LOCALISATION TECHNIQUES FOR DIVISION IN DOUGLAS ALGEBRAS

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ABSTRACT

In this paper we use uniform algebra techniques and recent results on separation properties to prove several division theorems for closed subalgebras of L^∞ containing H^∞. We also study ideals having the local approximation property and we show that not every ideal in such algebras is local.

1. Introduction

Let X be a compact Hausdorff space and let A be a uniform algebra on X, that is, a uniformly closed subalgebra of $C(X)$ separating the points and containing the constants. As usual, M_A denotes the maximal ideal space of A. For any closed subset E of M_A, a function $f \in C(M_A)$ is said to belong locally to A on E if for every $x \in E$ there exists a neighbourhood U of x in E such that $f|U$ belongs to $A|U$. The algebra A is said to be a *local algebra on* E if every function $f \in C(M_A)$ belonging locally to A on E actually belongs to A. Similarly, an ideal $I \subseteq A$ is called *local* when it contains all the functions $f \in A$ that belong locally to I on M_A.

Shilov [19] claimed that every function algebra is local on its maximal ideal space. Later, an error was discovered in this proof and it became a conjecture [1] that every function algebra was local on its maximal ideal space. In 1968, Eva Kallin [16] gave a counterexample to that conjecture. Her counterexample also exhibited two functions f and g in A such that f vanishes on an open subset containing the zeros of g, but f is not divisible by g in A; in other words, $f \notin gA$. Work in the area continued, with some interesting theorems and questions presented by S. Sidney [20]. More recent work can be found in [9].

In general, one expects an algebra to be local on its maximal ideal space. It is often very difficult to show that an algebra of functions is not a local algebra. Given

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the importance of this work to questions about division and factorisation, it makes sense to look at *local ideals* and, in particular, principal local ideals.

Now we look at a closely related idea. Let $I \subseteq A$ be an ideal, and $f \in C(M_A)$. A function f is said to be *I-holomorphic* at some point $x \in M_A$ if there is a neighbourhood V of x such that f can be uniformly approximated on V by functions in I. When this happens for every $x \in E \subseteq M_A$ we say that f *is I-holomorphic on E*, or simply I-holomorphic if $E = M_A$. In particular, when $A = I$ we have the notion of A-holomorphic given in [6]. An ideal I is said to have the *local approximation property* with respect to $C(M_A)$ or A if every I-holomorphic function $f \in C(M_A)$ (respectively $f \in A$) is in I. Clearly, when the algebra A itself has the local approximation property (with respect to $C(M_A)$) then both localisation notions agree for ideals of A.

In this paper, we study local approximation properties for closed subalgebras of $L^\infty = L^\infty(\partial D)$, the algebra of essentially bounded Lebesgue measurable functions on the unit circle. The most interesting subalgebras include the disk algebra and the algebra of bounded analytic functions H^∞. However, the maximal ideal spaces of both these algebras contain the open unit disk D as a dense subset. This fact, combined with the fact that the functions involved are holomorphic, means that the study of local properties of the algebra is not particularly interesting, yet the local properties of ideals are quite interesting. This is, in part, due to the fact that closed ideals in H^∞ have not been completely described.

In order to study the closed ideals in H^∞, it turns out to be helpful to study the local properties of certain related algebras. When one studies Douglas algebras—that is, closed subalgebras of L^∞ properly containing H^∞—the local nature of the algebra is no longer clear. In this paper we show that all closed subalgebras of L^∞ containing H^∞ have the local approximation property and we show the implications of this result for division and factorisation in the algebra. Our theorems yield a unified approach to the study of division in Douglas algebras and we are able to obtain all division theorems known thus far. For many, we are able to obtain a stronger formulation than the original. We complete this paper by describing a wide class of principal ideals having the local approximation property.

2. Preliminaries

In this section we include most of the preliminaries necessary to read this paper. The reader is referred to [7, chapter IX] for more detailed information on this subject.

Let A be a uniform algebra and $f \in C(M_A)$. The Shilov boundary of A is denoted by ∂A and $[A, f]$ denotes the closed subalgebra of $C(M_A)$ generated by A and f. As usual, the zero set of a function $f \in C(M_A)$ is denoted by $Z(f) = \{x \in M_A : f(x) = 0\}$.

We will be most concerned with Douglas algebras. These algebras have been the subject of much study in recent years. The most important theorem in this area is the Chang–Marshall theorem [5], [17], which tells us that Douglas algebras are determined by their maximal ideal spaces. In addition, it is a consequence of this theorem that every closed subalgebra containing H^∞ is generated as a uniform subalgebra of L^∞ by H^∞ and the conjugates of Blaschke products b such that $\overline{b}$ is in the algebra.

