A NOTE ON POSITIVE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS AND RELATED BOUNDARY VALUE PROBLEMS

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ABSTRACT

Using Krasnoselskii's fixed point theorem, the existence of positive solutions of the possibly singular integral equation $y(t) = \int_0^T k(t,s)[f(s,y(s)) + g(s,y(s))]\,ds$, $t \in [0,T]$ is discussed. An application to certain types of boundary value problems is also considered.

1. Introduction

In this paper we discuss the existence of continuous, positive solutions of the second-order boundary value problem

$$\begin{cases} -y'' = f(t,y) + g(t,y), \ t \in (0,1) \\ \alpha y(0) - \beta y'(0) = 0 \\ \gamma y(1) + \delta y'(1) = 0 \end{cases} \tag{1.1}$$

where $f : [0,1] \times [0,\infty) \to [0,\infty)$, $g : [0,1] \times (0,\infty) \to [0,\infty)$ and α, β, γ, $\delta \geq 0$ with $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$. In addition, we assume that $g(t,0)$ is possibly undefined. We note that (1.1) with $g \equiv 0$ has over the past fifteen years received some attention in the literature (see [1]–[5] and the references therein). The boundary value problem is of course equivalent to the integral equation

$$y(t) = \int_0^1 G(t,s)[f(s,y(s)) + g(s,y(s))]\,ds, \ t \in [0,1], \tag{1.2}$$

where the Green's function is given by

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{1.3}$$

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Thus the question of proving the existence of positive, continuous solutions of (1.1) reduces to proving the existence of positive, continuous solutions of a Fredholm integral equation—a subject which has recently been of interest to the authors (see [6]–[10]). Therefore, keeping the boundary value problem (1.1) in mind, in Section 2 we examine the possibly singular Fredholm integral equation

$$y(t) = \int_0^T k(t,s)[f(s,y(s)) + g(s,y(s))]\, ds, \ t \in [0,T]. \tag{1.4}$$

By singular we mean that $g : [0,T] \times (0,\infty) \to [0,\infty)$ is such that $g(t,0)$ is undefined. We place quite natural conditions on k, f and g to ensure that (1.4) has at least one positive solution $y \in C[0,T]$. In the first result we assume that (1.4) is singular, while in the second result it is assumed that $g(t,0)$ is defined (allowing less restrictive conditions to be placed on the kernel k).

An immediate application of our result will be to show in Section 3 that under certain conditions on f and g the integral equation (1.2), where G is as defined in (1.3), has at least one positive solution $y \in C[0,1]$. Hence we also have an existence result for the boundary value problem (1.1).

Finally, for our existence theorem we will use the fixed point theorem of Krasnoselskii, which we now state along with the following definition which will be required in the note.

Definition 1.1. Let I_1, I_2 be intervals in $\mathbf{R}$ and let q be such that $1 \leq q \leq \infty$. A function $h : [0,T] \times I_1 \to I_2$ is *L^q-Carathéodory* if the following conditions hold:

(i) the map $t \mapsto h(t,y)$ is measurable for all $y \in I_1$,

(ii) the map $y \mapsto h(t,y)$ is continuous for almost all $t \in [0,T]$,

(iii) for any $r > 0$, there exists $\mu_r \in L^q[0,T]$ such that $|y| \leq r$ implies that $|h(t,y)| \leq \mu_r(t)$ for almost all $t \in [0,T]$.

We say that a function $h : [0,T] \times I_1 \to I_2$ is *Carathéodory* if conditions (i) and (ii) hold.

Theorem 1.1 [Krasnoselskii's fixed point theorem]. *Let E be a Banach space and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$K : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that either

$$\text{(i)} \quad ||Ku|| \leq ||u||, \ u \in C \cap \partial\Omega_1 \ \textit{and} \ ||Ku|| \geq ||u||, \ u \in C \cap \partial\Omega_2$$

or

$$\text{(ii)} \quad ||Ku|| \geq ||u||, \ u \in C \cap \partial\Omega_1 \ \textit{and} \ ||Ku|| \leq ||u||, \ u \in C \cap \partial\Omega_2$$

is true. Then K has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2. Continuous positive solutions of an integral equation

Consider the singular non-linear integral equation

$$y(t) = \int_0^T k(t,s)[f(s,y(s)) + g(s,y(s))]\,ds,\ \ t \in [0,T]. \tag{2.1}$$

We place conditions on k, f and g and apply Krasnoselskii's fixed point theorem to ensure that (2.1) has at least one positive solution $y \in C[0,T]$.

