GENERALISED MATRIX C^*-ALGEBRAS AND REPRESENTATIONS OF HILBERT MODULES

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Abstract

This paper attempts to provide an introduction to Hilbert modules easily accessible to anyone with an elementary background in operator algebras. Our goal is to point out that von Neumann modules are those Hilbert modules which behave best compared with Hilbert spaces. Most results in the theory of operators on Hilbert spaces remain valid when translated into the language of operators on von Neumann modules.

1. Introduction

The concept of Hilbert modules was introduced by Paschke [9] and Rieffel [11] for the first time in a non-commutative context. Hilbert modules are a straightforward generalisation of Hilbert spaces where the scalar field $\mathbb{C}$ is replaced by a C^*-algebra (see Definition 3.1). The origin of Hilbert modules is in operator theory, where they constitute an important tool in areas like KK-theory, quantum groups and several other areas; see e.g. [6], the first book devoted entirely to Hilbert modules.

Hilbert modules arise naturally by generalising the GNS-construction based on a state to a GNS-construction based on a completely positive mapping (see Section 7). In this context Hilbert modules also entered quantum probability in different contexts. On the one hand, there is Voiculescu's *operator-valued free probability* [19], which generalises the notion of free independence of states to the notion of free independence of conditional expectations. The (operator-valued) central limit distribution is described best by moments of creation and annihilation operators on a full Fock module (instead of a full Fock space) introduced by Pimsner [10] (see [18]). On the other hand, any system where a small system (e.g. an electron or some initial space) is coupled to a big system (e.g. a field or a noise space) with a distinguished state on the big system (e.g. the vacuum state) gives rise to a natural conditional expectation (e.g. the vacuum conditional expectation). It turns out (see [16]) that the GNS-modules of such conditional expectations belong to a particularly well-behaved subcategory of the two-sided Hilbert modules, namely the so-called *centred Hilbert modules*. (In [16] we introduced this subcategory because the construction of a *symmetric* Fock module is, in general, possible only within this subcategory.) In Section 6 we will have a glance at the well-behavedness of the centred Hilbert modules. The two areas of operator-valued probability and centred Hilbert modules are glued together by the *stochastic limit* for a free electron in quantum electrodynamics investigated by Accardi and Lu [1]. In [16] we pointed out that the stochastic limit of the distribution of the field operators (being operators on a centred Hilbert module) may be interpreted as the distribution in the vacuum conditional expectation of creators and annihilators on a (non-centred) full Fock module.

The goal of this paper is to introduce Hilbert modules as subspaces of C^*-algebras. To this end, in Section 2 we introduce a generalisation of the algebra $M_n(\mathscr{B})$ of matrices with entries in a fixed C^*-algebra $\mathscr{B}$. In this generalisation the entries of a fixed place in the diagonal still form a C^*-algebra. This C^*-algebra, however, may depend on the place in the diagonal. Consequently, the off-diagonal entries form two-sided modules over the diagonal entries and carry a natural Hilbert module structure.

In Section 3 we show that any Hilbert module may be recovered as an off-diagonal element of a suitable generalised 2×2-matrix C^*-algebra. It is not difficult to find the algebraic structure of this generalised matrix C^*-algebra and, in fact, as mentioned by Blecher [4], it is easy to find a complete C^*-norm on this generalised matrix algebra; see Remark 8.6 below. However, since we have in mind a generalisation of this result (Theorem 3.11), we prefer to show existence of this C^*-norm by constructing a faithful representation. As in the case of $M_n(\mathscr{B})$, the faithful representation is constructed by extending a faithful representation of a diagonal entry to the whole of the matrix algebra.

Identifying a Hilbert module with its image under the faithful representation, we recover a result obtained by Murphy [8] (by an application of the Kolmogorov decomposition for a positive definite kernel) according to which any Hilbert module E may be realised as a submodule of the module $\mathscr{B}(G, H)$ of bounded operators *between* two Hilbert spaces G and H. (This generalises the well-known fact that any C^*-algebra may be represented as an algebra of bounded operators *on* a Hilbert space.) Going beyond Murphy (and Blecher), we show that H carries a faithful representation not only of the C^*-algebra of adjointable module homomorphisms, but also of the Banach algebra of all bounded module homomorphisms (Theorem 3.11). As an application we find that the set of *all* $\mathscr{B}$-functionals (not only those determined by elements of E) may be represented in $\mathscr{B}(H, G)$ (Corollary 3.14). This result is essential for all statements in this paper concerning von Neumann modules.

Even nowadays von Neumann algebras are usually introduced as strongly closed operator algebras. The abstract characterisation due to Sakai [13] (based on the σ-weak topology) in practice is used rather rarely. Having at hand that any Hilbert module has a representation as a concrete submodule of $\mathscr{B}(G, H)$, it is natural to introduce von Neumann modules as strongly closed submodules of $\mathscr{B}(G, H)$. Embedding $\mathscr{B}(G, H)$ into $\mathscr{B}(G \oplus H)$ results well known for von Neumann algebras such as *polar decomposition* and the *Kaplansky density theorem* are immediate also for von Neumann modules.

In Section 4 we establish the main properties of von Neumann modules. In particular, von Neumann modules are self-dual (i.e. the Riesz representation theorem for $\mathscr{B}$-functionals holds). In Section 5 we supplement this by the result that a Hilbert module over a von Neumann algebra is self-dual, if and only if it admits a representation as a von Neumann module (Theorem 5.5). Theorems of Hahn–Banach type and a spectral theorem are also established. The criterion for self-duality of Hilbert modules over von Neumann algebras seems to be new and presents a considerable extension of well-known results, whereas the Hahn–Banach theorems are special cases of results already due to Paschke [9]. We remark that all these results are obtained by establishing a suitable substitute for an orthonormal base in a usual Hilbert space and then applying the well-known Hilbert space techniques. In this way von Neumann modules are shown to be those Hilbert modules which are most similar to Hilbert spaces.

Section 6 is devoted to centred Hilbert modules. In Section 7 we point out that the Stinespring construction is a corollary of Paschke's GNS-construction [9] and our represention of Hilbert modules. In Appendices A and B we investigate representations of Hilbert modules intrinsically and recall a trivial but useful result on semi-norms.

Throughout this paper we intended to be as elementary as possible. Also for this reason we did not hesitate to use such 'unabstract' things as orthonormal bases. We only assume the knowledge of a first course in C^*-algebra and operator theory (for instance, [7, chapters 1–5]). We avoided the use of σ-topologies, and some interesting results concerning tensor products are only quoted in Section 8 because they require further knowledge of completely positive mappings. We postponed to Section 8 any quotation of the contributions of other authors. We apologise in advance to any authors whom we did not mention sufficiently. It seems impossible to give a list of references which would be even approximately complete.

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2. Generalised matrix C^*-algebras

Definition 2.1. Let $\mathscr{M}$ be a *-algebra with subspaces B_{ij} $(i, j = 1, 2)$ such that

$$\mathscr{M} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \left(\text{i.e. } \mathscr{M} = \bigoplus_{i,j=1}^{2} B_{ij} \right).$$

We say that $\mathscr{M}$ is a *generalised matrix *-algebra* (of order 2) if multiplication and involution of $\mathscr{M}$ are compatible with the usual matrix operations, i.e. if

$$(bb')_{ij} = \sum_{k=1}^{2} b_{ik} b'_{kj}$$

and

$$(b^*)_{ij} = b^*_{ji}$$

for all elements $b = (b_{ij})$ and $b' = (b'_{ij})$ in $\mathscr{M}$. If $\mathscr{M}$ is also a (pre-)C^*-algebra, then we call $\mathscr{M}$ a *generalised matrix (pre-)C^*-algebra* (of order 2).

A *generalised matrix *-subalgebra* of $\mathscr{M}$ is a collection of subspaces $C_{ij} \subset B_{ij}$, such that $\mathscr{N} = (C_{ij})$ is a *-subalgebra of $\mathscr{M}$.

Example 2.2. If $\mathscr{B}$ is a C^*-algebra and $B_{ij} = \mathscr{B}$, we recover the usual matrix C^*-algebra $M_2(\mathscr{B})$.

In what follows we will omit the word 'generalised' and speak simply of matrix algebras. If we refer to $M_n(\mathscr{B})$, we say a 'usual matrix algebra'.

Example 2.3. Let $H = H_1 \oplus H_2$ be a Hilbert space. We may decompose $\mathscr{B}(H)$ according to the subspaces H_1 and H_2. Clearly,

$$\mathscr{B}(H_1 \oplus H_2) = \begin{pmatrix} \mathscr{B}(H_1, H_1) & \mathscr{B}(H_2, H_1) \\ \mathscr{B}(H_1, H_2) & \mathscr{B}(H_2, H_2) \end{pmatrix}$$

is a matrix C^*-algebra.

On the other hand, if Π is a (non-degenerate) *-representation of a matrix *-algebra $\mathscr{M}$ by bounded operators on a Hilbert space H, then it is easy to check that H decomposes into the subspaces $H_i = \overline{\text{span}}(\Pi(B_{ii})H)$ and that $\Pi(B_{ij}) \subset \mathscr{B}(H_j, H_i)$. Clearly,

$$\Pi(\mathscr{M}) = \begin{pmatrix} \Pi(B_{11}) & \Pi(B_{12}) \\ \Pi(B_{21}) & \Pi(B_{22}) \end{pmatrix}$$

is a matrix pre-C^*-subalgebra of $\mathscr{B}(H_1 \oplus H_2)$.