Let x and y denote two points in the maximal ideal space M_{H^∞}. The pseudo-hyperbolic distance between x and y is defined to be $\rho(x, y)$ where

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty, ||f||_\infty \le 1, f(x) = 0\}.$$

The relation $x \sim y$ if and only if $\rho(x, y) < 1$ is an equivalence relation, and the equivalence class of a point x is denoted by $P(x)$ and is called the Gleason part of x. The equivalence class of a point in the disk is the unit disk itself. For points outside the disk such that the Gleason part is non-trivial (consists of more than one point), there is a bijective map L_x mapping the disk onto the Gleason part $P(x)$ such that $f \circ L_x$ is analytic for every $f \in H^\infty$ and $L_x(0) = x$. Since $f \circ L_x$ is an analytic function, if $f(x) = 0$, we will (as usual) define the order $\operatorname{ord}(f, x)$ of the zero of f at x to be the order of the analytic function $f \circ L_x$ at zero. If the point x has a trivial Gleason part and $f(x) = 0$, then the order of the zero of f at x is infinite. The same can be said about a function f in a Douglas algebra A when $x \in M_A$. The set of all points x in the maximal ideal space of the algebra under study such that f has a zero of infinite order at x is denoted $Z_\infty(f)$. The interior of the zero set of a function f in an algebra A is denoted by $Z(f)^\circ$. The interior is always taken within M_A.

Each point x in M_{H^∞} has a unique representing measure on M_{L^∞}; that is, there is a unique probability measure μ_x supported on M_{L^∞} such that $f(x) = \int_{M_{L^\infty}} f \, d\mu_x$ for every $f \in H^\infty$. Since the Shilov boundary for every Douglas algebra A is M_{L^∞}, each point $x \in M_A$ also has such a representing measure. The support sets of these measures will be denoted by $\operatorname{supp} \mu_x$.

3. Local properties

We begin this paper with the following result of Rickart ([18] or [6, 93]).

Theorem 3.1. *Let A be a uniform algebra and suppose that $f \in C(M_A)$ is A-holomorphic on $M_A \setminus Z(f)$; then*

$$\partial[A, f] = \partial A \quad \textit{and} \quad M_{[A,f]} = M_A. \tag{1}$$

We obtain several localisation results involving Douglas algebras. Before we begin we show that if u is an inner function and $\bar{u}$ is locally in the Douglas algebra A on the Shilov boundary, then $\bar{u} \in A$. To see this, suppose that $x \in M_A$. Choose a point $y \in \operatorname{supp}\mu_x$. By assumption, there is an open set U of the Shilov boundary M_{L^∞} containing y such that $\bar{u}|U = f|U$ for some $f \in A$. Then multiplying by u we see that $1 - uf = 0$ on U. By [12, 190], this means that $(1 - uf)(x) = 0$. In particular, then, $u(x)f(x) = (uf)(x) = 1$ so $u(x) \neq 0$. Therefore u is invertible in A and consequently $\bar{u} \in A$.

Of course this is not true for functions in L^∞ in general. For example, any characteristic function in L^∞ is locally in $H^\infty + C$ on the Shilov boundary, but $H^\infty + C$ does not contain any characteristic functions (see [12, 188]). Nevertheless, we do have the following corollary of Theorem 3.1 which shows that a Douglas algebra A is local on its maximal ideal space.

Corollary 3.2. *If A is a Douglas algebra and $f \in C(M_A)$ is A-holomorphic on $M_A \setminus Z(f)$, then $f \in A$.*

Proof. Consider the subalgebra $[A, f]$ of $C(M_A)$. By Theorem 3.1 we know that $M_{[A,f]} = M_A$ and $\partial[A, f] = \partial A = M_{L^\infty}$. Since M_{L^∞} is the Shilov boundary of the algebra $[A, f]$, we see that $[A, f]$ is a closed subalgebra of L^∞. Since $H^\infty \subset A$, we see that $[A, f]$ is a Douglas algebra. Now, $M_{[A,f]} = M_A$, so the Chang–Marshall theorem tells us that $[A, f] = A$. ■

The above remarks make the next well-known theorem (see [6, 94]) particularly significant when A is a Douglas algebra.

