LOCALNESS OF $A(\Psi)$ ALGEBRAS

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Abstract

Let d and r be positive integers. Given $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \ldots, \psi_r]$ generated by $\{\psi_1, \ldots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem, to provide an explicit description of Ψ, is local to the level sets of Ψ.

1. Introduction

Let d and r be positive integers throughout. Given $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \ldots, \psi_r]$ generated by $\{\psi_1, \ldots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We recall that $A(\Psi)$ is also the closure of the algebra

$$C^\infty(\Psi) = \{g \circ \Psi : g \in C^\infty(\mathbf{R}^r, \mathbf{R})\}$$

(cf. [1; 2] for background).

This paper is about the properties of $A(\Psi)$ *qua* Fréchet algebra, and the approximation problem: describe $A(\Psi)$ explicitly.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem is local to the level sets of Ψ.

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2. Associated topologies

We shall only use the Euclidean topology on $\mathbf{R}^r$, but we need to consider some other, *a priori* distinct topologies on $\mathbf{R}^d$.

Definition. We define the *Ψ-hull-kernel topology* on $\mathbf{R}^d$ as that corresponding to the Kuratowski closure operation

$$E \mapsto \mathrm{HK}(E) = \{b \in \mathbf{R}^d : f(b) = 0 \text{ whenever } f \in A(\Psi) \text{ and } f|E = 0\}.$$

Remark. This is in general finer than the pull-back topology

$$\{\Psi^{-1}(U) : U \text{ open in } \mathbf{R}^r\}.$$

It may happen that $\mathrm{clos}\Psi(E) \cap \mathrm{clos}\Psi(F) \neq 0$ for disjoint Ψ-hull-kernel closed sets E, F.

We abbreviate Ψ-hull-kernel topology to HK-topology when convenient. We denote the HK-closure of a set E by $\mathrm{HK}(E)$. This is consistent with the following (more or less standard) notation:

$$H(F) = \{a \in \mathbf{R}^d : f(a) = 0, \forall f \in F\}, \quad \forall F \subset A(\Psi);$$

$$K(E) = \{f \in A(\Psi) : f(a) = 0, \forall a \in E\}, \quad \forall E \subset \mathbf{R}^d.$$

Note that

$$HKH(F) = H(F), \quad \forall F \subset A(\Psi),$$

$$KHK(E) = K(E), \quad \forall E \subset \mathbf{R}^d.$$

In particular, $H(F)$ is HK-closed, for each $F \subset A(\Psi)$.

We note that each HK-closed set is a union of level sets of Ψ, that $\mathrm{HK}(\{a\}) = \Psi^{-1}(\Psi(a))$ (the level set through the point a), and that the minimal non-empty HK-closed sets are these level sets of Ψ. We abbreviate $\mathrm{HK}(\{a\})$ to $\mathrm{HK}(a)$.

Definition. We define the *Ψ-weak-star topology* on $\mathbf{R}^d$ as the pull-back topology corresponding to the weak-star topology on the dual $A(\Psi)^*$ and the natural injection of $\mathbf{R}^d$ into $A(\Psi)^*$.

In other words, the set N is a weak-star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exist a finite number of functions $f_1, \ldots, f_n$ belonging to $A(\Psi)$, such that

$$\{x \in \mathbf{R}^d : |f_j(x) - f_j(a)| < 1, \forall j\} \subset N.$$

Since we are dealing here with real-valued functions, it is evident that it makes no difference if we insist that n always equal 1. In fact, it is easy to see that the set N is a weak-star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exists a function

$f \in A(\Psi)$, such that $f(a) = 0$ and

$$\{x \in \mathbf{R}^d : f(x) < 1\} \subset N.$$

We abbreviate Ψ-weak-star topology to WS-topology, when convenient.

Unqualified topological terms (open, closed, ...) refer to the Euclidean topology. It is clear that the WS-topology is at least as fine as the HK-topology, and the Euclidean topology is at least as fine as the WS-topology. We shall discover more below.

We denote the closed ball with centre $x \in \mathbf{R}^d$ and radius $r \geq 0$ by $\mathbf{B}(x, r)$, and the corresponding open ball by $\mathbf{U}(x, r)$.