Proposition 2.4. *Let $\mathscr{M}$ be a matrix pre-C^*-algebra. $\mathscr{M}$ is complete if and only if each B_{ij} is complete with respect to the norm induced by $\mathscr{M}$.*

PROOF. Let $b \in \mathscr{M}$ with components $b_{ij} \in B_{ij}$. Then bb^* has the components $\sum_{k=1}^{2} b_{ik}b_{jk}^* \in B_{ij}$. On the other hand, if $c \in \mathscr{M}$ and $b_{ij} \in B_{ij}$, then $b_{ij}^* c b_{ij} = b_{ij}^* c_{ii} b_{ij} \in B_{jj}$, where c_{ii} is the component of c in B_{ii}. Combining both we find

$$b_{ij}^* bb^* b_{ij} = \sum_{k=1}^{2} b_{ij}^* b_{ik} b_{ik}^* b_{ij} = \sum_{k=1}^{2} (b_{ik}^* b_{ij})^* b_{ik}^* b_{ij} \geq (b_{ij}^* b_{ij})^* b_{ij}^* b_{ij},$$

so that

$$||b||\,||b_{ij}|| \geq ||b^* b_{ij}|| \geq ||b_{ij}^* b_{ij}|| = ||b_{ij}||^2.$$

This implies that $||b|| \geq ||b_{ij}||$ whether $b = 0$ or not, and, of course, $\sum_{i,j=1}^{2} ||b_{ij}|| \geq ||b||$. We conclude that a sequence $\big(b_n\big)_{n\in\mathbb{N}}$ in $\mathscr{M}$ is a Cauchy sequence if and only if each component $\big((b_n)_{ij}\big)_{n\in\mathbb{N}}$ forms a Cauchy sequence in B_{ij}. Obviously, if any B_{ij} is complete, then so is $\mathscr{M}$. On the other hand, if a Cauchy sequence in B_{ij} converges in $\mathscr{M}$, then the limit must be an element of B_{ij}, so that completeness of $\mathscr{M}$ implies completeness of B_{ij}. ■

Corollary 2.5. *The projections $\mathcal{P}_{ij}\colon \mathscr{M} \to B_{ij}, \mathcal{P}_{ij}(b) = b_{ij}$ have norm 1.*

Corollary 2.6. *Let $(u_\lambda)_{\lambda\in\Lambda}$ denote an approximate unit for $\mathscr{M}$. Set $u_\lambda^{ij} = \mathcal{P}_{ij}(u_\lambda)$. Then*

$$\lim_\lambda u_\lambda^{k\ell} b_{ij} = \delta_{k\ell}\delta_{\ell i} b_{ij} \quad \text{and} \quad \lim_\lambda b_{ij} u_\lambda^{k\ell} = \delta_{jk}\delta_{k\ell} b_{ij} \tag{2.1}$$

for all $b_{ij} \in B_{ij}$. Moreover, $(u_\lambda^{11})_{\lambda\in\Lambda}$, $(u_\lambda^{22})_{\lambda\in\Lambda}$ and $(u_\lambda^{11} + u_\lambda^{22})_{\lambda\in\Lambda}$ form approximate units for B_{11}, B_{22} and $\mathscr{M}$ respectively.

PROOF. Equation (2.1) follows from $(bc)_{ij} = \sum_{k=1}^{2} \mathcal{P}_{ik}(b)\mathcal{P}_{kj}(c)$ $(b, c \in \mathscr{M})$ and continuity of $\mathcal{P}_{ij}$. To see also that the net $(u_\lambda^{ii})_{\lambda\in\Lambda}$ is increasing we observe that $\mathcal{P}_{ii}$ is a positive mapping, because by (2.1) we find $\mathcal{P}_{ii}(b) = \lim_\lambda u_\lambda^{ii} b u_\lambda^{ii}$. ■

Definition 2.7. A *matrix von Neumann algebra on $H_1 \oplus H_2$* is a strongly (or weakly) closed *-subalgebra $\mathscr{M}$ of $\mathscr{B}(H_1 \oplus H_2)$. Clearly, $\mathscr{M}$ is a von Neumann algebra on $H_1 \oplus H_2$. In particular, $\mathscr{M}$ is unital and the unit of $\mathscr{M}$ is the sum of the units p_i of the diagonal von Neumann subalgebras B_{ii}.

Proposition 2.8. *Let $\mathscr{M} = (B_{ij})$ be a matrix pre-C^*-subalgebra of the von Neumann algebra $\mathscr{B}(H_1 \oplus H_2)$. Then $\mathscr{M}$ is strongly (weakly) closed if and only if each B_{ij} is strongly (weakly) closed in $\mathscr{B}(H_j, H_i)$.*

PROOF. Obvious, because $\mathscr{B}(H_j, H_i)$ is strongly and weakly closed in $\mathscr{B}(H_1 \oplus H_2)$. ■

Proposition 2.9. *Let $\mathscr{M} = (B_{ij})$ be a strongly dense matrix pre-C^*-subalgebra of a matrix von Neumann algebra $\mathscr{M}^{vN} = (B_{ij}^{vN})$. Then the unit-ball of B_{ij} is strongly dense in the unit-ball of B_{ij}^{vN}.*

PROOF. Let b denote an element of B_{ij}^{vN}. By the *Kaplansky density theorem*, we may approximate b strongly by a net $(b_n)_{n\in\mathcal{N}}$ of elements in the unit-ball of $\mathscr{M}$. Then $(\mathcal{P}_{ij}(b_n))_{n\in\mathcal{N}} = (p_i b_n p_j)_{n\in\mathcal{N}}$ is a net consisting of elements in the unit-ball of B_{ij} which converges strongly to b. ■

Proposition 2.10. *Let $\mathscr{M}$ be a matrix von Neumann algebra on $H_1 \oplus H_2$ and b an element of B_{ij}. Denote $|b| = \sqrt{b^*b}$. There exists a unique partial isometry v in B_{ij} such that*

$$b = v|b| \quad \text{and} \quad \ker(v) = \ker(b).$$

PROOF. By *polar decomposition* there exists a unique partial isometry v in $\mathscr{M}$ with the claimed properties. Obviously, v vanishes on $H_j^\perp$ and its range is contained in H_i. This means that $v = \mathcal{P}_{ij}(v) \in B_{ij}$. ■

Remark 2.11. We observe that B_{21} is a right B_{11}-module. The elements of B_{22} act on B_{21} from the left as right module homomorphisms. Moreover, they are *adjointable* in the sense that $b_{21}^*(b_{22}b'_{21}) = (b_{22}^*b_{21})^*b'_{21}$. We will see that, if B_{11} is a pre-C^*-algebra, then B_{21} is a pre-Hilbert module (see Definition 3.1). Our aim is to recover an arbitrary Hilbert module as an off-diagonal element B_{21} of a suitable matrix C^*-algebra. Therefore, in the study of matrix *-algebras we put some emphasis on the element B_{21} and use the asymmetric notation

$$\mathscr{M} = \begin{pmatrix} \mathscr{B} & E^* \\ E & \mathscr{A} \end{pmatrix}.$$

The elements of E^* are denoted by x^* where x is a unique element of E. We call E the *lower submodule* of $\mathscr{M}$.

3. Faithful representations of Hilbert modules

Definition 3.1. Let $\mathscr{B}$ denote a pre-C^*-algebra. A *pre-Hilbert $\mathscr{B}$-module* is a right $\mathscr{B}$-module E with a sesquilinear inner product $\langle \bullet, \bullet \rangle : E \times E \to \mathscr{B}$ fulfilling

$$\langle x, yb \rangle = \langle x, y \rangle b \tag{3.1a}$$

$$\langle x, x \rangle \geq 0 \tag{3.1b}$$

$$\langle x, x \rangle = 0 \Longleftrightarrow x = 0 \tag{3.1c}$$

for all $x, y \in E$ and $b \in \mathscr{B}$.

Proposition 3.2. *We have*

$$\langle x, y \rangle = \langle y, x \rangle^* \tag{3.2a}$$

$$\langle xb, y \rangle = b^* \langle x, y \rangle. \tag{3.2b}$$

Proof. (3.2b) follows from (3.2a) and (3.2a) follows from an investigation of $\langle x + \lambda y, x + \lambda y \rangle$ for $\lambda = 1, i, -1, -i$. ∎

Proposition 3.3. *We have*

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \, \langle x, x \rangle. \tag{3.3a}$$

Consequently,

$$\|x\| = \sqrt{\|\langle x, x \rangle\|} \tag{3.3b}$$

defines a norm on E.

Proof. Suppose that $y \neq 0$. Then (3.3a) follows by an investigation of the length $|z| = \sqrt{\langle z, z \rangle}$ of the element $z = x - \frac{y\langle y, x \rangle}{\|\langle y, y \rangle\|}$. If $y = 0$, (3.3a) is fulfilled trivially. ∎

Definition 3.4. A *Hilbert module* is a pre-Hilbert module E which is complete in the norm (3.3b).

By $\mathscr{B}^r(E)$ we denote the algebra of *bounded right linear mappings* on E. A mapping $T : E \to E$ is called *adjointable* if there exists a (unique) adjoint mapping

$T^*: E \to E$ fulfilling $\langle x, Ty\rangle = \langle T^*x, y\rangle$ $(x, y \in E)$. Observe that $T \in \mathscr{B}^r(E)$ automatically. (Boundedness follows by an application of the *closed graph theorem.*) By $\mathscr{B}^a(E)$ we denote the C^*-algebra of adjointable operators on E.

By $\mathcal{F}_{\mathscr{B}}(E)$ we denote the algebra of *$\mathscr{B}$-finite rank operators*, i.e. the linear span of all operators $|x\rangle\langle y|: z \mapsto x\langle y, z\rangle$. Defining an involution by $(|x\rangle\langle y|)^* = |y\rangle\langle x|$, we turn $\mathcal{F}_{\mathscr{B}}(E)$ into a *-subalgebra of $\mathscr{B}^a(E)$. The C^*-algebra $\mathcal{K}_{\mathscr{B}}(E)$ of *$\mathscr{B}$-compact operators* is defined as the norm closure of $\mathcal{F}_{\mathscr{B}}(E)$ in $\mathscr{B}^a(E)$.

We use the same notions if E is only a pre-Hilbert $\mathscr{B}$-module. However, notice that in this case $\mathscr{B}^r(E)$, $\mathscr{B}^a(E)$ and $\mathcal{K}_{\mathscr{B}}(E)$ do not need to be norm-complete. In addition, adjointable mappings do not need to be bounded. If we want to allow also for unbounded mappings, we use the notation $\mathscr{L}^r(E)$ and $\mathscr{L}^a(E)$.

Example 3.5. The subspace B_{21} of a matrix (pre-)C^*-algebra $\mathscr{M}$ with the inner product $\langle b_{21}, b'_{21}\rangle = b^*_{21}b'_{21}$ is a (pre-)Hilbert B_{11}-module. B_{22} has a *-homomorphic image in $\mathscr{B}^a(B_{21})$ which contains $\mathcal{F}_{\mathscr{B}}(B_{21})$ (even $\mathcal{K}_{\mathscr{B}}(B_{21})$, if B_{22} is complete). Of course, B_{12} is a (pre-)Hilbert B_{22}-module. The $*$ is an anti-isomorphism between B_{21} and B_{12} so that any element b_{12} in B_{12} may be written as b^*_{21} with a unique element b_{21} in B_{21}. The inner product $\langle b^*_{21}, {b'_{21}}^*\rangle$ on B_{12} acts on B_{21} as the operator $|b_{21}\rangle\langle b'_{21}|$.

