $\mathscr{C}$-anti($\mathscr{P}$). Conversely, if we can select $\lambda \in \mathscr{C}$-ind($\mathscr{P}$), we may choose a $\mathscr{C}$ and $\mathscr{P}$ space X on λ-many points. As X is a $\mathscr{C}$ and $\mathscr{P}$ subspace of itself with $|X| \in \mathscr{C}$-ind($\mathscr{P}$), Lemma 6 implies that X is not $\mathscr{C}$-anti($\mathscr{P}$).

(ii) Clearly $\mathscr{C}$-ind($\mathscr{C}$) $= \emptyset$, whence the result follows from (i). ■

Lemma 12. *Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties such that $\mathscr{C} \cap \mathscr{P} \neq \emptyset$. Then*

$$\mathscr{C}\text{-anti}(\mathscr{P}) = \mathscr{C} \cap \mathscr{P} \Leftrightarrow \mathscr{C} \cap \mathscr{P} = \mathscr{C}.$$

Proof. [$\Leftarrow$] See the preceding result.

[$\Rightarrow$] If the first equality holds but not the second, then $\mathscr{C}$-anti($\mathscr{P}$) $\neq \mathscr{C}$. From Lemma 6 we can pick a cardinal $\lambda \in \mathscr{C}$-ind($\mathscr{P}$). We can therefore select a $\mathscr{C}$ and $\mathscr{P}$ space X on λ-many points. Now X is also $\mathscr{C}$-anti($\mathscr{P}$) and so $\lambda \in \mathscr{C}$-spec($\mathscr{P}$), contradicting our choice of λ. ■

Lemma 13. *Let $\mathscr{C}$ and $\mathscr{P}$ be topological properties; then*

$$\mathscr{C}\text{-spec}(\mathscr{C}\text{-anti}(\mathscr{P})) \cap \mathscr{C}\text{-ind}(\mathscr{P}) = \emptyset.$$

Proof. Supposing not, we choose $\lambda \in \mathscr{C}$-spec($\mathscr{C}$-anti($\mathscr{P}$)) $\cap\, \mathscr{C}$-ind($\mathscr{P}$). Therefore we can choose a $\mathscr{C}$ and $\mathscr{P}$ space on λ-many points which is also $\mathscr{C}$-anti($\mathscr{P}$), forcing λ to belong to $\mathscr{C}$-spec($\mathscr{P}$), a contradiction. ■

Lemma 14. *Let $\mathscr{C}$ be a hereditary property, and let $\mathscr{P}$ be a $\mathscr{C}$-hereditary property. Then $\mathscr{C}$-anti^2($\mathscr{P}$) $= \mathscr{C}$-anti^4($\mathscr{P}$).*

Proof. From Lemma 9 we know that $\mathscr{C}$-anti^2($\mathscr{P}$) $\Rightarrow \mathscr{C}$-anti^4($\mathscr{P}$) as $\mathscr{C}$ is hereditary. We suppose that there is a $\mathscr{C}$ space X which is $\mathscr{C}$-anti^4($\mathscr{P}$) but which is not $\mathscr{C}$-anti^2($\mathscr{P}$). It follows that X has a $\mathscr{C}$ subspace Y which is $\mathscr{C}$-anti($\mathscr{P}$) but such that $|Y| \in \mathscr{C}$-ind($\mathscr{C}$-anti($\mathscr{P}$)). Therefore we can select a $\mathscr{C}$ space Z such that $|Y| = |Z|$ and Z is not $\mathscr{C}$-anti($\mathscr{P}$), and Z consequently has a $\mathscr{C}$ and $\mathscr{P}$ subspace W such that $|W| \in \mathscr{C}$-ind($\mathscr{P}$). We also pick a $\mathscr{C}$ subspace U of Y such that $|U| = |W|$.

The space U must be $\mathscr{C}$-anti($\mathscr{P}$) as it is a subspace of Y, while the space W cannot be $\mathscr{C}$-anti($\mathscr{P}$) as W is a $\mathscr{C}$ and $\mathscr{P}$ subspace of itself even though $|W| \notin \mathscr{C}$-spec($\mathscr{P}$). We must conclude that $|W| = |U| \in \mathscr{C}$-ind($\mathscr{C}$-anti($\mathscr{P}$)), and so $|W| = |U| \notin \mathscr{C}$-spec($\mathscr{C}$-anti^2($\mathscr{P}$)) as otherwise Lemma 13 would be contradicted. Additionally, U is $\mathscr{C}$-anti^3($\mathscr{P}$) and $\mathscr{C}$-anti^4($\mathscr{P}$), because U is $\mathscr{C}$-anti($\mathscr{P}$) (using Lemma 9) and U is a $\mathscr{C}$ subspace of the $\mathscr{C}$-anti^4($\mathscr{P}$) space X. Therefore $|W| = |U| \in \mathscr{C}$-spec($\mathscr{C}$-anti^3($\mathscr{P}$)) and so by appealing to Lemma 13 again we find that $|U| \notin \mathscr{C}$-ind($\mathscr{C}$-anti^2($\mathscr{P}$)). Therefore $|U| \in \mathscr{C}$-proh($\mathscr{C}$-anti^2($\mathscr{P}$)).

