SYSTEM-PERMUTABLE FISCHER SUBGROUPS ARE INJECTORS

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[Received 23 February 2000. Read 12 October 2000. Published 16 October 2001.]

Abstract

In 1973 Dark provided the first example of a group and a Fitting set D such that the D-injectors are not normally embedded, and the first example of a group with Fischer D-subgroups that are not D-injectors, though they are injectors for another Fitting set, F. In their 1992 book *Finite soluble groups*, Doerk and Hawkes point out that in this second example the F-injectors are not even system-permutable, a weaker condition than normally embedded. Here we work with system-permutable Fischer F-subgroups. First, we show that a system-permutable Fischer F-subgroup that is also pronormal must be an F-injector. Then we prove that we can drop the requirement of pronormality and reach the same conclusion. Thus in Dark's example the existence of Fischer D-subgroups that are not system-permutable is necessary for any Fischer D-subgroups not to be D-injectors.

Several authors have studied the question of when the Fischer F-subgroups and F-injectors of a finite solvable group are the same subgroups. Fischer [5] showed that if F is what is now called a Fischer set, then the F-subgroups of G are indeed F-injectors of G. A somewhat more general result of Anderson [1] is that when the Fischer F-subgroups of H are normally embedded in H for each subgroup H of G, the Fischer F-subgroups and F-injectors of G coincide. In [4], we came to the same conclusion if all the Fischer F subgroups of G are either subnormally embedded or locally pronormal in G. In [2], Dark provides the first example of a group and a Fitting set D such that the D-injectors are not normally embedded, and the first example of a group with Fischer D-subgroups that are not D-injectors, though they are injectors for another Fitting set, F. Doerk and Hawkes [3, 646] point out that in this latter example the F-injectors are not even system-permutable, a weaker condition than normally embedded. Doerk and Hawkes [3, VIII(4.9)] have also provided an example in which the Fischer F-subgroups are not injectors for any Fitting set. Here we work with system-permutable Fischer F-subgroups. First we show that a system-permutable Fischer F-subgroup that is also pronormal must be an F-injector. Then we prove that we can drop the requirement of pronormality and reach the same conclusion. Thus, in Dark's example, the existence of Fischer D-subgroups that are not system-permutable is necessary for the Fischer D-subgroups not to be D-injectors. We begin with some key definitions. Definitions and notation will be as in [3], and all groups will be finite and solvable.

A subgroup A is *pronormal* in G if, for each $g \in G$, A is conjugate to A^g by an element of $\langle A, A^g \rangle$.

A subgroup A is *system-permutable* in G if there exists a Hall system Σ such that, for every subgroup B of Σ, AB is a subgroup of G.

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A subgroup A is *normally embedded* in G if, for each prime r dividing A, a Sylow r-subgroup of A is a Sylow r-subgroup of some normal subgroup of G.

If F is a Fitting set of G, a subgroup U of G is a *Fischer F-subgroup* of G if U contains every F-subgroup of G that it normalises.

If F is a Fitting set of G and H is a subgroup of G, then F_H will denote the Fitting set $\{X \leq H : X \in F\}$. The subscript will be suppressed in all cases unless its inclusion clarifies the argument.

Lemma 1. *If U is a proper subgroup of a finite group G, then there exists a maximal subgroup M of G such that $M = UK$, where K is normal in G.*

Proof. Let $1 = N_0 < N_1 < N_2 < \ldots < G$ be a U-composition series for G, and let k be the smallest index such that $UN_k = G$. Thus $UN_{k-1} < G$. Let $M = UN_{k-1}$. Note that N_{k-1} is U-invariant and normal in N_k, so N_{k-1} is normal in $G = UN_k$. Now suppose $M \leq X \leq G$. Then $U \leq M \leq X$, so X is U-invariant. Also, $N_{k-1} \leq X$, so $N_{k-1} \leq X \cap N_k \leq N_k$. Because $X \cap N_k$ is U-invariant and N_k/N_{k-1} is U-irreducible, $X \cap N_k = N_k$ or $X \cap N_k = N_{k-1}$. Thus $X = X \cap G = X \cap UN_k = U(X \cap N_k) = G$ or M, and M is maximal in G. With $K = N_{k-1}$, M is the subgroup we seek. ■

Lemma 2. *Suppose that G is a finite solvable group, $U \leq H \leq G$, and U is system-permutable and pronormal in G. Then U is system-permutable and pronormal in H.*

Proof. U is pronormal in H by [3, I(6.3)(a)]. Now choose any Hall system of U. By [3, I(4.16)], it extends to a Hall system of H, which similarly extends to a Hall system Σ of G. Then Σ reduces into H by definition, and, because U is system-permutable and pronormal in G, U permutes with Σ by [3, I(6.7)]. Then, by [3, I(4.25)(c)], U permutes with $\Sigma \cap H$ in H. ■