In particular, considering the subspace $B_{21} = \mathscr{B}$ of a usual matrix C^*-algebra $M_2(\mathscr{B})$, we see that any C^*-algebra $\mathscr{B}$ is a Hilbert $\mathscr{B}$-module.

In what follows, in favour of a straight development we always assume that E is a Hilbert module over a C^*-algebra $\mathscr{B}$. For a more algebraic discussion including pre-Hilbert modules we refer to Appendix A. We define the subspace

$$E^* = \{x^*: E \longrightarrow \mathscr{B} | x^*y = \langle x, y\rangle (y \in E)\}$$

of the space $E' := \mathscr{B}^r(E, \mathscr{B})$ of $\mathscr{B}$-functionals on E (i.e. right linear bounded $\mathscr{B}$-valued mappings). If we want to embed a Hilbert $\mathscr{B}$-module E into a matrix C^*-algebra as a lower submodule, Example 3.5 and Remark 2.11 show that

$$\mathscr{M}(E, \mathscr{B}) = \begin{pmatrix} \mathscr{B} & E^* \\ E & \mathcal{K}_{\mathscr{B}}(E) \end{pmatrix}$$

is a minimal choice and that

$$\mathscr{M}^0(E, \mathscr{B}) = \begin{pmatrix} \mathscr{B} & E^* \\ E & \mathscr{B}^a(E) \end{pmatrix}$$

is a maximal choice (maximal in the sense that the elements of an algebra 'bigger' than $\mathscr{B}^a(E)$ will no longer be separated by elements of E: in other words, $\mathscr{B}^a(E)$ is the multiplier algebra of $\mathcal{K}_{\mathscr{B}}(E)$). The following can be checked easily.

Proposition 3.6. *$\mathscr{M}^0(E, \mathscr{B})$ is a *-algebra with multiplication defined by*

$$\begin{pmatrix} b & y^* \\ x & a \end{pmatrix} \begin{pmatrix} b' & y'^* \\ x' & a' \end{pmatrix} = \begin{pmatrix} bb' + \langle y, x'\rangle & (y'b^* + a'^*y)^* \\ xb' + ax' & |x\rangle\langle y'| + aa' \end{pmatrix}$$

and involution defined by

$$\begin{pmatrix} b & y^* \\ x & a \end{pmatrix}^* = \begin{pmatrix} b^* & x^* \\ y & a^* \end{pmatrix}.$$

*$\mathscr{M}(E, \mathscr{B})$ is a matrix *-subalgebra of $\mathscr{M}^0(E, \mathscr{B})$.*

In the remainder of this section we will show that $\mathscr{M}^0(E, \mathscr{B})$ has a C^*-norm extending the norm on $\mathscr{B}$. Assume for a moment that we have already constructed such a norm. From (3.3b) we see that the restrictions of this norm to E and E^* coincide with the original norms on E and E^*. Of course, the same is true for the C^*-algebra $\mathscr{B}^a(E)$, because a C^*-algebra has at most one C^*-norm. From Proposition 2.4 we see that such a norm must be complete and, of course, unique.

As in the case of usual matrix C^*-algebras, we find the norm by extending a faithful representation π of $\mathscr{B}$ on G (e.g. the *universal representation*) to a *-representation Π of $\mathscr{M}^0(E, \mathscr{B})$ on $G \oplus H$ where H is a suitable Hilbert space. However, unlike the case of $M_2(\mathscr{B})$ (where H just coincides with G), in our case H is not known *a priori*, but constitutes an essential part of the construction.

The restriction η of Π to E gives rise to a *representation of the Hilbert module E* by operators in $\mathscr{B}(G, H)$. We investigate this more systematically in Appendix A.

Definition 3.7. Let π be a *-representation of a C^*-algebra $\mathscr{B}$ on a Hilbert space G. A *cyclic decomposition* of G is a family $(G_\alpha, g_\alpha)_{\alpha \in A}$ where the G_α are a collection of subspaces of G invariant for π such that $G = \bigoplus_{\alpha \in A} G_\alpha$ and $g_\alpha \in G_\alpha$ are such that $\overline{\pi(\mathscr{B}) g_\alpha} = G_\alpha$. Of course, a cyclic decomposition always exists.

Let π denote a faithful *-representation of $\mathscr{B}$ on a Hilbert space G. We define a sesquilinear form on $E \otimes G$ by setting

$$\langle x \otimes g, x' \otimes g' \rangle = \langle g, \pi(\langle x, x' \rangle) g' \rangle. \tag{3.4}$$

Proposition 3.8. *The sesquilinear form defined by* (3.4) *is positive. Henceforth, $E \otimes G$ is a semi-Hilbert space.*

Proof. We have to show that $\sum_{i,j=1}^{n} \langle g_i, \pi(\langle x_i, x_j \rangle) g_j \rangle \geq 0$ for all choices of $n \in \mathbb{N}$, $x_i \in E$ and $g_i \in G$ $(i = 1, \ldots, n)$. Let $(G_\alpha, g_\alpha)_{\alpha \in A}$ denote a cyclic decomposition of π. Then each g_i is the (norm) limit of a sequence (g_i^m) whose members have the form $g_i^m = \sum_\alpha \pi(b_i^{m\alpha}) g_\alpha$ where the sum runs (for fixed i and m) only over finitely many

$\alpha \in A$. We find

$$\sum_{i,j=1}^{n} \langle g_i, \pi(\langle x_i, x_j \rangle) g_j \rangle = \lim_{m\to\infty} \sum_{i,j=1}^{n} \sum_{\alpha,\alpha'} \langle g_\alpha, \pi(\langle x_i b_i^{m\alpha}, x_j b_j^{m\alpha'} \rangle) g_{\alpha'} \rangle$$

$$= \lim_{m\to\infty} \sum_{\alpha} \left\langle g_\alpha, \pi\left(\left\langle \sum_{i=1}^{n} x_i b_i^{m\alpha}, \sum_{j=1}^{n} x_j b_j^{m\alpha} \right\rangle\right) g_\alpha \right\rangle \geq 0. \quad ■$$

By $H = \overline{E \otimes G}$ we denote the *Hilbert space associated with* $E \otimes G$, i.e. the completion of the pre-Hilbert space obtained from $E \otimes G$ by dividing out the null-space $\mathcal{N}$ of the inner product. We identify elements $x \otimes g \in E \otimes G$ with equivalence classes $x \otimes g + \mathcal{N} \in H$. With each $x \in E$ we associate a mapping $L_x \in \mathscr{B}(G, H)$ defined by $L_x g = x \otimes g$. This is, indeed, a bounded mapping, because $\|L_x g\|^2 = \langle g, \pi(\langle x, x \rangle) g \rangle \leq \|g\|^2 \, \|\langle x, x \rangle\|$. Moreover, $\langle L_x g, L_{x'} g' \rangle = \langle g, \pi(\langle x, x' \rangle) g' \rangle$ for all $g, g' \in G$ so that $L_x^* L_{x'} = \pi(\langle x, x' \rangle)$. Similarly, $L_x \pi(b) g = L_{xb} g$. We define $\eta : x \mapsto L_x$ and $\eta^* : x^* \mapsto L_x^*$. Of course, $H = \overline{\mathsf{span}}(\eta(E)G)$.

Let a be an element of $\mathscr{B}^a(E)$. We associate with a a mapping on $E \otimes G$ by $x \otimes g \mapsto ax \otimes g$. From $\langle x \otimes g, ax' \otimes g' \rangle = \langle a^* x \otimes g, x' \otimes g' \rangle$ we see that this mapping leaves invariant $\mathcal{N}$ so that it induces a mapping $\rho_0(a)$ on $E \otimes G/\mathcal{N}$. Clearly, $\rho_0 : a \mapsto \rho_0(a)$ is a (non-degenerate) faithful *-representation of $\mathscr{B}^a(E)$ by possibly unbounded operators on $E \otimes G/\mathcal{N}$. It remains to show that $\rho_0(a)$ is a bounded operator.

Lemma 3.9. *Let $a \in \mathscr{B}^a(E)$. Then*

$$\|\rho_0(a)\| = \|a\|.$$

*In particular, $\rho_0(a)$ is bounded so that ρ_0 extends to a *-representation ρ of $\mathscr{B}^a(E)$ by bounded operators on H.*

PROOF. For formal reasons we allow a norm to take the value ∞. By definition we have

$$\|a\| = \sup_{\|x\| \leq 1} \|ax\| = \sup_{\|L_x\| \leq 1} \|L_{ax}\| = \sup_{\substack{h = L_x g \\ \|x\| \leq 1, \|g\| \leq 1}} \|\rho_0(a) h\| \leq \|\rho_0(a)\|.$$

Let $(G_\alpha, g_\alpha)_{\alpha \in A}$ denote a cyclic decomposition of G. Observe that $\langle x \otimes g_\alpha, x' \otimes g_{\alpha'} \rangle$ is 0, if $\alpha \neq \alpha'$. We conclude also that H decomposes into orthogonal subspaces $H_\alpha = \overline{\mathsf{span}}(\eta(E) G_\alpha)$. As easily as before we check that H_α is invariant for ρ_0 so that

$$\|\rho_0(a)\| = \sup_{\alpha \in A} \|\rho_0(a) \upharpoonright H_\alpha\|.$$

An element h in H_α is a limit of elements $\sum_i x_i \otimes g_i$ where $g_i \in G_\alpha$ has the form $g_i = \pi(b_i) g_\alpha$. Denote $x = \sum_i x_i b_i$. One easily checks (modulo $\mathcal{N}$) that $\sum_i x_i \otimes g_i = x \otimes g_\alpha = L_x g_\alpha$. Therefore, g_α is cyclic for H_α in the sense that $H_\alpha = \overline{\eta(E) g_\alpha}$.