Proposition 1. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Suppose that E and F are disjoint closed subsets of $\mathbf{R}^d$, $\Psi^{-1}\Psi(E) = E$, and $\Psi^{-1}\Psi(F) = F$. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F.*

Proof. Let E and F be as in the hypothesis. Define

$$\begin{aligned} B_n &= \mathbf{B}(0, n), \quad B'_n = \Psi(B_n), \\ E_n &= E \cap B_n, \quad E'_n = \Psi(E_n), \\ F_n &= F \cap B_n, \quad F'_n = \Psi(F_n), \\ H_n &= B_{n-1} \cup E_n \cup F_n, \quad H'_n = \Psi(H_n). \end{aligned}$$

We observe that E'_n, F'_n, and H'_n are compact subsets of $\mathbf{R}^r$, and

$$\begin{aligned} E'_n \cap F'_n &= \emptyset, \\ E_{n+1} \cap B_n &= E_n, \\ E'_{n+1} \cap B'_n &= E'_n \quad (\text{because } E = \Psi^{-1}\Psi(E)), \\ F'_{n+1} \cap B'_n &= F'_n. \end{aligned}$$

Choose $g_1 \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_1 = 1$ near E'_1 and $g_1 = 0$ near F'_1.

Consider the function $g_2 : H'_2 \to \mathbf{R}$ given by

$$g_2(x) = \begin{cases} g_1(x), x \in B'_1, \\ 1, x \in E'_2, \\ 0, x \in F'_2. \end{cases}$$

This is well defined, since $g_1 = 1$ on $B'_1 \cap E'_2$ and $g_1 = 0$ on $B'_1 \cap F'_2$. Moreover, each point of H'_2 has a neighbourhood to which g_2 has a C^∞ extension; indeed, one of g_1, 1 or 0 will do as the extension. Since the existence of a global C^∞ extension is a local property, it follows that g_2 has an extension in $C^\infty(\mathbf{R}^r, \mathbf{R})$, and we denote such an extension by the same symbol, g_2. We may choose g_2 so that it is 1 *near* E'_2 and 0 *near* F'_2.

Continuing in this way, we find $g_{n+1} \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$g_{n+1}|B'_n = g_n|B'_n,$$

$g_{n+1} = 1$ near E'_{n+1} and $g_{n+1} = 0$ near F'_{n+1}.

Define $f : \mathbf{R}^d \to \mathbf{R}$ by setting

$$f|B_n = g_n \circ \Psi|B_n, \quad \forall n.$$

Evidently, f is well defined, $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$, and $g_n \circ \Psi \to f$ in C^∞ topology as $n \uparrow +\infty$. Thus $f \in A(\Psi)$. Finally, it is clear that $f = 1$ on E and $f = 0$ on F, so we are done. ■

Corollary 2. *Suppose $E \subset \mathbf{R}^d$. Then E is HK-closed if and only if E is closed and $\Psi^{-1}\Psi(E) = E$.*

Proof. The 'only if' part is obvious. For the converse, suppose that E is closed and $\Psi^{-1}\Psi(E) = E$. Let $a \notin E$. Then $F = \Psi^{-1}\Psi(a)$ is closed and disjoint from E, and $\Psi^{-1}\Psi(F) = F$. By Proposition 1, there exists $f \in A(\Psi)$ such that $f|E = 0$ and $f|F = 1$. Thus $a \notin \mathrm{HK}(E)$. This shows that $\mathrm{HK}(E) \subset E$. Evidently $E \subset \mathrm{HK}(E)$, so $\mathrm{HK}(E) = E$, and we are done. ■

Corollary 3. *Let $E \subset \mathbf{R}^d$. Then $HK(E)$ is the least set $F \subset \mathbf{R}^d$ such that $E \subset F$, F is closed, and $\Psi^{-1}\Psi(F) \subset F$.* ■

This fact implies that the HK-closure of a set E may be obtained by forming $E_0 = E$, and proceeding by transfinite induction:

$$E_{\alpha+1} = \mathrm{clos}\Psi^{-1}\Psi(E_\alpha), \quad \forall \text{ ordinals } \alpha,$$
$$E_\alpha = \bigcup_{\beta<\alpha} E_\beta, \quad \forall \text{ limit ordinals } \alpha,$$

until the first ordinal having cardinal greater than the continuum, at the worst.