However, W is a $\mathscr{C}$ and $\mathscr{P}$ space and, as $\mathscr{C}$ is hereditary (and via Lemma 9), a $\mathscr{C}$-anti($\mathscr{P}$) space, implying that $|W| = |U| \notin \mathscr{C}$-proh($\mathscr{C}$-anti$^2\mathscr{P}$)), contradicting our choice of W and hence of X. ■

This leads to the following corollary.

Corollary 15. *Let $\mathscr{C}$ be a hereditary topological property, and let $\mathscr{P}$ be a topological invariant. Then* $\mathscr{C}\text{-anti}^3(\mathscr{P}) = \mathscr{C}\text{-anti}^5(\mathscr{P})$.

Lemma 16. *Let $\mathscr{C}$ be a hereditary property, and let $\mathscr{P}$ be a $\mathscr{C}$-hereditary property. Then either* $\mathscr{C}\text{-anti}^2(\mathscr{P}) = \mathscr{C}$, *or* $\mathscr{C}\text{-anti}(\mathscr{P}) = \mathscr{C}\text{-anti}^3(\mathscr{P})$.

Proof. Suppose that both $\mathscr{C}\text{-anti}^2(\mathscr{P}) \neq \mathscr{C}$ and that $\mathscr{C}\text{-anti}(P) \neq \mathscr{C}\text{-anti}^3(\mathscr{P})$. From Lemma 9 we may then select a $\mathscr{C}$ space X which is $\mathscr{C}\text{-anti}^3(\mathscr{P})$ but not $\mathscr{C}\text{-anti}(\mathscr{P})$. Thus we can select a subspace Y of X such that Y is both $\mathscr{C}$ and $\mathscr{P}$ but $\lambda = |Y| \in \mathscr{C}\text{-ind}(\mathscr{P})$. Therefore Y is $\mathscr{C}\text{-anti}^2(\mathscr{P})$ by Lemma 9, and is $\mathscr{C}\text{-anti}^3(\mathscr{P})$ because $\mathscr{C}\text{-anti}^3(\mathscr{P})$ is $\mathscr{C}$-hereditary. It follows that $\lambda \in \mathscr{C}\text{-spec}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$. By Lemma 13, $\lambda \notin \mathscr{C}\text{-spec}(\mathscr{C}\text{-anti}(\mathscr{P}))$, and $\lambda \notin \mathscr{C}\text{-ind}(\mathscr{C}\text{-anti}(\mathscr{P}))$. Therefore $\lambda \in \mathscr{C}\text{-proh}(\mathscr{C}\text{-anti}(\mathscr{P}))$.

As $\mathscr{C}\text{-anti}^2(\mathscr{P}) \neq \mathscr{C}$ we can select $\mu \in \mathscr{C}\text{-ind}(\mathscr{C}\text{-anti}(\mathscr{P}))$. By Lemma 10, $\mu < \lambda$ as $\lambda \in \mathscr{C}\text{-proh}(\mathscr{C}\text{-anti}(\mathscr{P}))$ which is an increasing subclass of $\text{spec}(\mathscr{C}) \cup \text{ind}(\mathscr{C})$. As $\mu \in \mathscr{C}\text{-ind}(\mathscr{C}\text{-anti}(\mathscr{P}))$, $\mu \notin \mathscr{C}\text{-spec}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$ by Lemma 13. Also $\mu \notin \mathscr{C}\text{-proh}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$ as Y is $\mathscr{C}\text{-anti}^2(\mathscr{P})$ and $\mu < \lambda = |Y|$ and clearly $\lambda \notin \mathscr{C}\text{-proh}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$. Therefore $\mu \in \mathscr{C}\text{-ind}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$.

Choose a $\mathscr{C}$ subspace Z of Y such that $|Z| = \mu$. Then Z is a $\mathscr{C}$ and $\mathscr{P}$ space as $\mathscr{P}$ is $\mathscr{C}$-hereditary, and so Z is $\mathscr{C}\text{-anti}^2(\mathscr{P})$ by Lemma 9. Therefore Z cannot be $\mathscr{C}\text{-anti}^3(\mathscr{P})$ as it is a $\mathscr{C}\text{-anti}^2(\mathscr{P})$ subspace of itself and $|Z| = \mu \notin \mathscr{C}\text{-spec}(\mathscr{C}\text{-anti}^2(\mathscr{P}))$. Now Z is a $\mathscr{C}$ subspace of X and thus is $\mathscr{C}\text{-anti}^3(\mathscr{P})$, contradicting our choice of Z and thus of X. ■

Theorem 17. *Let $\mathscr{C}$ be a hereditary topological property, and let $\mathscr{P}$ be a topological property. Then in the constrained Bankston iteration sequence, no new terms appear after the fourth term. Additionally, either*

$\mathscr{C}\text{-anti}^n(\mathscr{P}) = \mathscr{C}$ *for all* $n \geq 3$; *or*

$\mathscr{C}\text{-anti}^{2n}(\mathscr{P}) = \mathscr{C}\text{-anti}^2(\mathscr{P})$ *and* $\mathscr{C}\text{-anti}^{2n+1}(\mathscr{P}) = \mathscr{C}\text{-anti}^3(\mathscr{P})$, *for all* $n \geq 1$.