Let $h = L_x g_\alpha \in H_\alpha$. Let $L_x = v\sqrt{\pi(\langle x, x\rangle)}$ be the polar decomposition of L_x according to Proposition 2.10. Set $g = \sqrt{\pi(\langle x, x\rangle)}g_\alpha$. Then $||h|| = ||g||$ and $h = vg$. By Proposition 2.9, v may be approximated by operators L_y where y is in the unit-ball of E. We find

$$||\rho_0(a) \upharpoonright H_\alpha|| = \sup_{\substack{h=L_x g, g\in G_\alpha \\ ||x||\leq 1, ||g||\leq 1}} ||\rho_0(a)h||$$

so that

$$||\rho_0(a)|| = \sup_{\substack{\alpha\in A \\ h=L_x g, g\in G_\alpha \\ ||x||\leq 1, ||g||\leq 1}} ||\rho_0(a)h|| \leq ||a||. \quad \blacksquare$$

Thus we have shown that

$$\Pi = \begin{pmatrix} \pi & \eta^* \\ \eta & \rho \end{pmatrix}$$

defines a faithful *-representation of $\mathscr{M}^0(E, \mathscr{B})$ by bounded operators on $G \oplus H$. We collect the results.

Theorem 3.10. *$\mathscr{M}^0(E, \mathscr{B})$ and $\mathscr{M}(E, \mathscr{B})$ are matrix C^*-algebras and the embedding of any matrix element is an isometry.*

Theorem 3.11. *The *-representation ρ of $\mathscr{B}^a(E)$ extends uniquely to an isometric representation of $\mathscr{B}^r(E)$ by bounded operators on H.*

Proof. Uniqueness is clear, because $\mathfrak{F}_{\mathscr{B}}(E)$ is an essential left ideal in $\mathscr{B}^r(E)$ (i.e. $ak = 0 \forall k \in \mathfrak{F}_{\mathscr{B}}(E) \Rightarrow a = 0$). The crucial point is that in the definition of ρ_0 we do not have at hand an adjoint, i.e. we cannot show directly that the mapping $x \otimes g \mapsto ax \otimes g$ leaves invariant $\mathfrak{N}$. It is, however, obvious that this mapping respects the relations $xb \otimes g - x \otimes bg$. These relations are the only ones which are needed in the proof of Lemma 3.9 so that Lemma 3.9 remains true for the operator semi-norm of the operator $x \otimes g \mapsto ax \otimes g$ on the semi-Hilbert space $E \otimes G$. Now the statement follows from Appendix B. (But cf. also Remark 8.1.) $\blacksquare$

The proof of the following generalisation to mappings between different Hilbert modules differs from its precedessors merely by notation.

Theorem 3.12. *Let E_1, E_2 denote two Hilbert $\mathscr{B}$-modules and define H_1, H_2 as above by means of the representation π. For any $a \in \mathscr{B}^r(E_1, E_2)$ the mapping $\rho(a): x \otimes g \mapsto ax \otimes g$ extends to a well-defined element of $\mathscr{B}(H_1, H_2)$. The mapping $\rho: \mathscr{B}^r(E_1, E_2) \to \mathscr{B}(H_1, H_2)$ defined by $a \mapsto \rho(a)$ is an isometry; ρ is functorial in the sense that $\rho(aa') = \rho(a)\rho(a')$. If a is adjointable, we have $\rho(a^*) = \rho(a)^*$.*

Corollary 3.13. *For any $a \in \mathscr{B}^r(E_1, E_2)$ we have*

$$\langle ax, ax\rangle \leq ||a||^2 \langle x, x\rangle.$$

PROOF. This follows by considering both x and a as elements of $\mathscr{B}(G \oplus H_1 \oplus H_2)$. ■

By considering $\mathscr{B}$ as a Hilbert module we obtain the following application to $\mathscr{B}$-functionals most important for us.

Corollary 3.14. *There is a unique mapping* $\eta' : E' \to \mathscr{B}(H, G)$ (*extending* η^* *and*) *fulfilling*

$$\pi(\Phi x) = \eta'(\Phi) L_x$$

for all $\Phi \in E'$ *and* $x \in E$. *In particular,*

$$(\Phi x)^*(\Phi x) \leq \|\Phi\|^2 \langle x, x \rangle.$$

PROOF. This follows from Theorem 3.12 with $E_1 = E$ and $E_2 = \mathscr{B}$ and from the observation that $H_2 = \overline{\mathscr{B} \otimes G} \subset G$ via the isometry defined by setting $b \otimes g \mapsto \pi(b)g$. ■

4. Self-dual Hilbert modules and von Neumann modules

Definition 4.1. Let $\mathscr{B}$ denote a C^*-algebra. A Hilbert $\mathscr{B}$-module E is called *self-dual*, if $E' = E^*$.

Proposition 4.2. *Let* E *denote a self-dual Hilbert* $\mathscr{B}$*-module and* F *denote a Hilbert* $\mathscr{B}$*-module. Then*

$$\mathscr{B}^r(E, F) = \mathscr{B}^a(E, F).$$

PROOF. Precisely as in the case of Hilbert spaces. (Cf. also proof of Corollary 5.3.) ■

Before we define von Neumann modules we characterise general self-dual Hilbert modules.

Theorem 4.3. *A Hilbert* $\mathscr{B}$*-module* E *is self-dual if and only if the unit-ball of* E *is complete with respect to the topology defined by the semi-norms* $\|\langle x, \bullet \rangle\|$ $(x \in E)$.

PROOF. Obviously, the unit-ball of a self-dual Hilbert module E must be complete with respect to this topology. So, let E be an arbitrary Hilbert $\mathscr{B}$-module and $\Phi \in E'$. Observe that $E' \mathcal{F}_{\mathscr{B}}(E) \subset E^*$. By Corollary 2.6 there exists an approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ for $\mathcal{K}_{\mathscr{B}}(E)$ such that $\lim_\lambda u_\lambda x = x$ $(x \in E)$. We also may assume that $\|u_\lambda\| < 1$. For any λ we may choose $k_\lambda \in \mathcal{F}_{\mathscr{B}}(E)$ such that $\|u_\lambda - k_\lambda\| < 1 - \|u_\lambda\|$. Setting $\Phi_\lambda = \Phi k_\lambda \in E^*$, we see that $\lim_\lambda \Phi_\lambda x = \Phi x$. Thus, if E is complete in the topology mentioned above, then $\Phi \in E^*$. ■

In what follows, $\mathscr{B} \subset \mathscr{B}(G)$ is always a von Neumann algebra of operators on a Hilbert space G, unless explicitly stated otherwise. For a Hilbert module E over $\mathscr{B}$ we use the notations of Section 3. In particular, we always identify $x \in E$ with $L_x \in \mathscr{B}(G, H)$ and we always identify $a \in \mathscr{B}^r(E)$ with an element in $\mathscr{B}(H)$.

Definition 4.4. A *von Neumann $\mathscr{B}$-module* is a Hilbert $\mathscr{B}$-module E for which $\mathscr{M}^0(E, \mathscr{B})$ is a matrix von Neumann algebra on $G \oplus H$.

Proposition 4.5. *E is a von Neumann module if and only if E is strongly closed in $\mathscr{B}(G, H)$. In particular, if E is a von Neumann module, then $\mathscr{B}^a(E)$ is a von Neumann algebra.*

Proof. We only need to show one direction. Assume that E is strongly closed in $\mathscr{B}(G, H)$. By Proposition 2.8, we see that closing $\mathscr{M}^0(E, \mathscr{B})$ actually means closing $\mathscr{B}^a(E)$ in $\mathscr{B}(H)$, because all other 'matrix entries' are already strongly closed. Of course, an element a of the closure of $\mathscr{B}^a(E)$ acts as an adjointable mapping on E. (This follows from the fact that the strong closure of $\mathscr{M}^0(E, \mathscr{B})$ is also a matrix *-algebra.) We conclude that $a \in \mathscr{B}^a(E)$, and hence $\mathscr{B}^a(E)$ is strongly closed. ■

Proposition 4.6. *The $\mathscr{B}$-functionals are strongly continuous mappings $E \to \mathscr{B}$. For all $x \in E$ the mapping $\mathscr{B}^a(E) \to E, a \mapsto ax$ is strongly continuous. For all $a \in \mathscr{B}^a(E)$ the mapping $E \to E, x \mapsto ax$ is strongly continuous.*

Proof. All assertions follow from the fact that multiplication in $\mathscr{B}(G \oplus H)$ is separately strongly continuous. ■

Proposition 4.7. *A bounded net $(a_\alpha)_{\alpha \in A}$ of elements in $\mathscr{B}^r(E)$ converges strongly if and only if $a_\alpha xg$ is a Cauchy net in H for all $x \in E$ and $g \in G$.*

Proof. For a bounded net it is sufficient to check strong convergence on the dense subset $\mathsf{span}(EG)$ of H. ■

Definition 4.8. A *quasi-orthonormal system* is a family $(e_\beta, p_\beta)_{\beta \in B}$ of pairs consisting of an element $e_\beta \in E$ and a projection $p_\beta \in \mathscr{B}$ such that

$$\langle e_\beta, e_{\beta'} \rangle = p_\beta \delta_{\beta\beta'}.$$

Proposition 4.9. *Let $(e_\beta, p_\beta)_{\beta \in B}$ be a quasi-orthonormal system. Then the increasing net*

$$\left(\sum_{\beta \in B'} |e_\beta\rangle\langle e_\beta| \right)_{B' \subset B, \# B' < \infty}$$

of projections converges strongly to a projection in $\mathscr{B}^a(E)$. We call this projection the projection associated with $(e_\beta, p_\beta)_{\beta \in B}$.