Corollary 4. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Then the following three conditions are equivalent:*
(1) *Ψ is injective;*
(2) *the HK-topology is Hausdorff;*
(3) *the HK-topology is the same as the Euclidean topology.*

Proof. Obviously (3) ⇒ (2) ⇒ (1). The only delicate point is (1) ⇒ (3), and this is immediate from Corollary 2. ■

Finally, we note that by combining Proposition 1 and Corollary 2 we have the following corollary.

Corollary 5. *Let E and F be disjoint HK-closed sets. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F.* ■

Now we consider the C^∞ action and its consequences.

Proposition 6. *Let $h \in C^\infty(\mathbf{R}^r, \mathbf{R}^p)$. Then the (usually non-linear) map*

$$h\circ : \begin{cases} C^\infty(\mathbf{R}^d, \mathbf{R}^r) \to C^\infty(\mathbf{R}^d, \mathbf{R}^p), \\ \qquad f \mapsto h \circ f, \end{cases}$$

is continuous.

Proof. This is immediate from Faa di Bruno's formula [4, 222]. ■

Corollary 7. *$C^\infty(\mathbf{R}, \mathbf{R})$ acts by composition on $A(\Psi)$, i.e. $f \mapsto h \circ f$ maps $A(\Psi)$ into itself, whenever $h \in C^\infty(\mathbf{R}, \mathbf{R})$.*

Proof. Let $f \in A(\Psi)$. Choose $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_n \circ \Psi \to f$ in $C^\infty(\mathbf{R}^d, \mathbf{R})$ topology. By Proposition 6,

$$(h \circ g_n) \circ \Psi = h \circ (g_n \circ \Psi) \to h \circ f$$

in C^∞ topology. Thus $h \circ f \in A(\Psi)$. ■

Corollary 8. *Let $h \in C^\infty(\mathbf{R}, \mathbf{R})$, with $h(0) = 0$. Then $h\circ$ maps each ideal $I \subset A(\Psi)$ into itself.*

Proof. We may factorise $h(x)$ as $xk(x)$, where $k \in C^\infty(\mathbf{R}, \mathbf{R})$[5].

Let $f \in I$. Then, using Corollary 7, we get

$$h \circ f = f \cdot (k \circ f) \in f \cdot A(\Psi) \subset I,$$

as required. ■

Corollary 9. *Let $g \in A(\Psi)$. Then the set*

$$U = \{a \in \mathbf{R}^d : g(a) > 0\}$$

is HK-open.

Proof. Choose $h \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $h(x) > 0$ whenever $x > 0$ and $h(x) = 0$ whenever $x \le 0$.

By Corollary 7, $h \circ g \in A(\Psi)$, and evidently

$$\mathbf{R}^d \sim U = H(\{h \circ g\})$$

is HK-closed. This suffices. ■

Corollary 10. *The HK-topology is the same as the WS-topology.*

Proof. It follows readily from Corollary 9 that the set

$$\{x \in \mathbf{R}^d : f(x) < 1\}$$

is HK-open, whenever $f \in A(\Psi)$. Since the sets of this form, corresponding to $f \in A(\Psi)$ with $f(a) = 0$, form a neighbourhood base for the point $a \in \mathbf{R}^d$, we conclude that each WS-open set is HK-open, and this suffices. ■

We summarise our characterisations of the HK-topology, adding a useful converse to Corollary 9.

Theorem 11. *Let $U \subset \mathbf{R}^d$. Then the following are equivalent:*
(1) *U is HK-open;*
(2) *U is WS-open;*
(3) *U is open and $\Psi^{-1}\Psi(U) = U$;*
(4) *there exists $g \in A(\Psi)$ such that*

$$U = \{x \in \mathbf{R}^d : g(x) > 0\}.$$

Proof. In view of Corollaries 2, 9 and 10, it only remains to prove that (1) implies (4).

Let $F = \mathbf{R}^d \sim U$.