Theorem 1. *Suppose that G is a finite solvable group and F is a Fitting set of G. If V is a pronormal, system-permutable Fischer F-subgroup of G, then V is an F-injector of G.*

Proof. Suppose that G is a counterexample to the theorem of minimal order, and V is a pronormal, system-permutable Fischer F-subgroup that is not an F-injector of G. If $V \leq H < G$, then V is a Fischer F-subgroup of H, and, by Lemma 2, V is system-permutable and pronormal in H. Hence V is an F-injector of H by minimality of G. Also, if N is normal in G, VN/N is pronormal and system-permutable in G/N by [3, I(4.25)(b), I(6.3)(c)]. These facts make it possible to follow exactly steps (1)–(5) of the proof of Anderson's result related in [3, VIII(4.7)].

Thus we know the following: the radical, G_F, is trivial; $G = VSN$, where N is minimal normal in G; S is an F-injector of SN; V is an F-injector of VN; SN is normal in G; VN *does not* contain SN; and G/N is primitive with unique minimal normal subgroup SN/N and core-free maximal subgroup VN/N.

Now let Σ be the Hall system of G with which V permutes, let p be the prime dividing $|N|$, and let Y be the Hall p'-subgroup of G in Σ. Note that $S \cap N$ is an F-injector of N, and is normal in the abelian N, so $S \cap N = 1$ because, by [3, VIII(2.4)(d)], $N_F = N \cap G_F = 1$. Thus S is isomorphic to SN/N and is an elementary abelian q-group. If $q = p$, then S is subnormal in the p-group SN, which is normal in G, and so, by [3, VIII(2.4)(c)], $S \le G_F = 1$, a contradiction. Hence $q \ne p$. Because SN is normal in G, $Y \cap SN$ is some Sylow q-subgroup of SN, which is of the form S^x, where $x \in N$.

Now because V is Σ-permutable, VY is a subgroup of G, and $VY \cap SN = VY \cap S^xN = S^x(VY \cap N)$. Clearly, $VY \cap N \le O_p(VY) \le V$. But $V \cap N = 1$ for the same reason that $S \cap N = 1$, so $VY \cap N = 1$. Hence $VY \cap SN = S^x$. But this means that the F-subgroup S^x is normalised by VY and therefore by V, so $S^x \le V$ because V is a Fischer F-subgroup. Hence $SN = S^xN \le VN$, contradicting our assumption and establishing the theorem. ∎

Theorem 2. *Suppose that G is a finite solvable group and F is a Fitting set of G. If U is a system-permutable Fischer F-subgroup of G, then U is an F-injector of G.*

PROOF. Suppose that G is a minimal counterexample to the theorem, and U is a system-permutable Fischer F-subgroup of G that is not an F-injector of G. Then, by Lemma 1, there exists a maximal subgroup M of G such that $M = UK$, where K is normal in G. Now we know that U is Σ-permutable for some Hall system Σ of G, so Σ reduces into U by [3, I(4.25)(a)], and Σ reduces into $M = UK$ by [3, I(4.17)]. Thus U is $\Sigma \cap M$-permutable in M by [3, I(4.25)(c)], and U is a Fischer F-subgroup of M. By minimality of G, $U \in Inj_F(M)$. If M is not normal in G, $N_G(M) = M$, so U is pronormal in $N_G(M)$ because it is an F-injector of M by [3, VIII(2.14)(a)]. But $M = UK$ is pronormal in G because M is maximal in G, so U being pronormal in $M = N_G(UK)$ implies that U is pronormal in G by [3, I(6.4)]. But then $U \in Inj_F(G)$ by Theorem 1, a contradiction. Hence M is normal in G.

Let $g \in G$ and consider $\langle U, U^g \rangle$. Because $U \in Inj_F(M)$, $U^g \in Inj_F(M^g) = Inj_F(M)$ because M is normal in G. But this means that $U^g = U^m$ for some $m \in M$ by [3, VIII(2.9)]. But U is pronormal in M, so $U^m = U^x$ for some $x \in \langle U, U^m \rangle = \langle U, U^g \rangle$. Hence $U^g = U^x$ for $x \in \langle U, U^g \rangle$, and U is pronormal in G. Thus, by Theorem 1, $U \in Inj_F(G)$, the final contradiction. ∎

ACKNOWLEDGEMENTS

The author wishes to thank Franklin and Marshall College for its generous sabbatical leave policy and the National University of Ireland, Galway, for its warm welcome.

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