Proof. Clear, since $\mathscr{B}^a(E)$ is a von Neumann algebra. ■

Definition 4.10. A quasi-orthonormal system $\left(e_\beta, p_\beta\right)_{\beta\in B}$ is called *complete* if

$$\sum_{\beta\in B} |e_\beta\rangle\langle e_\beta| = \mathbf{1}.$$

Theorem 4.11. *Any von Neumann $\mathscr{B}$-module E admits a complete quasi-orthonormal system.*

PROOF. An application of *Zorn's lemma* tells us that the set consisting of all quasi-orthonormal systems has a maximal element. Let $\left(e_\beta, p_\beta\right)_{\beta\in B}$ be a maximal quasi-orthonormal system. If $\left(e_\beta, p_\beta\right)_{\beta\in B}$ is not complete, then $E_B^\perp = \left(\mathbf{1} - \sum_{\beta\in B} |e_\beta\rangle\langle e_\beta|\right)E$ is non-trivial. We choose $x \in E_B^\perp$ different from 0. By Proposition 4.6, $\langle x, e_\beta\rangle = (\langle e_\beta, x\rangle)^* = 0$ for all $\beta \in B$. By Proposition 2.10, $x = v|x|$ where $v \in E$ is a partial isometry. Then also $\left(e_\beta, p_\beta\right)_{\beta\in B}$ enlarged by $(v, |v|)$ is a quasi-orthonormal system. This contradicts the maximality of $\left(e_\beta, p_\beta\right)_{\beta\in B}$. ■

Corollary 4.12. *Let $\left(e_\beta, p_\beta\right)_{\beta\in B}$ be a complete quasi-orthonormal system for E. Let $x \in E$. Then $b_\beta = \langle e_\beta, x\rangle$ are unique elements in $p_\beta\mathscr{B}$ such that*

$$x = \sum_{\beta\in B} e_\beta b_\beta.$$

Conversely, if $b_\beta \in \mathscr{B}$ and $M > 0$ such that

$$\sum_{\beta\in B'} b_\beta^* p_\beta b_\beta < M$$

for all finite subsets B' of B, then

$$\sum_{\beta\in B} e_\beta b_\beta$$

exists and is an element of E.

PROOF. This is an immediate consequence of Proposition 4.6 and of the order-completeness of the von Neumann algebra $\mathscr{B}$. ■

Definition 4.13. Let F be a submodule of a (pre-)Hilbert module E. The *orthogonal complement* of F in E is the set $F^\perp = \left\{x \in E : \langle y, x\rangle = 0 \ (y \in F)\right\}$. We say that F is *complemented* in E if E decomposes into the (pre-)Hilbert module direct sum $F \oplus F^\perp$.

Theorem 4.14. *Let E denote a von Neumann $\mathscr{B}$-module with a complete quasi-orthonormal system $\left(e_\beta, p_\beta\right)_{\beta\in B}$. Denote by H_B a Hilbert space with an orthonormal* *basis* $(e'_\beta)_{\beta\in B}$. *For* $h \in H_B$ *and* $b \in \mathscr{B}$ *identify* $h \otimes b$ *with the mapping* $g \mapsto h \otimes bg$ *in* $\mathscr{B}(G, H_B \otimes G)$. *Then* E *is a complemented submodule of the strong closure of* $H_B \otimes \mathscr{B}$.

Proof. E is the closure of the direct sum over all $p_\beta\mathscr{B}$. (Notice that this is the von Neumann module direct sum and not the von Neumann algebra direct sum. The former is a subset of $\mathscr{B}\Big(G, \bigoplus_{\beta\in B} G\Big)$, whereas the latter is a subset of $\mathscr{B}\Big(\bigoplus_{\beta\in B} G, \bigoplus_{\beta\in B} G\Big)$.) We consider the right ideal $p_\beta\mathscr{B}$ as a subset of $\mathscr{B}$ so that $\bigoplus_{\beta\in B} p_\beta\mathscr{B}$ is contained in $\bigoplus_{\beta\in B}\mathscr{B}$ and $\bigoplus_{\beta\in B}(\mathbf{1} - p_\beta)\mathscr{B}$ is its complement. We establish the claimed isomorphism by sending the βth summand to $e'_\beta \otimes \mathscr{B}$. ■

Remark 4.15. Notice that the cardinality of a complete quasi-orthonormal system is not unique. For instance, for the von Neumann module $\mathscr{B}$ we may choose $(\mathbf{1}, \mathbf{1})$ as well as $(p_\beta, p_\beta)_{\beta\in B}$ for an arbitrary decomposition of $\mathbf{1}$ into orthogonal projections p_β. This example also shows that the number of coefficients with respect to a quasi-orthonormal system (for overcountable B) does not need to be countable.

Theorem 4.16. *Any von Neumann* $\mathscr{B}$*-module* E *is self-dual.*

Proof. Recall from Corollary 3.14 that E' may be identified as a subspace of $\mathscr{B}(H, G)$ containing E^*. The matrix element B_{21} of the von Neumann matrix subalgebra of $\mathscr{B}(G \oplus H)$ generated by E' is a von Neumann module (not necessarily over $\mathscr{B}$) containing $E'^* \supset E$. Clearly, a complete quasi-orthonormal system $(e_\beta, p_\beta)_{\beta\in B}$ for E is also a quasi-orthonormal system for B_{21}. This implies that

$$\sum_{\beta\in B}(\Phi e_\beta)(\Phi e_\beta)^* < \infty$$

for all $\Phi \in E'$. In particular, if we set $b_\beta = (\Phi e_\beta)^*$, then $x_\Phi = \sum_{\beta\in B} e_\beta b_\beta$ is an element of E.

Taking into account Proposition 4.6, we find that

$$\langle x_\Phi, x\rangle = \left(\sum_{\beta\in B}\langle x, e_\beta\rangle b_\beta\right)^* = \sum_{\beta\in B}\Phi e_\beta\langle e_\beta, x\rangle = \Phi x$$

for all $x \in E$. (The equation is to be understood weakly, because the $*$ is only weakly continuous.) Hence, $\Phi = x_\Phi^* \in E^*$. ■

5. Miscellaneous

Proposition 5.1. *Let* $\mathscr{B}$ *be a von Neumann algebra of operators on a Hilbert space* G. *Let* F *be a strongly dense* $\mathscr{B}$*-submodule of a von Neumann* $\mathscr{B}$*-module* E. *Then any* $\mathscr{B}$*-functional* Φ *on* F *extends to a (unique)* $\mathscr{B}$*-functional* $\bar{\Phi}$ *on* E. *Moreover,* $\|\bar{\Phi}\| = \|\Phi\|$.

Proof. Clearly, Φ extends to the norm closure of F so that we may assume F to be complete. The closed subspace of H generated by FG is H. (Otherwise, F was not strongly dense in E.) By Corollary 3.14, Φ may be identified with an element of $\mathscr{B}(H, G)$. Of course, Φ acts strongly continuously on F (see proof of Proposition 4.6) so that also the range of the strong extension $\bar{\Phi}$ of Φ to E is $\mathscr{B}$. Clearly, $\|\bar{\Phi}\| = \|\Phi\|$. ∎

Theorem 5.2. *Any $\mathscr{B}$-functional Φ on a $\mathscr{B}$-submodule F of a von Neumann $\mathscr{B}$-module E may be extended norm-preserving and uniquely to a $\mathscr{B}$-functional on E vanishing on $F^\perp$.*

PROOF. By the preceding proposition Φ extends uniquely to a $\mathscr{B}$-functional $\bar{\Phi}$ on the strong closure $\bar{F}$ of F. Since $\bar{F}$ is a von Neumann module, we may choose a complete quasi-orthonormal system $(e_\beta, p_\beta)_{\beta\in B}$ for $\bar{F}$. This system is also a quasi-orthonormal system for E so that the projection p associated with $(e_\beta, p_\beta)_{\beta\in B}$ is a projection in E onto $\bar{F}$. The $\mathscr{B}$-functional $\bar{\Phi}p$ has all the claimed properties and is, of course, uniquely determined. ∎

Corollary 5.3. *Let E_1, E_2 be von Neumann $\mathscr{B}$-modules and F a $\mathscr{B}$-submodule of E_1. An arbitrary mapping a in $\mathscr{B}^r(F, E_2)$ extends uniquely to a mapping in $\mathscr{B}^a(E_1, E_2)$ having the same norm and vanishing on $F^\perp$.*

PROOF. We proceed as in the case of Hilbert spaces. We observe that $\|\langle x, ay\rangle\| \leq \|x\|\|a\|\|y\|$ for all $x \in E_2$ and $y \in F$. Hence, for fixed $x \in E_2$ the $\mathscr{B}$-functional $y \mapsto \langle x, ay\rangle$ extends uniquely to an element Φ_x in E_1^* with $\|\Phi_x\| \leq \|x\|\|a\|$ and vanishing on $F^\perp$.

Conversely, for fixed $y \in E_1$ the mapping $\Phi_y : x \mapsto (\Phi_x y)^*$ is a $\mathscr{B}$-functional on E_2. The mapping $y \mapsto \Phi_y^*$ is the claimed extension of a. ∎

Observe that $x \mapsto \Phi_x^*$ defines an operator a^* in $\mathscr{B}^a(E_2, E_1)$. Obviously, $y \mapsto \Phi_y^*$ is the adjoint of a^*. The only ingredient which actually enters the proof is the sesquilinear form $A(x, y) = \langle x, ay\rangle$.

Corollary 5.4. *Let E_1, E_2 be von Neumann $\mathscr{B}$-modules and F_1, F_2 be $\mathscr{B}$-submodules of E_1, E_2, respectively. Let $A : F_1 \times F_2 \to \mathscr{B}$ denote a bounded $\mathscr{B}$-valued $\mathscr{B}$-sesquilinear form. This means that $A(xb, y) = b^*A(x, y)$ and $A(x, yb) = A(x, y)b$ and that there exists a smallest number $\|A\| \geq 0$ fulfilling $\|A(x, y)\| \leq \|x\|\|A\|\|y\|$. Then there exists a unique operator a in $\mathscr{B}^a(E_1, E_2)$ such that $A(x, y) = \langle x, ay\rangle$ for $x \in F_2, y \in F_1$ and $\langle x, ay\rangle = 0$ for $x \in F_2^\perp$ or $y \in F_1^\perp$. Moreover, $\|a\| = \|A\|$.*

Theorem 5.5. *Let E be a Hilbert module over a von Neumann algebra $\mathscr{B}$. Take any faithful normal representation π of $\mathscr{B}$ on G and construct the representation Π of $\mathscr{M}^0(E, \mathscr{B})$ on $G \oplus H$ as in Section 3. Then E is self-dual if and only if $\eta(E)$ is a von Neuman $\pi(\mathscr{B})$-module.*

PROOF. If $\eta(E)$ is strongly closed, then $\eta(E)$ is a von Neumann module and therefore self-dual. Let Φ be a $\mathscr{B}$-functional on E. Then $\varphi = \pi \circ \Phi$ is a $\pi(\mathscr{B})$-functional on $\eta(E)$. Since $\eta(E)$ is self-dual, we find a unique $x \in E$, such that $\varphi = L_x^*$. Clearly, $\Phi = x^*$ so that E is also self-dual.