Given $n \in \mathbf{N}$, consider $B_n = \mathbf{B}(0, n)$ and

$$E_n = \left\{ x \in B_n : \operatorname{dist}(x, F) \geq \frac{1}{n} \right\}.$$

For each $a \in E_n$, there exists $f_a \in A(\Psi)$ such that $f(a) = 1$ and $f = 0$ on F. Thus the set

$$N_a = \{x \in \mathbf{R}^d : f_a(x) > 0\}$$

is a HK-neighbourhood of a. Since E_n is compact, we may choose $a_1, a_2, \ldots, a_m \in E_n$ such that $N_{a_1}, \ldots, N_{a_m}$ cover E_n. Let

$$g_n = f_{a_1}^2 + \ldots + f_{a_m}^2.$$

Then $g_n \in A(\Psi)$, $g_n = 0$ on F, $g_n \geq 0$ on $\mathbf{R}^d$, and $g_n > 0$ on E_n. Let

$$M_n = 1 + \max_{|i| \leq n} \sup_{B_n} |\partial^i g_n|.$$

Define

$$g = \sum_{n=1}^{\infty} \frac{g_n}{2^n M_n}.$$

For any given $m \in \mathbf{N}$ and $k \in \mathbf{Z}_+$, we have

$$\sup_{B_m} \left| \partial^i \left(\frac{g_n}{2^n M_n} \right) \right| \leq 2^{-n}, \qquad \forall |i| \leq k \quad \forall n \geq m.$$

Thus the series for g converges in C^∞ topology, and $g \in A(\Psi)$.

Evidently, $g > 0$ on U and $g = 0$ on F, so we are done. ■

3. Maximal ideals

Proposition 12. *Suppose $f \in A(\Psi)$ and $f(a) \neq 0$, $\forall a \in \mathbf{R}^d$. Then $1/f \in A(\Psi)$.*

PROOF. Fix K compact in $\mathbf{R}^d$ and $k \in \mathbf{Z}_+$. It suffices to show that there exist $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i(g_n \circ \Psi) \to \partial^i\left(\frac{1}{f}\right), \quad \forall |i| \leq k,$$

uniformly on K.

We may assume that K is a ball, without loss of generality.

Choose $h_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i(h_n \circ \Psi) \to \partial^i(f), \quad \forall |i| \leq k,$$

uniformly on K. Since $f \neq 0$ on K, we may assume, without loss of generality, that $f > 0$ on K (since K is connected). Let $\kappa = \inf_K f$.

Discarding some initial terms of the sequence (if need be), we may assume that $h_n > \kappa/2$ on $\Psi(K)$, for each n.

Choose $k_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $k_n = \log h_n$ near K, and let $r_n = \exp(-k_n)$. Then $g_n = 1/h_n$ on $\Psi(K)$, and

$$\partial^i(g_n \circ \Psi) = \partial^i\left(\frac{1}{h_n \circ \Psi}\right) \to \partial^i\left(\frac{1}{f}\right), \quad \forall |i| \leq k,$$

uniformly on K, as required. ■

Theorem 13. *Let $M \subset A(\Psi)$. Then the following are equivalent:*
(1) *M is a closed maximal ideal in $A(\Psi)$;*
(2) *there exists $a \in \mathbf{R}^d$ such that $M = K(a)$.*

PROOF. It is easy to see that (2) implies (1).

To prove that (1) implies (2), fix a closed maximal ideal M. We wish to show that M is the HK-closure of some point. Since (2) implies (1), it suffices to show that $\mathrm{H}(M)$ is non-empty.

Suppose that $\mathrm{H}(M) = \emptyset$.

For each $a \in \mathbf{R}^d$, we may choose $f_a \in M$ such that $f_a(a) = 1$. Using compactness, as in the proof of Proposition 12, we may choose for each $n \in \mathbf{N}$ a function $g_n \in M$ such that $g_n \geq 0$ on $\mathbf{R}^d$ and $g_n > 0$ on B_n. Adding these up with suitable weights, we get $g \in \mathrm{clos}(M) = M$ such that $g > 0$ on $\mathbf{R}^d$. By Proposition 12, $1/g \in A(\Psi)$, so $1 \in M$, so $M = A(\Psi)$, contradicting the maximality of M. This contradiction shows that $\mathrm{H}(M) \neq \emptyset$, and we are done. ■

Remark. The argument of this proof actually shows that each ideal having empty hull is dense in $A(\Psi)$. From this observation, it is not hard to deduce that the maximal closed ideals are the same as the closed maximal ideals.