Notation. We use $|.|_0$ to denote the norm on $C[0,T]$, that is, for $y \in C[0,T]$, $|y|_0 := \max_{t\in[0,T]} |y(t)|$. For $y \in L^p[0,T]$ the norm is given by

$$||y||_p := \left(\int_0^T |y(t)|^p\,dt\right)^{\frac{1}{p}} \text{ for } 1 \le p < \infty \text{ and } ||y||_p := \text{ess}\sup_{t\in[0,T]} |y(t)| \text{ for } p = \infty.$$

However, to avoid repetition, throughout this paper for $p = \infty$ we will use $(\int_0^T |y(t)|^p\,dt)^{1/p}$ to denote $||y||_\infty$.

Theorem 2.1. *Let $1 \le p \le \infty$ be a constant and q be such that $1/p+1/q = 1$. Suppose that*

$$\begin{cases} \text{there exists } a \in C[0,T] \text{ and } t^\star \in [0,T] \text{ with } a(t) > 0 \text{ for} \\ \text{a.e. } t \in [0,T] \text{ and } a(t^\star) > 0, \text{ in addition to } \kappa \in L^p[0,T] \\ \text{with } \kappa(t) \ge 0 \text{ a.e. } t \in [0,T] \text{ and } ||\kappa||_p > 0, \text{ such that} \\ a(t)\kappa(s) \le k(t,s) \text{ for all } t \in [0,T],\ \text{a.e. } s \in [0,T], \end{cases} \tag{2.2}$$

$$k_t(s) := k(t,s) \le \kappa(s) \text{ for all } t \in [0,T],\ \text{a.e. } s \in [0,T], \tag{2.3}$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,T] \text{ to } L^p[0,T], \tag{2.4}$$

$$f : [0,T] \times [0,\infty) \to [0,\infty) \text{ is Carathéodory, with } f(t,y) \text{ non-decreasing in } y, \tag{2.5}$$

$$\begin{cases} g : [0,T] \times (0,\infty) \to [0,\infty) \text{ is Carathéodory, with} \\ g(t,0) \text{ undefined, and } g(t,y) \text{ non-increasing in } y, \end{cases} \tag{2.6}$$

$$f(t,y) + g(t,y) > 0 \text{ for } y > 0,\ t \in [0,T], \tag{2.7}$$

$$\int_0^T [f(s,R) + g(s,\tilde{R}a(s))]^q\,ds < \infty \text{ for any constants } R, \tilde{R} > 0, \tag{2.8}$$

$$\text{there exists } R_1 > 0 \text{ such that } 1 < \frac{R_1}{\int_0^T \kappa(s)[f(s,R_1) + g(s,R_1 a(s))]\,ds} \tag{2.9}$$

and

$$\text{there exists } R_2 > 0,\ R_2 \neq R_1,\ \text{such that } 1 > \frac{R_2}{a(t^\star)\int_0^T \kappa(s)[f(s,R_2a(s)) + g(s,R_2)]\,ds} \tag{2.10}$$

hold. Then (2.1) *has at least one* **positive** *solution* $y \in C[0,T]$ *and either*

(A) $0 < R_1 < |y|_0 < R_2$ *and* $y(t) \geq a(t)R_1 \geq 0,\ t \in [0,T]$ *if* $R_1 < R_2$

or

(B) $0 < R_2 < |y|_0 < R_1$ *and* $y(t) \geq a(t)R_2 \geq 0,\ t \in [0,T]$ *if* $R_2 < R_1$

holds.