If $\eta(E)$ is not strongly closed, then there exists an element φ^* in the strong closure of $\eta(E)$ which is not an element of $\eta(E)$. Clearly, $\varphi = (\varphi^*)^*$ is an element of $\eta(E)'$ which gives rise to a $\mathscr{B}$-functional $\Phi = \pi^{-1} \circ \varphi$ on E. If Φ is in E^*, then φ is in $\eta(E)^*$, which contradicts our assumption. Consequently, E is not self-dual. ■

The property of E to be self-dual or not is an intrinsic property and cannot depend on the normal representation π. The following theorem is proved just by collecting the preceding results.

Theorem 5.6. *Let E be a Hilbert module over a von Neumann algebra $\mathscr{B}$. Then E' is a self-dual Hilbert $\mathscr{B}$-module. Moreover, if F is a self-dual Hilbert $\mathscr{B}$-module, then any element in $\mathscr{B}^r(E,F)$ extends uniquely to an element in $\mathscr{B}^a(E',F)$.*

Theorem 5.7. *Let E be a von Neumann $\mathscr{B}$-module and a a self-adjoint element of $\mathscr{B}^a(E)$. There exists a projection-valued function $E_\lambda\colon \mathbb{R} \to \mathscr{B}^a(E)$ fulfilling $\lambda \leq \mu \Rightarrow E_\lambda \leq E_\mu$, $E_{\lambda+0} = E_\lambda$ (strongly), $E_{-\|a\|-0} = 0$, $E_{\|a\|} = \mathbf{1}$ and*

$$\int \lambda \, dE_\lambda = a.$$

The integral is the norm limit of Riemann sums. E_λ is called the spectral resolution of identity.

Proof. This is a direct translation of the corresponding statement for an operator in $\mathscr{B}(H)$. We only have to recognise that E_λ is a strong limit of polynomials in a. This guarantees that $E_\lambda \in \mathscr{B}(H)$ may be interpreted as an element of $\mathscr{B}^a(E)$. ■

Corollary 5.8. *Let E be a von Neumann $\mathscr{B}$-module and a a self-adjoint element of $\mathscr{B}^a(E)$ with spectral resolution of identity E_λ. Moreover, let $\Omega\colon \mathscr{A} \to \mathscr{B}$, a completely positive mapping (see Section 7). Then with the operator-valued measure $\mu(d\lambda) = \Omega(E_{\lambda+d\lambda} - E_\lambda)$ the moments $\Omega(a^n)$ of a may be computed by*

$$\Omega(a^n) = \int \lambda^n \mu(d\lambda).$$

6. Centred Hilbert modules

Definition 6.1. Let $\mathscr{A}$ denote a *-algebra and $\mathscr{B}$ denote a pre-C^*-algebra. A *two-sided pre-Hilbert $\mathscr{A}$–$\mathscr{B}$-module E* is an $\mathscr{A}$–$\mathscr{B}$-module and a pre-Hilbert $\mathscr{B}$-module such that

$$\langle x, ay \rangle = \langle a^* x, y \rangle. \tag{6.1}$$

This means that $\mathscr{A}$ has a *-homomorphic image in $\mathscr{L}^a(E)$. If E is a Hilbert module, then this image is in $\mathscr{B}^a(E)$.

We also say that E is a **-representation of $\mathscr{A}$ over $\mathscr{B}$.*

Definition 6.2. The *$\mathcal{B}$-centre* of a $\mathcal{B}$–$\mathcal{B}$-module E is the set

$$C_{\mathcal{B}}(E) = \{x \in E : xb = bx \ (b \in \mathcal{B})\}.$$

In particular, $C_{\mathcal{B}}(\mathcal{B})$ is the *centre* of $\mathcal{B}$.

Proposition 6.3. *Let E be a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module. Then*

$$\langle C_{\mathcal{B}}(E), C_{\mathcal{B}}(E)\rangle \subset C_{\mathcal{B}}(\mathcal{B}).$$

Proof. Let $x, y \in C_{\mathcal{B}}(E)$ and $b \in \mathcal{B}$. Then $\langle x, y\rangle b = \langle x, yb\rangle = \langle x, by\rangle = \langle b^* x, y\rangle = \langle xb^*, y\rangle = b\langle x, y\rangle$. ■

Definition 6.4. A pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module E is called a *centred* pre-Hilbert $\mathcal{B}$-module, Hilbert $\mathcal{B}$-module and von Neumann $\mathcal{B}$-module E, if E is generated by $C_{\mathcal{B}}(E)$ as a pre-Hilbert $\mathcal{B}$-module, a Hilbert $\mathcal{B}$-module and a von Neumann $\mathcal{B}$-module respectively.

Example 6.5. Let H denote a Hilbert space and $\mathcal{B}$ a unital C^*-algebra. Then $H \otimes \mathcal{B}$ is a pre-Hilbert $\mathcal{B}$–$\mathcal{B}$-module in an obvious manner. The $\mathcal{B}$-centre of $H \otimes \mathcal{B}$ contains $H \otimes \mathbf{1}$ so that $H \otimes \mathcal{B}$ is a centred pre-Hilbert $\mathcal{B}$-module. The norm completion of $H \otimes \mathcal{B}$ is a centred Hilbert $\mathcal{B}$-module. If $\mathcal{B}$ is a von Neumann algebra on a Hilbert space G, then the strong closure of $H \otimes \mathcal{B}$ in $\mathcal{B}(G, H \otimes G)$ (cf. Theorem 4.14) is a centred von Neumann $\mathcal{B}$-module.

Proposition 6.6. *Let j be a $\mathcal{B}$–$\mathcal{B}$-linear mapping on a $\mathcal{B}$–$\mathcal{B}$-module E. Then the $\mathcal{B}$-centre of E is mapped to the $\mathcal{B}$-centre of the range of j. Consequently, if E is generated by its $\mathcal{B}$-centre, then so is the range of j.*

Proof. Obvious. ■

Corollary 6.7. *Let $(e_\beta, p_\beta)_{\beta \in B}$ be a quasi-orthonormal system in a centred von Neumann $\mathcal{B}$-module E. Furthermore, suppose that all e_β are in $C_{\mathcal{B}}(E)$. Then both the range of $\sum_{\beta \in B} |e_\beta\rangle\langle e_\beta|$ and its complement are centred von Neumann $\mathcal{B}$-modules.*

Proof. $\sum_{\beta \in B} |e_\beta\rangle\langle e_\beta|$ and, consequently, $\mathbf{1} - \sum_{\beta \in B} |e_\beta\rangle\langle e_\beta|$ are $\mathcal{B}$–$\mathcal{B}$-linear mappings. ■

Theorem 6.8. *Let E be a centred von Neumann $\mathcal{B}$-module. Then E admits a complete quasi-orthonormal system $(e_\beta, p_\beta)_{\beta \in B}$ consisting of elements $e_\beta \in C_{\mathcal{B}}(E)$ and central projections p_β.*

Proof. Suppose that $(e_\beta, p_\beta)_{\beta \in B}$ is not complete. By Corollary 6.7 we may choose a non-zero x in $C_{\mathcal{B}}(E)$ which is orthogonal to all e_β. Then $|x|$ is in the centre of $\mathcal{B}$. Let $v|x|$ be the polar decomposition of x. Then v is also in $C_{\mathcal{B}}(E)$. (Indeed, let $|x|g$ be an element in the range of $|x|$ and $b \in \mathcal{B}$. Then $vb|x|g = v|x|bg = bv|x|g$, hence

$vb = bv$.) The pair $(v, |v|)$ extends $(e_\beta, p_\beta)_{\beta \in B}$ to a bigger quasi-orthonormal system so that we are ready for an application of Zorn's lemma. ■

Theorem 6.9. *Let E be a centred von Neumann $\mathscr{B}$-module. Then E may be identified as a complemented von Neumann $\mathscr{B}$-submodule of the strong completion of $H \otimes \mathscr{B}$ where H is a suitable Hilbert space in such a way that left multiplication is preserved.*

Proof. We choose a complete orthonormal system for E which consists of elements of $C_{\mathscr{B}}(E)$ and perform the construction according to Theorem 4.14. On the $\mathscr{B}$-centre left multiplication, clearly, is preserved. By Proposition 4.6, left multiplication is strongly continuous on E so that any extension is determined uniquely by its values on the $\mathscr{B}$-centre. ■

Corollary 6.10. *E is the strong closure of the pre-Hilbert $\mathscr{B}$–$\mathscr{B}$-module direct sum of ideals $p_\beta \mathscr{B}$ of $\mathscr{B}$.*

Remark 6.11. Any (pre-)Hilbert $\mathscr{B}$-module E may be considered as a submodule of a von Neumann module. (For instance, we may embed E into the strong completion of any faithful representation of $\mathscr{M}^0(E, \mathscr{B})$.) Similarly, if E is a centred (pre-)Hilbert $\mathscr{B}$-module, then E may be considered as a $\mathscr{B}$–$\mathscr{B}$-submodule of a suitable completion of $H \otimes \mathscr{B}$ for a suitable Hilbert space H.

Definition 6.12. Let $\mathscr{A}$ and $\mathscr{B}$ denote von Neumann algebras where $\mathscr{B}$ acts on a Hilbert space G. A *two-sided von Neumann $\mathscr{A}$–$\mathscr{B}$-module* E is a von Neumann $\mathscr{B}$-module and a pre-Hilbert $\mathscr{A}$–$\mathscr{B}$-module such that the mapping $a \mapsto \rho(a)$ ($\rho(a) \in \mathscr{B}(H)$) on $\mathscr{A}$ is normal.