Theorem 3.3. *Let A be a uniform algebra, $f \in C(M_A)$ and $g_j \in A$ $(0 \leq j \leq n-1)$ such that on M_A the following equation holds:*

$$f^n + g_{n-1}f^{n-1} + \cdots + g_1 f + g_0 = 0. \tag{2}$$

Then $\partial[A, f] = \partial A$ and $M_{[A,f]} = M_A$.

When A is a Douglas algebra and f is a function in $C(M_A)$ satisfying (2), we use the same argument as in Corollary 3.2 to conclude that $f \in A$. We isolate this result for future reference.

Corollary 3.4. *If A is a Douglas algebra and $f \in C(M_A)$ is such that there exist functions $g_j \in A$ with*

$$f^n + g_{n-1}f^{n-1} + \cdots + g_1 f + g_0 = 0,$$

then $f \in A$.

We now wish to remove the monic hypothesis in the equation above. The proof is necessarily somewhat different from well-known proofs (see [6]) and will require some definitions and elementary lemmas.

Let A be a uniform algebra and $E \subseteq M_A$ be a closed set. Consider the uniform algebra A_E defined as the closure in $C(E)$ of $A|_E$ and the uniform algebra $R_A(E)$, which is the closure of $\{f|_E / g|_E : f, g \in A, g \text{ is zero-free on } E\}$ in $C(E)$. In [21, 359, 369–71] it is proved that

$$M_{A_E} = \hat{E} \overset{\text{def}}{=} \{x \in M_A : |f(x)| \leq \sup_E |f| \text{ for all } f \in A\}$$

and

$$M_{R_A(E)} = \tilde{E} \overset{\text{def}}{=} \{x \in M_A : f(x) \in f(E) \text{ for all } f \in A\}.$$

The sets $\hat{E}$ and $\tilde{E}$ are called the A-convex and the A-rational convex hull of E respectively. Consequently, E is said to be A-convex (or A-rationally convex) if $E = \hat{E}$ (resp. $E = \tilde{E}$).

Lemma 3.5. *Let A be a uniform algebra and $x \in M_A$. The set of closed A-convex neighbourhoods of x is a base of neighbourhoods.*

PROOF. Let $U \subset M_A$ be an open neighbourhood of x. Then for each $y \in M_A \setminus U$ there exists $f_y \in A$ such that $f_y(x) = 0$ and $f_y(y) \neq 0$. By a compactness argument there exist $f_1, \ldots, f_n \in A$ such that $f_1(x) = \cdots = f_n(x) = 0$ and

$$\inf_{y \in M_A \setminus U} \max\{|f_1(y)|, \ldots, |f_n(y)|\} = \alpha > 0.$$

Hence, the set $\bigcap_{1 \le j \le n}\{|f_j| \le \alpha/2\}$ is a closed A-convex neighbourhood of x contained in U. ∎

A uniform algebra A is called separating if $\tilde{E} = E$ for every closed set $E \subset M_A$. In [22] it is proved that H^∞ is a separating algebra, and therefore so is every Douglas algebra.

Lemma 3.6. *Let A be a (uniform) separating algebra and $E \subseteq M_A$ be a closed set. Then $R_A(E) = A_E$ if and only if E is A-convex.*

PROOF. Suppose first that $R_A(E) = A_E$. Since A is separating, we see that $M_{R_A(E)} = E$. Thus E is A-convex. Now suppose that E is A-convex and let $a, b \in A$ with b zero-free on E. Since functions of the form a/b are dense in $R_A(E)$, it will be enough to prove that $a/b \in A_E$. Clearly, we need only show that $1/b \in A_E$. But $b \in A_E$ does not vanish on $E = M_{A_E}$, so b is invertible in A_E. ∎

Theorem 3.7. *Let A be a Douglas algebra and $f \in C(M_A)$. Suppose that there are a positive integer n and $g_j \in A$, with $0 \le j \le n$, such that on M_A*

$$g_n f^n + \cdots + g_1 f + g_0 = 0. \tag{3}$$

Then $f \in A$ if and only if $f^n|_{Z(g_n)} \in A|_{Z(g_n)}$.

PROOF. If $f \in A$, then we clearly have $f^n|_{Z(g_n)} \in A|_{Z(g_n)}$.

Suppose now that (3) holds and $f^n|_{Z(g_n)} \in A|_{Z(g_n)}$. Choose $g \in A$ such that $f^n = g$ on $Z(g_n)$.