Proof. Define the operator K by

$$Ky(t) := \int_0^T k(t,s)[f(s,y(s)) + g(s,y(s))]\,ds,\ t \in [0,T]$$

and let the cone C_a be given by

$$C_a := \{y \in C[0,T] : y(t) \geq a(t)|y|_0 \text{ for all } t \in [0,T]\}.$$

In addition, define Ω_{R_1} and Ω_{R_2} by

$$\Omega_{R_1} := \{y \in C[0,T] : |y|_0 < R_1\}$$

and

$$\Omega_{R_2} := \{y \in C[0,T] : |y|_0 < R_2\}$$

respectively, and suppose in what follows that $R_2 < R_1$. (A similar argument holds if $R_1 < R_2$.) Since we intend to apply Krasnoselskii's fixed point theorem we first show that

$$K : C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \to C_a \text{ is a compact operator} \tag{2.11}$$

(and hence completely continuous).

Let $y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$. Then

$$0 < R_2 \leq |y|_0 \leq R_1 \text{ and } 0 < R_2a(t) \leq y(t) \leq R_1,\ \text{a.e. } t \in [0,T]$$

and (2.5) and (2.6) yield

$$0 < f(t,y(t)) + g(t,y(t)) \leq f(t,R_1) + g(t,R_2a(t)) \text{ a.e. } t \in [0,T].$$

(Note that since $a(t) > 0$, a.e. $t \in [0,T]$ and hence $0 < R_2a(t) \leq y(t)$, a.e. $t \in [0,T]$, we have that $g(t,y(t))$ is measurable for a.e. $t \in [0,T]$.) From this inequality and

(2.8) we obtain

$$\int_0^T [f(s,y(s)) + g(s,y(s))]^q \, ds \le \int_0^T [f(s,R_1) + g(s,R_2 a(s))]^q \, ds < \infty, \tag{2.12}$$

that is, $f(t,y(t)) + g(t,y(t)) \in L^q[0,T]$ for $y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$. Now for $t, t' \in [0,T]$ we have from (2.3) and (2.12) that

$$|Ky(t) - Ky(t')| \le \left(\int_0^T |k_t(s) - k_{t'}(s)|^p \, ds \right)^{\frac{1}{p}} \left(\int_0^T [f(s,R_1) + g(s,R_2 a(s))]^q \, ds \right)^{\frac{1}{q}}$$

holds, and hence from (2.4) we see that

$$|Ky(t) - Ky(t')| \to 0 \text{ as } t \to t' \text{ for } y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \tag{2.13}$$

is true. Consequently

$$K : C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \to C[0,T] \text{ is well defined.}$$

In addition, for $y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$, condition (2.2) yields

$$Ky(t) \ge a(t) \int_0^T \kappa(s)[f(s,y(s)) + g(s,y(s))] \, ds, \; t \in [0,T],$$

while (2.3) ensures that

$$|Ky|_0 \le \int_0^T \kappa(s)[f(s,y(s)) + g(s,y(s))] \, ds, \; t \in [0,T].$$

Combining both of these inequalities we see that for $y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$,

$$Ky(t) \ge a(t)|Ky|_0$$

and therefore

$$K : C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \to C_a \text{ is well defined.}$$

We next use the Arzéla–Ascoli Theorem to prove that the operator $K : C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \to C_a$ is compact. For $y \in C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$, we obtain from (2.3) and (2.12)

$$|Ky|_0 \le ||\kappa||_p \left(\int_0^T [f(s,R_1) + g(s,R_2 a(s))]^q \, ds \right)^{\frac{1}{q}} \equiv M < \infty$$

—thus $K(C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}))$ is uniformly bounded. In addition, (2.13) immediately guarantees the equicontinuity of $K(C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}))$. Therefore the compactness of $K : C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2}) \to C_a$ now follows from the Arzéla–Ascoli Theorem. In conclusion we have shown that (2.11) is true.

To apply Krasnoselskii's fixed point theorem it remains to show that

$$|Ky|_0 < |y|_0 \text{ for } y \in C_a \cap \partial\Omega_{R_1} \tag{2.14}$$

and

$$|Ky|_0 > |y|_0 \text{ for } y \in C_a \cap \partial\Omega_{R_2} \tag{2.15}$$

hold. Let $y \in C_a \cap \partial\Omega_{R_1}$. Then $|y|_0 = R_1$ and

$$0 \le a(t)R_1 \le y(t) \le R_1, \ t \in [0, T],$$

therefore by (2.3), (2.5), (2.6) and (2.9)

$$|Ky|_0 \le \int_0^T \kappa(s)[f(s, y(s)) + g(s, y(s))]\, ds \le \int_0^T \kappa(s)[f(s, R_1) + g(s, R_1 a(s))]\, ds < R_1 = |y|_0.$$

Hence (2.14) is true.