Theorem 6.13. *Any centred von Neumann module is a two-sided von Neumann module.*

Proof. E may be identified as a subset of $\mathscr{B}(G, H \otimes G)$ where b acts from the left as an operator on $H \otimes G$. Clearly, $b \mapsto \rho(b)$ is nothing but the normal *-representation $\mathbf{1} \otimes \mathsf{id}$ on $H \otimes G$. ■

7. GNS-construction versus Stinespring construction

Definition 7.1. Let $\mathscr{B}$ be a pre-C^*-algebra. A pre-Hilbert $\mathscr{B}$-module E in which (3.1c) does not hold necessarily is called a *semi-Hilbert $\mathscr{B}$-module.*

Proposition 7.2. *Equation* (3.3a) *is also valid in a semi-Hilbert module. In particular, $\langle x, x \rangle = 0$ implies that $\langle y, x \rangle = 0$ for all y.*

Proof. If $\langle y, y \rangle \neq 0$ the proof works as well. If $\langle y, y \rangle = 0$ but $\langle x, x \rangle \neq 0$, then the statement follows by exchanging x and y. Only the case $\langle x, x \rangle = \langle y, y \rangle = 0$ requires additional work. As in the proof of (3.2) we investigate $\langle x + \lambda y, x + \lambda y \rangle$ for $\lambda = 1, i, -1, -i$. From $\lambda = 1, -1$ we conclude that the real part of $\langle x, y \rangle$ is positive and negative, hence 0. From $\lambda = i, -i$ we conclude that the imaginary part of $\langle x, y\rangle$ is positive and negative, hence 0. This implies that $\langle x, y\rangle = 0$. ■

Corollary 7.3. *Let $\mathscr{A}$ be a *-algebra and $\mathscr{B}$ a pre-C^*-algebra. Let E be a two-sided semi-Hilbert $\mathscr{A}$–$\mathscr{B}$-module (i.e. E is an $\mathscr{A}$–$\mathscr{B}$-module and a semi-Hilbert $\mathscr{B}$-module such that (6.1) is fulfilled). Then the null-space $\mathcal{N} = \{x \in E : \langle x, x\rangle = 0\}$ of E is an $\mathscr{A}$–$\mathscr{B}$-submodule of E. The quotient $E/\mathcal{N}$ is turned into a pre-Hilbert $\mathscr{A}$–$\mathscr{B}$-module with inner product*

$$\langle x + \mathcal{N}, y + \mathcal{N}\rangle = \langle x, y\rangle.$$

Definition 7.4. Let $\mathscr{A}$ be a *-algebra and $\mathscr{B}$ a pre-C^*-algebra. A mapping $\Omega : \mathscr{A} \to \mathscr{B}$ is called *completely positive* if $\sum_{i,j} b_i^* \Omega(a_i^* a_j) b_j \geq 0$ for all $a_i \in \mathscr{A}; b_i \in \mathscr{B}; i = 1, \ldots, n; n \in \mathbb{N}$.

Theorem 7.5. *To any completely positive mapping $\Omega : \mathscr{A} \to \mathscr{B}$ from a unital *-algebra $\mathscr{A}$ into a unital pre-C^*-algebra $\mathscr{B}$ there exists a pre-Hilbert $\mathscr{A}$–$\mathscr{B}$-module E and a cyclic vector $\xi \in E$ (i.e. $\mathsf{span}(\mathscr{A}\xi\mathscr{B}) = E$), such that*

$$\Omega(a) = \langle \xi, a\xi\rangle.$$

The pair (E, ξ) is determined up to two-sided pre-Hilbert module isomorphism (i.e. an isomorphism of two-sided modules which preserves inner products). (E, ξ) is called the GNS-representation of Ω.

PROOF. Consider $\mathscr{A} \otimes \mathscr{B}$ with its natural $\mathscr{A}$–$\mathscr{B}$-module structure. Since Ω is completely positive, we turn $\mathscr{A} \otimes \mathscr{B}$ into a semi-Hilbert $\mathscr{A}$–$\mathscr{B}$-module by setting $\left\langle \sum_i a_i \otimes b_i, \sum_j a'_j \otimes b'_j \right\rangle = \sum_{i,j} b_i^* \Omega(a_i^* a'_j) b'_j$ $(a_i, a'_j \in \mathscr{A}; b_i, b'_j \in \mathscr{B})$. By Corollary 7.3, $E = \mathscr{A} \otimes \mathscr{B}/\mathcal{N}$ is a pre-Hilbert $\mathscr{A}$–$\mathscr{B}$-module. Setting $\xi = \mathbf{1} \otimes \mathbf{1} + \mathcal{N}$, the pair (E, ξ) has the claimed properties. Uniqueness follows in the usual way. ■

Corollary 7.6. *If $\mathscr{A}$ and $\mathscr{B}$ are also C^*-algebras, then the elements of $\mathscr{A}$ act boundedly on E. In particular, $\overline{E}$ is also a representation of $\mathscr{A}$ over $\mathscr{B}$.*

PROOF. Unital C^*-algebras are spanned linearly by their unitaries which necessarily act as bounded operators. ■

Corollary 7.7. *Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{B}$ be a unital C^*-algebra of operators on a Hilbert space G. To any completely positive mapping $\Omega : \mathscr{A} \to \mathscr{B}$ there exists a Hilbert space H with a *-representation ρ_0 of $\mathscr{A}$ on H and a mapping $L_\xi \in \mathscr{B}(G, H)$, such that*

$$\Omega(a) = L_\xi^* \rho_0(a) L_\xi.$$

The pair (ρ_0, L_ξ) is the Stinespring construction of Ω.

Proof. Let (E, ξ) be the GNS-representation of Ω. Perform the construction of the representation Π of $\mathscr{M}^0(\overline{E}, \mathscr{B})$ on $G \oplus H$ as described in Section 3. The representation ρ of $\mathscr{B}^a(E)$ on H induces a representation ρ_0 of $\mathscr{A}$. Then (ρ_0, L_ξ) has the claimed properties. ■

Theorem 7.8. *Corollaries 7.6 and 7.7 remain true, even if $\mathscr{A}$ and $\mathscr{B}$ are not necessarily unital.*

Proof. The construction of $\overline{E} = \overline{\mathscr{A} \otimes \mathscr{B}}$ is still possible. (Observe that E allows an extension of the left multiplication from $\mathscr{A}$ to the unitisation $\widetilde{\mathscr{A}}$. Therefore, the elements of $\mathscr{A}$ still act boundedly on E.) However, the cyclic element $\xi \in E$ is still missing.

First, observe that Ω is (like all positive mappings $\mathscr{A} \to \mathscr{B}$) bounded. (This can be proved as in the case of positive functionals by investigating the series $c = \sum_{n=1}^{\infty} \frac{a_n}{n^2}$, where a_n are positive elements of $\mathscr{A}$ fulfilling $||a_n|| = 1$ and $||\Omega(a_n)|| > n^3$. If such a_n existed, we had $||\Omega(c)|| > ||\Omega\left(\frac{a_n}{n^2}\right)|| > n \to \infty$, so that $c \in \mathscr{A}$ could not be in the domain of Ω. We conclude that Ω must be bounded on positive elements, hence on all elements of the unit-ball of $\mathscr{A}$.) Furthermore, by Section 3 we may assume that $\mathscr{B} \subset \mathscr{B}(G)$ and that $E \subset \mathscr{B}(G, H)$ for suitable Hilbert spaces G and H. The proof is finished if we show that $\xi = \lim_\lambda \lim_\mu u_\lambda \otimes v_\mu + \mathcal{N}$ (for (u_λ) being some approximate unit for $\mathscr{A}$ and (v_μ) being some approximate unit for $\mathscr{B}$) exists strongly in $\mathscr{B}(G, H)$. (For if ξ belongs to $\overline{E}$, then $\Omega(a) = \langle \xi, a\xi \rangle$ follows by continuity of Ω. On the other hand, if ξ does not belong to $\overline{E}$, then consider the Hilbert $\mathscr{A}$–$\mathscr{B}$-submodule $\widetilde{E}$ of $\mathscr{B}(G, H)$ which is generated by ξ.)

So let us choose $g \in G$. Then $\lim_\mu (u_\lambda \otimes v_\mu + \mathcal{N})g$ exists in H and the limit is uniformly in λ, because

$$||(u_\lambda \otimes (v_\mu - v_{\mu'}) + \mathcal{N})g||^2 = \langle (v_\mu - v_{\mu'})g, \Omega(u_\lambda^2)(v_\mu - v_{\mu'})g \rangle \leq ||\Omega||||(v_\mu - v_{\mu'})g||^2$$

and v_μ is a strong Cauchy net. Denote this limit by $u_\lambda \otimes \mathbf{1}g$. Next observe that for any $b \in \mathscr{B}$ the limit $\lim_\lambda (u_\lambda \otimes b)g$ exists strongly in H. This is so because $\tau(a) = \langle g, b^*\Omega(a)bg \rangle$ defines a positive functional on $\mathscr{A}$ and by a standard argument (used, for instance, to show existence of a cyclic vector in usual GNS-construction) $\tau((u_\lambda - u_{\lambda'})^2)$ is close to 0 for sufficiently big λ, λ'. Putting both results together, we obtain that $\lim_\lambda u_\lambda \otimes \mathbf{1}g$ also exists in H. Obviously, $||\xi|| = ||\Omega||$ so that $\xi \in \mathscr{B}(G, H)$. ■

Corollary 7.9. *In addition, the unitisation $\widetilde{\Omega} \colon \widetilde{\mathscr{A}} \to \widehat{\mathscr{B}}$ defined by setting $\widetilde{\Omega}(a) = \Omega(a)$ for $a \in \mathscr{A}$ and $\widetilde{\Omega}(\mathbf{1}) = \langle \xi, \xi \rangle$ is a completely positive mapping, where $\widehat{\mathscr{B}}$ is the universal enveloping von Neumann algebra of $\mathscr{B}$.*

8. Remarks

Remark 8.1. The construction of the space $H = \overline{E \otimes G / \mathcal{N}}$ is a special case of the interior tensor product of two-sided pre-Hilbert modules; see [6]. Let E be a pre-

Hilbert $\mathscr{A}$–$\mathscr{B}$-module and F be a pre-Hilbert $\mathscr{B}$–$\mathscr{C}$-module (cf. Definition 6.1). Then the tensor product $E \odot F$ *over* $\mathscr{B}$ (i.e. the tensor product $E \otimes F$ of vector spaces divided out by all relations $xb \otimes y - x \otimes by$) is turned into a semi-Hilbert $\mathscr{A}$–$\mathscr{C}$-module (cf. Definition 7.1) by defining an inner product via

$$\langle x \odot y, x' \odot y' \rangle = \langle y, \langle x, x' \rangle y' \rangle.$$

To see positivity requires some arguments involving completely positive mappings. Therefore we preferred to show Proposition 3.8 in a more elementary fashion by use of the cyclic decomposition which had to be introduced also for later use in order to prove Lemma 3.9. Moreover, the cyclic decomposition is also available in a more general framework where $\mathscr{B}$ is a *-algebra of not necessarily bounded operators.