By (3), $g_n f^n + \cdots + g_1 f + g_0 = 0$. Multiplying through by g_n, we obtain ${g_n}^2 f^n + \cdots + g_n g_1 f + g_n g_0 = 0$. Therefore we may assume that there exist functions $h_j \in A$ such that $h_n f^n + \cdots + h_1 f + h_0 = 0$ and $Z(h_n) \subseteq \bigcap_{j=0}^{n-1} Z(h_j)$. Now we will complete the proof by showing recursively that f is A-holomorphic on $M_A \setminus Z(h_n)$.

Write $k_1 = n h_n f^{n-1} + \cdots + h_1$ for the formal derivative of $h_n f^n + \cdots + h_1 f + h_0$. If $x \in M_A \setminus Z(k_1)$, then Lemma 3.5 implies that there is a closed A-convex neighbourhood V of x where k_1 is zero-free. But $Z(h_n) \subseteq Z(k_1)$, so h_n is also zero-free on V. Therefore, on $V = M_{R_A(V)}$ we have

$$f^n + \frac{h_{n-1}}{h_n} f^{n-1} + \cdots + \frac{h_0}{h_n} = 0$$

and

$$nf^{n-1} + (n-1)\frac{h_{n-1}}{h_n}f^{n-2} + \cdots + \frac{h_1}{h_n} \neq 0.$$

By the implicit function theorem for Banach algebras applied to $R_A(V)$ ([6, 88]), $f \in R_A(V)$. Using the fact that V is A-convex and Lemma 3.6, we see that $f \in A_V$. Hence f is A-holomorphic on $M_A \setminus Z(k_1)$, and so k_1 is also. By Corollary 3.2 $k_1 \in A$, and the equality

$$h_n f^{n-1} + \frac{(n-1)}{n}h_{n-1}f^{n-2} + \cdots + \frac{h_1 - k_1}{n} = 0$$

reduces our problem to a polynomial of degree $n-1$. Repeating this process $n-2$ more times, we obtain a function $k_{n-1} \in A$ such that

$$h_n f + \frac{(n-1)!}{n!}h_{n-1} - k_{n-1} = 0.$$

Now we may divide by h_n and use the above argument to conclude that f is A-holomorphic on $M_A \setminus Z(h_n)$. But $Z(h_n) = Z(g_n) \subseteq Z(f^n - g)$, so $f^n - g$ is A-holomorphic on $M_A \setminus Z(g_n) \supseteq M_A \setminus Z(f^n - g)$. By Corollary 3.2, $f^n - g \in A$. Thus $f^n \in A$ and by Corollary 3.4, $f \in A$. ■

4. Division in Douglas algebras

In H^∞ it is, of course, true that if we have two Blaschke products f and g and the zeros of g in the disk are contained in the zeros of f (with appropriate multiplicity), then f is divisible by g in H^∞. Sarason and Guillory [10] first began the study of division in $H^\infty + C$ by showing that this need not be the case if we consider zeros in $M_{H^\infty+C}$. An easy example (due to Davidson and Luecking, presented in [10]) follows if we use a couple of well-known factorisation theorems due to Axler [2] and Wolff [23]. Let E be a non-trivial (measurable) subset of the unit circle. Consider the unimodular L^∞ function given by $2\chi_E - 1$. Using Axler and Wolff's results, there exist Blaschke products b_1 and b_2 and a unimodular function u invertible in $H^\infty + C$ such that $2\chi_E - 1 = b_1\overline{b_2}u$. Squaring, we see that $b_2{}^2 = b_1{}^2u^2$ on $M_{H^\infty+C}$. Since u is invertible in $H^\infty + C$, we see that the zero sets of b_1 and b_2 are equal on $M_{H^\infty} \setminus \mathbf{D}$. In fact, $|b_1| = |b_2|$ on $M_{H^\infty+C}$. Nevertheless b_1 is not divisible in $H^\infty + C$ by b_2, since $b_1\overline{b_2} = (2\chi_E - 1)\overline{u}$, and the right-hand side of this equation is not in $H^\infty + C$, so the left-hand side cannot be either.

What Guillory and Sarason did show is that there exists an integer N with the property that if f and u are two functions in $H^\infty + C$ and u is unimodular on ∂D and satisfies $|f| \leq |u|$ on $M_{H^\infty+C}$, then u divides f^N in $H^\infty + C$. The example presented above shows that N cannot be taken so that $N = 1$, and Guillory and Sarason posed the question of what was the best constant. This was answered, using techniques similar to those of Guillory and Sarason, by K. Izuchi and Y. Izuchi [15]. They showed that one can take $N = 2$. In fact, they showed that if A is a Douglas algebra, $f \in A$ and u is an inner function satisfying $|f| \leq |u|$ on M_A, then for every $n = 1, 2, \ldots$, one has $f^{n+1}\overline{u}^n \in A$.