Example. Let $\Psi(x) = x$, $\forall x \in \mathbf{R}$, so that $\mathbf{A}(\Psi) = C^\infty(\mathbf{R}, \mathbf{R})$. The subset of all functions having compact support is a proper ideal, and hence is contained in a maximal ideal M. Since $\mathrm{H}(M) = \emptyset$, M cannot be closed.

In fact, this $A(\Psi)$ has many dense maximals ideals, corresponding to some kind of ultrafilters.

Recall that a *character* of a real Fréchet algebra is, by definition, a non-zero algebra homomorphism from the algebra to $\mathbf{R}$. There is a bijective correspondence between characters and maximal ideals with quotient isomorphic to $\mathbf{R}$. It is known that all characters on real Fréchet algebras are necessarily continuous [3].

Corollary 14. *The characters of $A(\Psi)$ are the evaluations at the points of $\mathbf{R}^d$.*

Proof. Since characters are continuous, the kernel of a character is a closed maximal ideal, hence is $K(a)$ for some $a \in \mathbf{R}^d$. It follows easily that the character is evaluation at a. ■

Characters belong to $A(\Psi)^*$, so the space of characters inherits the weak-star topology. We may thus rephrase Corollary 5, as follows.

Theorem 15. *Let E and F be disjoint weak-star closed sets of characters on $A(\Psi)$. Then there exists a function $f \in A(\Psi)$ such that $\phi(f) = 1$ for all $\phi \in E$ and $\phi(f) = 0$ for all $\phi \in F$.*

This is the *regularity* referred to in the introduction.

4. Localness

Segal asked in 1949 whether $A(\Psi)$ has a local description analogous to the Stone–Weierstrass Theorem. Nachbin conjectured that membership of f in $A(\Psi)$ is determined by the behaviour of f on each level set of Ψ. This conjecture may be reformulated in terms of Taylor series, and some special cases have been proved by Tougeron and the authors. For a more detailed account of the history, see [1; 2]. To date, it has not even been established that membership in $A(\Psi)$ depends only on the behaviour of f *near* each level set of Ψ. This we shall now do.

First, we establish a preliminary fact.

Lemma 16. *Let $E \subset U \subset \mathbf{R}^d$, where E is HK-closed and U is (Euclidean) open. Let K be compact. Then there exists a HK-open set V such that $E \subset V$ and $K \cap V \subset K \cap U$.*

Proof. By Theorem 11, we may choose $h \in A(\Psi)$ such that $h = 0$ on E and $h > 0$

off E. Let $\eta = \inf_{K\sim U} h$. Then $\eta > 0$. Take $V = \{x \in \mathbf{R}^d : h(x) < \eta\}$. Then V is HK-open, by Corollary 9, $E \subset V$, and $K \cap V \subset K \cap U$, as required. ■

Theorem 17. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$ and $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$. Then the following four conditions are equivalent:*

(1) $f \in A(\Psi)$;

(2) $\forall a \in \mathbf{R}^d$, *there exists a HK-neighbourhood U of a and there exists a function $g \in A(\Psi)$ such that $g = f$ on U;*

(3) $\forall a \in \mathbf{R}^d$, *there exists a HK-open neighbourhood U of a and there exists a sequence of $g_n \in \mathbf{R}[\Psi]$ such that $g_n \to f$ in $C^\infty(U, \mathbf{R})$ topology;*

(4) $\forall a \in \mathbf{R}^d$, *and for each compact $K \subset \mathbf{R}^d$ there exists a (Euclidean) open neighbourhood U of $K \cap HK(a)$ and there exists a sequence of $g_n \in \mathbf{R}[\Psi]$ such that $g_n \to f$ in $C^\infty(U, \mathbf{R})$ topology.*

PROOF. It is evident that (1) implies (2), (2) implies (3), and (3) implies (4).