Now let $y \in C_a \cap \partial\Omega_{R_2}$. Then $|y|_0 = R_2$ and in addition

$$0 \le a(t)R_2 \le y(t) \le R_2, \ t \in [0, T].$$

Then from (2.2), (2.5), (2.6) and (2.10) we obtain

$$\begin{aligned} |Ky|_0 \ge Ky(t^\star) &\ge a(t^\star)\int_0^T \kappa(s)[f(s, y(s)) + g(s, y(s))]\, ds \\ &\ge a(t^\star)\int_0^T \kappa(s)[f(s, R_2 a(s)) + g(s, R_2)]\, ds > R_2 = |y|_0. \end{aligned}$$

Thus (2.15) holds.

With the conditions of Krasnoselskii's fixed point theorem satisfied, we are guaranteed that the operator K has a fixed point in $C_a \cap (\overline{\Omega}_{R_1} \backslash \Omega_{R_2})$, that is, (2.1) has a positive solution $y \in C[0, T]$ with $0 < R_2 < |y|_0 < R_1$ and $y(t) \ge R_2 a(t) \ge 0$, $t \in [0, T]$. (It is clear that an analogous result holds if $R_1 < R_2$.) ∎

Remark 2.1. Note that the solution $y \in C[0, T]$ of (2.1) guaranteed by Theorem 2.1 is positive whenever $a \in C[0, T]$ is, since either

$$y(t) \ge R_1 a(t) \ge 0, \ t \in [0, T], \text{ or } y(t) \ge R_2 a(t) \ge 0, \ t \in [0, T]$$

is true. Clearly, then, if a is strictly positive on $[0, T]$, that is, $a(t) > 0$ for $t \in [0, T]$, then the solution too must be strictly positive on $[0, T]$.

The assumption that $g : [0, T] \times (0, \infty) \to [0, \infty)$ is such that $g(t, 0)$ is undefined in Theorem 2.1 forces $a \in C[0, T]$ in (2.2) to be strictly positive for a.e. $t \in [0, T]$. However, this assumption on $a \in C[0, T]$ can be relaxed if $g(t, 0)$ is defined for a.e. $t \in [0, T]$. We just state the result as the proof is almost identical to that of the proof of Theorem 2.1.

Theorem 2.2. *Let* $1 \leq p \leq \infty$ *be a constant and* q *be such that* $1/p + 1/q = 1$. *Suppose that* (2.3), (2.4), (2.9), (2.10),

$$\begin{cases} \text{there exists } a \in C[0,T] \text{ and } t^\star \in [0,T] \text{ with } a(t) \geq 0, \\ t \in [0,T] \text{ and } a(t^\star) > 0, \text{ in addition to } \kappa \in L^p[0,T] \\ \text{with } \kappa(t) \geq 0 \text{ a.e. } t \in [0,T] \text{ and } \|\kappa\|_p > 0, \text{ such that} \\ a(t)\kappa(s) \leq k(t,s) \text{ for all } t \in [0,T], \text{ a.e. } s \in [0,T], \end{cases} \tag{2.16}$$

$$f : [0,T] \times [0,\infty) \to [0,\infty) \text{ is } L^q\text{-Carathéodory, with } g(t,y) \text{ non-decreasing in } y, \tag{2.17}$$

$$g : [0,T] \times [0,\infty) \to [0,\infty) \text{ is } L^q\text{-Carathéodory, with } g(t,y) \text{ non-increasing in } y, \tag{2.18}$$

and

$$\begin{cases} \displaystyle\int_0^T \kappa(s)[f(s,R) + g(s,Ra(s))]\,ds > 0 \text{ and} \\ \displaystyle\int_0^T \kappa(s)[f(s,Ra(s)) + g(s,R)]\,ds > 0 \text{ for any } R > 0 \end{cases} \tag{2.19}$$

hold. Then (2.1) *has at least one* **positive** *solution* $y \in C[0,T]$ *and either*

(A) $0 < R_1 < |y|_0 < R_2$ *and* $y(t) \geq a(t)R_1 \geq 0,\ t \in [0,T]$ *if* $R_1 < R_2$

or

(B) $0 < R_2 < |y|_0 < R_1$ *and* $y(t) \geq a(t)R_2 \geq 0,\ t \in [0,T]$ *if* $R_2 < R_1$

holds.