Several results along these lines appeared. For example, Axler and Gorkin [3] and Guillory, Izuchi and Sarason [11] showed independently that if b is an interpolating Blaschke product with zeros contained in the zeros of a function $f \in H^\infty + C$, then f is divisible by b in $H^\infty + C$. Thus in this particular case one can take $N = 1$. Another case was handled by Gorkin and Mortini [9], who showed that if f vanishes on an open set containing the zeros of g, then f is divisible by g in $H^\infty + C$. Once again, then, this is a situation in which one can take $N = 1$.

Our interest in this paper is to handle these division theorems as a consequence of Rickart's result (Theorem 3.1), the Chang–Marshall theorem and the separating property of H^∞ [22]. The case $n = 1$ of Theorem 3.7 is particularly useful for the present section. We state it as the next corollary.

Corollary 4.1. *Let A be a Douglas algebra, $f \in C(M_A)$ and $h, g \in A$. Suppose that $h = fg$. Then $f \in A$ if and only if $f|_{Z(g)} \in A|_{Z(g)}$.*

We obtain the following result, which contains Izuchi and Izuchi's result (and therefore Sarason and Guillory's) discussed in the introduction to this section.

Corollary 4.2. *Let A be a Douglas algebra. Let $\alpha : [0, \infty) \to \mathbf{R}$ be a function satisfying $\alpha(0) = 0$ and $\alpha(t)/t \to 0$ as $t \to 0$. Suppose that $h, g \in A$ are such that $|h| \leq \alpha(|g|)$ on M_A. Then g divides h in A.*

PROOF. Define f on M_A as $f = h/g$ outside $Z(g)$ and $f = 0$ on $Z(g)$. By hypothesis f is continuous on M_A and $h = fg$. Then apply Corollary 4.1. ∎

The result of Gorkin and Mortini mentioned earlier is an immediate consequence of this corollary as well. One also obtains the second theorem of [9], as shown below.

Corollary 4.3. *Let A be a Douglas algebra properly containing H^∞, I an ideal in A and $h \in A$ a function vanishing in a neighbourhood of the hull of I. Then $h \in I$.*

PROOF. Let $h \equiv 0$ on U, where U is an open set containing $Z(I)$, the hull of the ideal. Since A is separating [22], there exists a function $g \in I$ such that g does not vanish on $M_A \setminus U$. By Corollary 4.2 we have $h \in gA \subset I$. ∎

Using the main results in [3] or [11], one can easily show that the following holds.

Proposition 4.4. *Let A be a Douglas algebra and let $f \in A$. If b is a finite product of interpolating Blaschke products and $Z(b) \subset Z_\infty(f)$, then $\{x \in M_A : |b(x)| < 1\} \subseteq Z_\infty(f)$. Consequently, we also have that $Z(b) \subset Z(f)^\circ$.*

Different versions and some generalisations of this result have appeared in the literature. Thanks to Corollary 4.2 we can give a very general version.

We will use Jensen's inequality for Douglas algebras [6, 33], stating that if A is a Douglas algebra, $g \in A$ and $x \in M_A$, then

$$\log|g(x)| \leq \int_{M_{L^\infty}} \log|g|\, d\mu_x. \tag{4}$$

Observe that, if g is invertible, the above inequality for g and g^{-1} forces equality in (4).

Theorem 4.5. *Let A be a Douglas algebra and $f, g \in A$ of norm at most 1. Suppose that there is $\alpha : [0,\infty) \to [0,\infty)$ such that $\alpha(0) = 0$ and $\alpha(t)/t^n \to 0$ as $t \to 0^+$ for every positive integer n, and $|f| \leq \alpha(|g|)$ on M_A. Then*

$$Z(g) \cup \{x \in M_A : \log|g(x)| < \int_{M_{L^\infty}} \log|g|\, d\mu_x\} \subset Z_\infty(f). \tag{5}$$

In particular, $x \in Z_\infty(f)$ if $|g(x)| < \min\{|g(y)| : y \in supp\, \mu_x\}$.