(4) $\Rightarrow$ (1): Suppose (4). Fix $K \subset \mathbf{R}^d$, compact, and $k \in \mathbf{Z}_+$.

For each $a \in K$, choose an open neighbourhood U_a of $K \cap \mathrm{HK}(a)$ and a sequence $g_{a,n} \in \mathbf{R}[\Psi]$ such that $g_{a,n} \to f$ in $C^\infty(U_a, \mathbf{R})$ topology.

For each $a \in K$, Lemma 16 allows us to choose a HK-open set V_a such that $\mathrm{HK}(a) \subset V_a$ and $K \cap V_a \subset K \cap U_a$.

By Corollary 5 we may choose $\rho_a \in A(\Psi)$ such that $\rho_a > 0$ precisely on V_a and $\rho = 1$ on $\mathrm{HK}(a)$. Let $W_a = \{x \in \mathbf{R}^d : \rho_a(x) > 1/2\}$. Then W_a is HK-open and $\mathrm{HK}(a) \subset W_a$.

By compactness, we may choose $a_1, \ldots, a_m \in K$ such that $K \subset W_{a_1} \cup \ldots \cup W_{a_m}$. Let us abbreviate W_{a_i} to W_i, V_{a_i} to V_i, and $g_{a_i,n}$ to $g_{i,n}$. Choose $h_i \in A(\Psi)$ such that $h_i > 0$ precisely on W_i.

Let $W = W_1 \cup \ldots \cup W_m$. Since W is HK-open, we may choose $h \in A(\Psi)$ such that $h > 0$ precisely on W. Since K is compact, the number $\eta = \inf_K h$ is strictly positive. Let $F = \{h \geq \eta/2\}$. Then F and $\mathbf{R}^d \sim W$ are disjoint HK-closed sets, and $K \subset F$. Choose $h_0 \in A(\Psi)$ such that $h_0 > 0$ precisely on $\mathbf{R}^d \sim F$. Then $s = h_0 + \ldots + h_m$ belongs to $A(\Psi)$ and $s > 0$ on $\mathbf{R}^d$, so $1/s \in A(\Psi)$, by Proposition 12. Let $k_i = h_i/s$. Then $k_i \in A(\Psi)$, $k_i \geq 0$, $\Sigma_0^m k_i = 1$, $k_0 = 0$ near K and $\mathrm{spt} k_i \subset W_i$ whenever $i \geq 1$.

Fix i, $1 \leq i \leq m$. Let $T_i = K \cap \{\rho_i \geq 1/2\}$. Then T_i is a compact subset of $K \cap U_i$. Since $k_i = 0$ on $K \sim T_i$, and $g_{i,n} \to f$ in $C^\infty(U_i, \mathbf{R})$, we see that

$$\partial^i(k_i \cdot g_{i,n} - k_i \cdot f) \to 0$$

uniformly on K, for each $|i| \leq k$, as $n \uparrow \infty$. Since $k_0 = 0$ near K, we conclude that the function $r_n = \Sigma_1^m k_i g_{i,n}$, which belongs to $A(\Psi)$, converges uniformly, along with all derivatives up to order k, uniformly on K, to $\Sigma_1^m k_i f$, which equals f on K. This suffices to show that (1) holds. ■

REFERENCES

[1] G. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Finitely-generated algebras of smooth functions, in one dimension, *Journal of Functional Analysis* **158** (1998), 458–74.

[2] G. Allan, G. Kakiko, A.G. O'Farrell and R.O. Watson, Algebras of smooth functions, in E. Albrecht and M. Mathieu (eds), *Banach Algebras '97*, de Gruyter, Berlin and New York, 1998, pp 43–53.
[3] S. Dao-Shing, On seminormed rings with involution [in Russian], *Izvestia Akademii Nauk-Seriia Matematitcheskaia* **23** (1959), 509–28.
[4] H. Federer, *Geometric measure theory*, Springer, New York and Heidelberg, 1969.
[5] J.-C. Tougeron, *Idéaux des fonctions différentiables*, Springer, Berlin and New York, 1972.