Further, the iteration pattern will be one of the following:

(1)	$(\mathscr{C})'$	(5)	$(\mathscr{P}, \mathscr{Q})'$
(2)	$\mathscr{P}, (\mathscr{C})'$	(6)	$\mathscr{P}, (\mathscr{Q}, \mathscr{R})'$
(3)	$\mathscr{P}, \mathscr{Q}, (\mathscr{C})'$	(7)	$\mathscr{P}, \mathscr{Q}, (\mathscr{R}, \mathscr{S})'$
(4)	$\mathscr{P}, \mathscr{Q}, \mathscr{R}, (\mathscr{C})'$		

Further, sequences (4) *and* (7) *cannot occur when $\mathscr{P}$ is $\mathscr{C}$-hereditary. Note that the properties $\mathscr{Q}$, $\mathscr{R}$ and $\mathscr{S}$ are all subfamilies of the constraint family of spaces $\mathscr{C}$.*

Proof. This follows easily from the lemmas proved previously. ■

It is apparent that this result is based upon the hereditary character of the constraint $\mathscr{C}$, but it is (as yet) unclear whether we can relax this requirement without also losing the result.

We shall now exhibit a certain non-hereditary constraint $\mathscr{C}$ from the general family of topological spaces $\mathscr{U}$ whose ordering under embeddability conforms to a qoset proposed by Matthews and McMaster in [3]. This counterexample shows that not only does the theorem above fail to hold without the hypothesis of hereditariness of the constraint, but that we can in fact form a never-repeating chain of 'negated' properties.

Example 18. For each positive integer n let us use $D(n)$ and $T(n)$ to denote the discrete and trivial topological space respectively on $\aleph_n$-many points. Further, we shall use the notation $A \oplus B$ to denote the topological direct sum of the spaces A and B. We now define a family of topological spaces X_n and Y_n for $n \geq 3$ as follows:

$X_n = D(n) \oplus T(n)$;
$Y_n = D(n) \oplus T(n-2)$ if n is odd;
$Y_n = D(n-2) \oplus T(n)$ if n is even.

Let $\mathscr{P}$ be the topological invariant comprising simply Y_3. Then it can easily be shown that $\mathscr{C}\text{-anti}^2(\mathscr{C} \cap \mathscr{P}) = \{X_3, Y_3, Y_5\}$, and that, in general, $\mathscr{C}\text{-anti}^n(\mathscr{C} \cap \mathscr{P}) = \{Y_{n+3}\} \cup \{X_m, Y_m : 3 \leq m \leq n+1\}$. Clearly, the Bankston iteration sequence here will never repeat itself, and the collapse of the iteration theorem is attributable to the loss of hereditariness in $\mathscr{C}$.

Example 19. It is noted in [2] that *connectedness* exhibits the Bankston iteration sequence $\mathscr{P}, \mathscr{Q}, (\mathscr{R}, \mathscr{S})'$ in the context of all topological spaces. In fact, $\mathscr{Q} =$ anti(compactness) $=$ totally disconnected, and thus the Bankston iteration sequence for $\mathscr{P} =$ totally disconnected is $\mathscr{P}, (\mathscr{Q}, \mathscr{R})'$. Now consider $\mathscr{P} =$ totally disconnected in the constraint $\mathscr{C} = T_2$. Observe first that T_2-spec$(\mathscr{P}) = (1, \aleph_0)$ and that T_2-ind$(\mathscr{P}) = [\aleph_0, \infty)$. Note that the natural numbers with the discrete topology form a totally disconnected space which is embedded in every infinite T_2 space. Thus every infinite T_2 space contains an 'indecisively' totally disconnected space, whereas every finite T_2 space only contains totally disconnected spaces of 'spectral' size. It follows that T_2-anti(totally disconnected) $= \{$finite T_2 spaces$\}$. From Lemma 11 above, the iteration pattern in T_2 is therefore

$\mathscr{C} \cap \mathscr{P}, \mathscr{Q}, \mathscr{C}, \ldots$.

Clearly, the imposition of this constraint, even though it is a hereditary one, has completely altered the Bankston iteration sequence of the property under scrutiny.

It may be noted that other properties are 'pre-antis' for totally disconnected. For example, anti(not totally disconnected) $=$ totally disconnected. It can be shown that if we take $\mathscr{P}$ to be 'not totally disconnected' and $\mathscr{C} = T_2$ then the absolute Bankston iteration sequence is

$\mathscr{P}, \mathscr{Q}, (\mathscr{R}, \mathscr{S})'$

and that the constrained sequence is

$\mathscr{C} \cap \mathscr{P}, \mathscr{Q}, \mathscr{R}, (\mathscr{C})'$.

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