Remark 2.2. It may not be immediately obvious what type of functions f and g satisfy conditions (2.9) and (2.10) of Theorem 2.1. In fact these conditions are quite general and are satisfied by a fairly large class of functions. We illustrate below with some examples. While we will discuss examples of functions f and g which satisfy (2.9) and (2.10), to simplify the examples and avoid getting lost in details we will assume that k satisfies the hypotheses of the following special case of Theorem 2.1.

The following result is a special case of Theorem 2.1 where $a \in C[0,T]$ is strictly positive and $f : [0,T] \times [0,\infty) \to [0,\infty)$ and $g : [0,T] \times (0,\infty) \to [0,\infty)$ are both continuous.

Theorem 2.3. *Suppose that* (2.3), (2.4) *with* $p = 1$, *and* (2.9) *hold and suppose also that there exists* $t^\star \in [0,T]$ *such that* (2.10) *is true. In addition, assume that*

$$\begin{cases} \text{there exists } a \in C[0,T] \text{ with } a(t) > 0 \text{ for } t \in [0,T], \text{ in addition to} \\ \kappa \in L^1[0,T] \text{ with } \kappa(t) \geq 0 \text{ a.e. } t \in [0,T] \text{ and } \|\kappa\|_1 > 0 \text{ such that} \\ a(t)\kappa(s) \leq k(t,s) \text{ for all } t \in [0,T], \text{ a.e. } s \in [0,T] \end{cases} \tag{2.20}$$

and

$$\begin{cases} f : [0,T] \times [0,\infty) \to [0,\infty) \text{ is continuous, with } f(t,y) \text{ non-decreasing in } y, \\ g : [0,T] \times (0,\infty) \to [0,\infty) \text{ is continuous, with } g(t,y) \text{ non-increasing in } y \\ \text{and } f(t,y) + g(t,y) > 0 \text{ for } y > 0,\ t \in [0,T] \end{cases} \tag{2.21}$$

hold. Then (2.1) *has at least one* **strictly positive** *solution* $y \in C[0,T]$ *and either*

(A) $0 < R_1 < |y|_0 < R_2$ *and* $y(t) \geq a(t)R_1 > 0,\ t \in [0,T]$ *if* $R_1 < R_2$

or

(B) $0 < R_2 < |y|_0 < R_1$ *and* $y(t) \geq a(t)R_2 > 0,\ t \in [0,T]$ *if* $R_2 < R_1$

holds.

Proof. Conditions (2.2)–(2.7) and (2.9)–(2.10) of Theorem 2.1 hold with $p = 1$. The strict positivity of $a \in C[0,T]$, along with the fact that f and g are continuous, implies that (2.8) holds with $q = \infty$. The result now follows from Theorem 2.1. ∎

In the following three examples we assume for simplicity that a and κ are as described in Theorem 2.3.

Example 2.1. Suppose that $f(t,y) = y^n$, $n \geq 0$ and $g \equiv 0$. Clearly (2.21) is satisfied. Fix $t^\star \in [0,T]$ and note that since $0 < a(t) \leq 1$, we have, for all $n \geq 0$,

$$a(t^\star) \int_0^T \kappa(s) a^n(s)\, ds \leq ||\kappa||_1 = \int_0^T \kappa(s)\, ds.$$

First, assume that $0 \leq n < 1$, that is, f exhibits sublinear growth. Then (2.9) and (2.10) are satisfied with $R_2 < R_1$ such that

$$0 < R_2^{1-n} < a(t^\star) \int_0^T \kappa(s) a^n(s)\, ds \leq ||\kappa||_1 < R_1^{1-n}.$$