Proof. By hypothesis there exists a sequence of positive numbers $\{\delta_n\}$ tending to 0 such that on M_A

$$|f| \leq \alpha(|g|) \leq |g|^n \quad \text{if} \quad |g| < \delta_n.$$

Since $\|f\| \leq 1$, we obtain that on M_A

$$|f| \leq \delta_n^{-n}|g|^n. \tag{6}$$

Let G_n be an outer function defined by condition

$$|G_n(e^{i\theta})| = \begin{cases} |g(e^{i\theta})| & \text{if } |g(e^{i\theta})| \geq \delta_n \\ 1 & \text{if } |g(e^{i\theta})| < \delta_n \end{cases}$$

almost everywhere on ∂D. Then G_n is invertible in H^∞ and on M_{L^∞} we have

$$|gG_n^{-1}| = \begin{cases} 1 & \text{if } |g| \geq \delta_n, \\ |g| & \text{if } |g| < \delta_n. \end{cases} \tag{7}$$

Note that by (6) $|f| \leq \alpha^*(|g|^{n-1})$, where $\alpha^*(t) = \delta_n^{-n} t^{n/(n-1)}$.

Hence, by Corollary 4.2, f is divisible by $(gG_n^{-1})^n$ in A for all $n \geq 1$. Furthermore, (6) and (7) imply that $|f/(gG_n^{-1})^n| \leq 1$ on M_{L^∞}. Therefore

$$|f| \leq |(gG_n^{-1})^n| \;\text{ on } M_A \quad \text{for all } \; n \geq 1. \tag{8}$$

The hypotheses of the theorem clearly imply that $Z(g) \subset Z_\infty(f)$. On the other hand, if $x \in M_A$ is not a zero of g, then Jensen's inequality implies that $\log|g| \in L^1(d\mu_x)$. Therefore it will be enough to prove that if $x \in M_A$ is such that $\log|g| \in L^1(d\mu_x)$ and the inequality (4) is strict, then $x \in Z_\infty(f)$.

Using the fact that $|g| \leq 1$ and that G_n is invertible we get

$$\int \log|g|\, d\mu_x \leq \int_{\{|g| \geq \delta_n\}} \log|g|\, d\mu_x = \int \log|G_n|\, d\mu_x = \log|G_n(x)|$$

for every $n \geq 1$. This inequality, together with the strictness in (4), implies that

$$|g(x)| < \beta_x \stackrel{\text{def}}{=} \exp \int \log |g| \, d\mu_x \leq |G_n(x)| \quad \text{for every} \quad n \geq 1.$$

Therefore $|g(x)G_n^{-1}(x)| \leq |g(x)|\beta_x^{-1} < 1$, a number independent of n. Fix some r with $0 < r < 1$ and consider all the points $y \in P(x)$ such that $\rho(y, x) \leq r$. Since $(gG_n^{-1})|_{P(x)}$ is an analytic function of norm bounded by 1, the Schwarz–Pick inequality [7, 2] tells us that $\rho(g(y)G_n^{-1}(y), g(x)G_n^{-1}(x)) \leq \rho(y, x) \leq r$ for every $n \geq 1$. Therefore

$$s \stackrel{\text{def}}{=} \sup\{|g(y)G_n^{-1}(y)| : \rho(y, x) \leq r, \ n \geq 1\} < 1.$$

So by (8) we conclude that for every y satisfying $\rho(y, x) \leq r$

$$|f(y)| \leq |g(y)G_n^{-1}(y)|^n \leq s^n \to 0 \quad \text{when} \quad n \to \infty.$$

That is, $x \in Z_\infty(f)$. ■

Before proceeding to the next proof, we show here the well-known fact that if A is a Douglas algebra and $f \in A$, then $Z_\infty(f)$ is closed. To see this, let $x_\alpha \in Z_\infty(f)$ be a net converging to $x \in M_A$. Then $f \circ L_{x_\alpha} \to f \circ L_x$ uniformly on $|z| \leq r < 1$, implying that $f \equiv 0$ on the part of x and hence $x \in Z_\infty(f)$.

We say that a Douglas algebra A is countably generated if it is generated as a uniform algebra by H^∞ and a finite or countably infinite set of complex conjugates of inner functions. (See acknowledgements.)

Lemma 4.6. *A Douglas algebra A is countably generated if and only if M_A is a G_δ set in M_{H^∞}.*

PROOF. First suppose that A is countably generated. Then there is a finite or countably infinite collection of Blaschke products $\{b_n\}$ with $\overline{b_n} \in A$ such that $A = H^\infty[\overline{b_n} : n = 1, \ldots]$. Since $M_A = \cap_n \{\phi \in M_{H^\infty} : |\phi(b_n)| = 1\}$, we see easily from this that M_A is a G_δ set.