Alternatively, assume that f is superlinear, that is, that $f(y) = y^n$ where $n > 1$. Now (2.9) and (2.10) are satisfied with $R_1 < R_2$ such that

$$R_1^{n-1} < \frac{1}{||\kappa||_1} \leq \frac{1}{a(t^\star) \displaystyle\int_0^T \kappa(s) a^n(s)\, ds} < R_2^{n-1}.$$

Example 2.2. Suppose that $f \equiv 0$ and $g(t,y) = y^{-n}$, $n \geq 0$. It is easy to see that (2.21) holds. Fix $t^\star \in [0,T]$ and observe that

$$a(t^\star)||\kappa||_1 \leq ||\kappa||_1 = \int_0^T \kappa(s)\, ds \leq \int_0^T \kappa(s) a^{-n}(s)\, ds.$$

On this occasion (2.9) and (2.10) are satisfied with $R_2 < R_1$ such that

$$R_2^{1+n} < a(t^\star)||\kappa||_1 \le \int_0^T \kappa(s)a^{-n}(s)\,ds < R_1^{1+n}.$$

Example 2.3. Let $f(t,y) = 1 + y^m + y^n$, $0 \le m < 1 < n$ and $g \equiv 0$. Fix $t^\star \in [0,T]$. Since

$$\frac{R}{a(t^\star)\displaystyle\int_0^T \kappa(s)[1 + a^m(s)R^m + a^n(s)R^n]\,ds} \to 0 \text{ as } R \to 0^+ \text{ and } R \to \infty,$$

there exists $0 < r < \tilde{r}$ such that (2.10) is satisfied with both $R_2 = R_2'$ and $R_2 = R_2''$ where $R_2' \in (0,r)$ and $R_2'' \in (\tilde{r},\infty)$. In addition, if

$$\sup_{R\in[0,\infty)} \frac{R}{||\kappa||_1[1 + R^m + R^n]} > 1$$

then there exists $R_1 > 0$ which satisfies (2.9). Here $0 < R_2' < R_1 < R_2''$.

Remark 2.3. One advantage of the theorems in this section is that in certain cases it may be possible to apply the theorem repeatedly to an integral equation to yield *multiple* positive solutions. For example, suppose that the kernel k satisfies the hypotheses of Theorem 2.3 and the functions f and g are as described above in Example 2.3. Then one application of Theorem 2.3 will yield a positive solution $y_1 \in C[0,T]$ of (2.1) where

$$0 < R_2' < |y_1|_0 < R_1 \text{ and } 0 < a(t)R_2' \le y_1(t) < R_1,\ t \in [0,T],$$

while a second application will guarantee the existence of a solution $y_2 \in C[0,T]$ such that

$$0 < R_1 < |y_2|_0 < R_2'' \text{ and } 0 < a(t)R_1 \le y_2(t) < R_2'',\ t \in [0,T].$$

3. An application

In this section we apply Theorem 2.1 to prove that, under certain conditions on f and g, the boundary value problem

$$\begin{cases} -y'' = f(t,y) + g(t,y),\ t \in (0,1) \\ \alpha y(0) - \beta y'(0) = 0 \\ \gamma y(1) + \delta y'(1) = 0, \end{cases} \tag{3.1}$$

similar to that discussed in [3]–[5], has one (or more) continuous, positive solution(s). Here α, β, γ, $\delta \ge 0$ and

$$\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0. \tag{3.2}$$

Since the boundary value problem is equivalent to the integral equation

$$y(t) = \int_0^1 G(t,s)[f(s,y(s)) + g(s,y(s))]\,ds, \tag{3.3}$$

where the Green's function is given by

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\gamma+\delta-\gamma t)(\beta+\alpha s), & 0 \le s \le t \le 1 \\ (\beta+\alpha t)(\gamma+\delta-\gamma s), & 0 \le t \le s \le 1, \end{cases} \tag{3.4}$$

we show that under certain hypotheses on f and g the conditions of Theorem 2.1 are satisfied with

$$k(t,s) := G(t,s),$$

thus giving the desired result for (3.1) and (3.3).