Now suppose that there is a countable collection of open sets V_n such that $M_A = \cap_1^\infty V_n$. By the Chang–Marshall theorem, A is generated by H^∞ and the complex conjugates of Blaschke products. Since each such Blaschke product b has modulus one on M_A, we see that $\cap\{|b| = 1\} = \cap_{n=1}^\infty V_n \subseteq V_m$ for each m. A compactness argument shows that there exist Blaschke products $b_1, \ldots, b_k$ such that $c_m = b_1 b_2 \cdots b_k$ is invertible in A and $\{|c_m| = 1\} \subseteq V_m$. Now $H^\infty[\overline{c_m} : m = 1, \ldots] \subseteq A$ and $M(H^\infty[\overline{c_m} : m = 1, \ldots]) = \cap\{|c_m| = 1\} \subseteq \cap V_m = M_A$. By the Chang–Marshall theorem, we conclude that $A = H^\infty[\overline{c_m} : m = 1, \ldots]$. ■

Lemma 4.7. *Let A be a countably generated Douglas algebra different from H^∞, and $f \in A$. Then $\overline{Z(f)^\circ} = Z_\infty(f)$.*

PROOF. First we note that, owing to the analytic structure of the Gleason parts in A, the inclusion $Z(f)^\circ \subset Z_\infty(f)$ is clear.

To prove the reverse inclusion, let $x \in Z_\infty(f)$ and U be a neighbourhood of x in M_A. Since A is countably generated, by Lemma 4.6 there exists a sequence $\{V_n\}$ of open sets in M_{H^∞} such that $M_A = \bigcap V_n$. Without loss of generality, we may also assume that $\overline{V}_{n+1} \subset V_n$. Choose an open set V in M_{H^∞} so that $x \in V$ and $M_A \cap \overline{V} \subset U$. By the corona theorem there exists a net (z_α) in $V \cap D$ converging to x. By [13], $g \circ L_{z_\alpha} \to g \circ L_x$ uniformly on $\{z \in D : |z| \leq r\}$ for every r such that $0 < r < 1$ and every $g \in C(M_{H^\infty})$. The harmonic extension $\tilde{f}$ (defined by $\tilde{f}(x) = \int_{M_{L^\infty}} f d\mu_x$) of f is continuous on M_{H^∞} and coincides with f on M_A. So, fixing r with $0 < r < 1$, we get $\sup_{|z|<r} |\tilde{f} \circ L_{z_\alpha}| \to 0$. Moreover, we see that we may choose a sequence z_n consisting of points of the net (z_α) such that

(1) $\sup_{|z|\leq r} |\tilde{f} \circ L_{z_n}| \to 0$,

(2) $z_n \in V \cap V_n \cap D$, and

(3) $\{z_n\}$ is an interpolating sequence.

Now every accumulation point of $\{z_n\}$ is in $\overline{V} \cap \bigcap_{n\geq 1} \overline{V}_n \subset U \cap M_A$. Let b be the associated interpolating Blaschke product. Then $Z(b) = \overline{\{z_n\}}$ [7, 314] (here $Z(b)$ denotes the zero set of b in M_{H^∞}), and consequently $Z(b) \cap U \neq \emptyset$. By (1) f has a zero of infinite order on the zeros of b. Thus $Z(b) \subset Z_\infty(f)$. By Proposition 4.4, $Z(b) \subset Z(f)^\circ$. Therefore $Z(f)^\circ$ is dense in $Z_\infty(f)$. ■

We remark that Lemma 4.7 is not true for arbitrary Douglas algebras. To see this, let $A = \{f \in L^\infty : f|_{M_1} \in H^\infty|_{M_1}\}$, where $M_1 = \{m \in M_A : m(z) = 1\}$ is the fibre over 1. Let $f(z) = 1 + z$. Then $Z(f)^\circ = \emptyset$, but $Z(f) = Z_\infty(f) = \{m \in M_{L^\infty} : m(z) = -1\}$.

The equivalences from (c) to (e) in our next theorem were established by Izuchi for general Douglas algebras in corollary 4.5 of [14].

Theorem 4.8. *Let A be a countably generated Douglas algebra and $u \in A$ such that $|u| = 1$ on M_{L^∞}. Let $I = uA$ be the ideal generated by u. Then I is closed and the following assertions are equivalent:*

(a) *I has the local approximation property,*

(b) $Z(u)^\circ = \emptyset$,

(c) $Z_\infty(u) = \emptyset$,

(d) $\operatorname{ord}(u, x) \leq N$ *for some integer N and all $x \in Z(u)$,*

(e) *$u = bv$, where b is a finite product of interpolating Blaschke products and v is a unimodular function invertible in A.*

If (e) holds, then $I = bA$.

Proof. The ideal uA is closed because $|u| \equiv 1$ on $\partial A = M_{L^\infty}$. First we show that (a) fails when (b) fails. Suppose that there is $x \in Z(u)^\circ$ (hence $x \notin M_{L^\infty}$). Let U be an open set in M_{H^∞} such that $x \in U$ and $\overline{U} \cap M_A \subseteq Z(u)^\circ$. If x is a trivial point then we may choose (see [8]) a non-trivial point $m \in U$ satisfying $\operatorname{supp} \mu_m \subset \operatorname{supp} \mu_x$. If x itself is non-trivial we choose $m = x$ below. Take an interpolating Blaschke product b with zero sequence contained in $U \cap D$ such that $b(m) = 0$. The inclusion of support sets implies that $|b(x)| < 1$. Furthermore, if $Z(b)$ denotes the zero set of b in M_A then $Z(b) \subset \overline{U} \cap M_A \subset Z(u)^\circ$. By [3] or [11] u is divisible by b in A, which we denote by $u/b \in A$.

We will see that u/b is locally approximable by functions in I. In fact, let $x \in M_A \setminus Z(u)^{\circ}$. By Lemma 3.5 we can choose a closed A-convex neighbourhood V of x in M_A which does not meet $Z(b)$. Thus on V we have $1/b \in R_A(V) = A_V$, where the equality of algebras follows from Lemma 3.6. Therefore, $1/b$ can be uniformly approximated on V by elements of $A|_V$. Consequently, u/b can be approximated on V by elements in $I|_V$, as claimed.

We now show that, despite this fact, u/b is not in I. Suppose, to the contrary, that there are $f_n \in A$ such that $u/b = \lim uf_n$. Multiplying through by b we see that $u(1 - bf_n) \to 0$. Thus, for n sufficiently large, bf_n is invertible in A. Since b is not invertible in A, this is not possible. This finishes the proof of (a) implies (b). (We note that we have not used here that A is countably generated.)

Owing to the assumption that A is countably generated, Lemma 4.7 shows that (b) implies (c). As we mentioned earlier, the equivalences from (c) to (e) were given by Izuchi.

It is clear that if (e) holds then $I = bA$. But this implies that (a) holds. Indeed, let $f \in C(M_A)$ be a function that is I-holomorphic. Then clearly $\operatorname{ord}(f, x) \geq \operatorname{ord}(b, x)$ for every $x \in Z(I)$. Hence, by [3] or [11], f is divisible by b in A, that is $f \in I$. ■

We remark that the proof of (e) implies (a) in the theorem above is valid for arbitrary Douglas algebras. The theorem contains a description of a wide class of ideals having the local approximation property. Of course, every such ideal is local. We don't know if the converse holds. With the aid of some classical theory, the results of this paper easily produce an ideal in $H^\infty + C$ that is not local, as shown in the example below. The sets of the example are taken in the maximal ideal space of $H^\infty + C$.

Example. Let E be a clopen subset of M_{L^∞}. Let b be a Blaschke product such that $b = 0$ on the set $\{0 < \chi_E < 1\}$. Then $b(H^\infty + C)$ is not a local ideal.

First we remark that such a Blaschke product always exists. To see this, note that Marshall has shown [7, 398] that there is an interpolating Blaschke product c such that $H^\infty[\chi_E] = H^\infty[\bar{c}]$. By the Chang–Marshall theorem and Shilov's idempotent theorem,

$$\{|c| = 1\} = M_{H^\infty[\bar{c}]} = M_{H^\infty[\chi_E]} = \{\chi_E(x) = \text{ 0 or } 1\}.$$

A classical construction of Newman [7, 195] then provides a Blaschke product b that vanishes on $\{|c| < 1\} = \{0 < \chi_E < 1\}$. Consider the function $f \in C(M_{H^\infty + C})$ defined by $f(x) = b(x)\chi_E(x)$. Clearly, $f = 0$ on $\{\chi_E < 1\}$ and $f = b$ on $\{0 < \chi_E\}$. Therefore, f is locally in the ideal, which together with Corollary 3.2 implies that $f \in H^\infty + C$. If $f = b\chi_E \in b(H^\infty + C)$, then $\chi_E \in H^\infty + C$, which is impossible. ■

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