We concentrate on the kernel and show that $k(t,s) = G(t,s)$ satisfies (2.2)–(2.4) with $p = \infty$ and $T = 1$. As in [3], for convenience we let

$$\phi(t) := \gamma+\delta-\gamma t \quad \text{and} \quad \psi(t) := \beta+\alpha t,\ t \in [0,1].$$

Since ϕ is non-increasing, while ψ is non-decreasing, we obtain

$$k(t,s) \le k(s,s),\ 0 \le t,s \le 1.$$

Define $\kappa \in C[0,1]$ by

$$\kappa(s) := k(s,s) = G(s,s),\ s \in [0,1] \tag{3.5}$$

and observe that

$$\kappa(s) = k(s,s) > 0 \text{ for } s \in (0,1).$$

Moreover, for $t \in (0,1)$ we see that

$$\frac{k(t,s)}{\kappa(s)} = \frac{k(t,s)}{k(s,s)} = \begin{cases} \dfrac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s}, & 0 \le s \le t < 1 \\ \dfrac{\beta+\alpha t}{\beta+\alpha s}, & 0 < t \le s \le 1 \end{cases} \ge \begin{cases} \dfrac{\gamma+\delta-\gamma t}{\gamma+\delta}, & 0 \le s \le t < 1 \\ \dfrac{\beta+\alpha t}{\beta+\alpha}, & 0 < t \le s \le 1. \end{cases} \tag{3.6}$$

Therefore, defining $a \in C[0,1]$ by

$$a(t) := \frac{(\gamma+\delta-\gamma t)(\beta+\alpha t)}{(\gamma+\delta)(\beta+\alpha)} = \frac{\phi(t)\psi(t)}{\eta},\ t \in [0,1],\ (\text{where } \eta := (\gamma+\delta)(\beta+\alpha)), \tag{3.7}$$

we see from (3.6) that

$$k(t,s) \ge a(t)\kappa(s),\ 0 \le t,s \le 1.$$

Furthermore, one can verify that

$$a(t) > 0 \text{ for } t \in (0,1).$$

We have thus shown that

$$\begin{cases} \text{there exists } a \in C[0,1] \text{ with } a(t) > 0 \text{ for } t \in (0,1), \\ \text{in addition to } \kappa \in C[0,1] \text{ with } \kappa(t) > 0 \text{ for } t \in (0,1), \\ \text{such that } a(t)\kappa(s) \le k(t,s) \le \kappa(s) \text{ for all } t \in [0,1],\ s \in [0,1]. \end{cases}$$

Thus $k(t,s) = G(t,s)$ satisfies (2.2) and (2.3) with $p = \infty$ and $T = 1$. Finally, it is clear that (2.4) holds (again with $p = \infty$ and $T = 1$).

In addition, if we assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times (0,\infty) \to [0,\infty)$ are such that (2.5)–(2.10) with $q = 1$ and $T = 1$ are true (see previous section for examples), then we have the following result for (3.3) (and hence (3.1)).

Theorem 3.1. *Suppose that*

$$\begin{cases} G : [0,1] \times [0,1] \to [0,\infty) \textit{ is as defined in } (3.4) \\ \textit{where } \alpha,\ \beta,\ \gamma,\ \delta \ge 0 \textit{ and } \rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0 \end{cases} \tag{3.8}$$

holds and $f : [0,1] \times [0,\infty) \to [0,\infty)$ *and* $g : [0,1] \times (0,\infty) \to [0,\infty)$ *satisfy* (2.5)–(2.10) *with* $q = 1$ *and* $T = 1$, *where* $\kappa \in C[0,1]$ *and* $a \in C[0,1]$ *are as described in* (3.5) *and* (3.7) *respectively. Then* (3.3) *has at least one* **positive** *solution* $y \in C[0,T]$ *and either*

(A) $0 < R_1 < |y|_0 < R_2$ *and* $y(t) \ge a(t)R_1 \ge 0,\ t \in [0,T]$ *if* $R_1 < R_2$

or

(B) $0 < R_2 < |y|_0 < R_1$ *and* $y(t) \ge a(t)R_2 \ge 0,\ t \in [0,T]$ *if* $R_2 < R_1$

holds.

PROOF. The result follows from the above analysis and Theorem 2.1. ■

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