Lance [6] shows that if E is a Hilbert module, then $E \odot F$ is a pre-Hilbert module. In our case, this means that $H = \overline{E \odot G}$ where G is a $\mathscr{B}$–$\mathbb{C}$-module in an obvious way. In this context it is clear that an operator a on E defines an operator $(a \odot \mathsf{id})$ on $E \odot G$. However, it remains open if $(a \odot \mathsf{id})$ is bounded. We are in a position to recover this result by an obvious extension of our technique of cyclic decomposition (cf. also Remark 8.4).

Remark 8.2. The second part of Proposition 4.5 and Theorem 4.14 are special cases of results already due to Paschke [9], who shows that the statements are true for all self-dual Hilbert modules. Then he shows that, if E is a Hilbert module over a von Neumann algebra, then E' is a self-dual Hilbert module over the same von Neumann algebra. We proceed somewhat conversely. Theorem 5.6 is a summary of Paschke's main results on Hilbert modules over von Neumann algebras [9].

Theorem 4.3 is due to Frank [5]. We supplement this by the criterion Proposition 4.5 and Theorem 4.16 for Hilbert modules over von Neumann algebras. Frank also shows that a Hilbert $\mathscr{B}$-module E, which is generated as a Hilbert module by an at most countable subset, is self-dual if and only if it is isomorphic to a direct sum of at most countably many copies of a finite-dimensional two-sided unital ideal of $\mathscr{B}$ and of a finitely generated submodule. (The condition to be countably generated replaces the condition of being separable in the context of Hilbert modules. Clearly, if $\mathscr{B}$ is separable, then the two conditions coincide.) This result corresponds to the fact that a separable von Neumann algebra is finite-dimensional. The examples in this paper tell us that the requirement for a self-dual Hilbert module to be countably generated might be felt to be too restrictive.

The proof of Theorem 4.16 is accomplished by means of a suitable substitute for orthonormal bases in usual Hilbert spaces. It would be interesting to mimic Riesz's original proof of its representation theorem; see [12]. He shows, for a given functional Φ on a Hilbert space H, the existence of an element in the unit-ball of H on which Φ assumes its maximal value. We suspect, however, that in the case of Hilbert modules we first have to 'maximise' Φ itself, in the sense that we have to find Φ' with $\|\Phi'\| \leq \|\Phi\|$ but $|\Phi x|^2 \leq |\Phi' x|^2$ for all x. (In this case there should exist a $b \in \mathscr{B}$ such that $\Phi = b\Phi'$.) Another possibility would be to show the existence of a (self-adjoint) projection onto the kernel of Φ.

Remark 8.3. The category of centred modules was introduced in [16], because the centred modules behave particularly well with respect to module tensor products; see Remark 8.1. To see this, consider two centred Hilbert $\mathscr{B}$–$\mathscr{B}$-modules E and F. Obviously, $x \odot y \mapsto y \odot x$ for x in the centre of E and y in the centre of F defines an inner product preserving 'flip' $E \odot F \to F \odot E$. Thus, in particular, the flip is a well-defined isomorphism. In [16] we show (purely algebraic without using the existence of an inner product) that the flip exists and is an isomorphism for the tensor product of arbitrary centred $\mathscr{B}$–$\mathscr{B}$-modules. This good behaviour under building tensor products enables us to formulate in [15] the Bose analogue of Voiculescu's *operator-valued free probability theory* [19].

Let H_1, H_2 be Hilbert spaces. It is easy to check that $(H_1 \otimes \mathscr{B}) \odot (H_2 \otimes \mathscr{B}) = (H_1 \otimes H_2) \otimes \mathscr{B}$ (up to suitable completions). Since an arbitrary centred Hilbert module may be identified as a submodule of $H \otimes \mathscr{B}$, one might expect that the tensor product of centred modules is a simple object. However, if we consider the decomposition of centred von Neumann modules into ideals of $\mathscr{B}$, and if we also take into account $(p\mathscr{B}) \odot (q\mathscr{B}) = pq\mathscr{B}$ (for central projections p, q), then we find that

$$\left(\bigoplus_{\beta \in B} p_\beta \mathscr{B}\right) \odot \left(\bigoplus_{\beta' \in B'} p'_{\beta'} \mathscr{B}\right) = \bigoplus_{\beta \in B, \beta' \in B'} p_\beta p_{\beta'} \mathscr{B}.$$

This shows that the decomposition into ideals for the tensor product can be quite complicated.

Remark 8.4. The GNS-construction for a completely positive mapping $\Omega : \mathscr{A} \to \mathscr{B}$ in the case of a U^*-algebra $\mathscr{A}$ (i.e. a *-algebra which is generated by its quasi-unitaries) is due to Paschke [9]. The Stinespring construction where $\mathscr{A}$ is a C^*-algebra and $\mathscr{B} \subset \mathscr{B}(G)$ for some Hilbert space admits a generalisation (the so-called Kasparov–Stinespring construction) to the case when $\mathscr{B} = \mathscr{B}^a(F)$ for some Hilbert $\mathscr{C}$-module F; see Lance [6] and Murphy [8]. We indicate briefly how this generalisation can easily be obtained easily within our framework.

Let us assume that $\mathscr{C} \subset \mathscr{B}(G)$ for some Hilbert space G. First, we perform the GNS-construction and obtain a Hilbert $\mathscr{A}$–$\mathscr{B}^a(F)$-module E. Then we perform a construction as in Section 3, firstly for F (yielding a Hilbert space H_1 such that $F \subset \mathscr{B}(G, H_1)$), and secondly for $E \odot F$ (yielding a Hilbert space H_2 such that $E \odot F \subset \mathscr{B}(G, H_2)$); cf. Remark 8.1. Using our technique of cyclic decomposition, it is not difficult to establish two analogues of Lemma 3.9 which assert that $E \subset \mathscr{B}^a(F, E \odot F) \subset \mathscr{B}(H_1, H_2)$ and $\mathscr{B}^a(E) \subset \mathscr{B}^a(E \odot F) \subset \mathscr{B}(H_2)$, isometrically. Thus we obtain a Hilbert $\mathscr{A}$–$\mathscr{B}$-module $\overline{E \odot F}$ and a mapping $\ell_\xi \in \mathscr{B}^a(F, E \odot F)$ (the image of the cyclic element $\xi \in E$), such that $\Omega(a) = \ell_\xi^* a \ell_\xi$.

Remark 8.5. The notion of a generalised matrix *-algebra can be extended to order $n > 2$ in an obvious manner. In this context the tensor product of Hilbert modules has a nice interpretation (see Remark 8.1). So let $\mathscr{M} = (B_{ij})$ be a generalised $n \times n$-matrix C^*-algebra. It is easy to see that the mapping $b_{ik} \odot b_{kj} \mapsto b_{ik} b_{kj}$ extends to an inner product preserving mapping into B_{ik}. This means that $B_{ij} \odot B_{kj}$

and $\mathrm{span}(B_{ik}B_{kj})$ are isomorphic pre-Hilbert modules. In particular, we see that $B_{ij} \odot B_{kj}^* = \mathfrak{F}_{B_{jj}}(B_{kj}, B_{ij})$.

In order to embed a Hilbert $\mathscr{B}$-module E, and a Hilbert $\mathscr{B}$–$\mathscr{C}$-module F and their tensor product $E \odot F$ into a generalised 3×3-matrix algebra, some restrictions have to be posed. In Section 3 we always considered E as a $\mathscr{B}^a(E)$–$\mathscr{B}$-module. In particular, we observe that the C^*-algebra $\mathscr{B}^a(E)$ acts faithfully on E. This enabled us to equip E^* with a natural Hilbert $\mathscr{B}$–$\mathscr{B}^a(E)$-module structure. Of course, the closure of the range of the inner product of E^* is $\mathcal{K}_{\mathscr{B}}(E)$. Thus, in order to equip F^* with a Hilbert $\mathscr{C}$–$\mathscr{B}$-module structure, $\mathscr{B}$ has to act faithfully on F so that we may identify $\mathscr{B}$ as a subset of $\mathscr{B}^a(F)$, and this subset must contain $\mathcal{K}_{\mathscr{C}}(F)$. (We can always achieve this by replacing E with the tensor product $E \odot \mathscr{B}^a(F)$ over $\mathscr{B}$ where we consider $\mathscr{B}^a(F)$ as a Hilbert $\mathscr{B}$–$\mathscr{B}^a(F)$-module in the obvious manner.) Using the technique of cyclic decomposition in a manner completely analogous to Section 3, one can show that

$$\mathscr{M} = \begin{pmatrix} \mathscr{C} & F^* & \overline{F^* \odot E^*} \\ F & \mathscr{B} & E^* \\ \overline{E \odot F} & E & \mathcal{K}_{\mathscr{B}}(E) \end{pmatrix}$$

is a generalised matrix C^*-algebra. Therefore $\overline{E \odot F} = \mathcal{K}_{\mathscr{B}}(F^*, E)$. Of course, here we are also able to represent $\mathscr{M}$ as a C^*-subalgebra of $\mathscr{B}(G \oplus H_1 \oplus H_2)$ where G carries a representation of $\mathscr{C}$, $H_1 = \overline{F \odot G}$ and $H_2 = \overline{E \odot F \odot G} = \overline{E \odot H_1}$.

Remark 8.6. We remind the reader of Blecher's paper [4]. Many statements, particularly those concerning tensor products, have beautiful interpretations when in his terminology they become statements on abstract operator spaces. He also observes that $\mathscr{M}(E, \mathscr{B})$ (and also $\mathscr{M}^0(E, \mathscr{B})$) have an obvious interpretation as *-algebras of bounded operators on the Hilbert $\mathscr{B}$-module $\mathscr{B} \oplus E$. Taking into account Proposition 2.4, we see that the operator norm coincides with the unique complete C^*-norm. In this way Blecher finds a proof of Theorem 3.10 much quicker than ours. (Notice that this technique also shows the existence of a unique C^*-norm on $M_n(\mathscr{B})$ without reference to a faithful representation of $\mathscr{B}$.) We emphasise, however, that it does not seem to be possible to prove Theorem 3.11 in this manner because $\mathscr{B}^r(E)$, in general, is not a C^*-algebra. Thus there is no simple reason why a complete norm should be unique.

Remark 8.7. Any Hilbert $\mathscr{B}$-module may be identified as a subset of $\mathscr{B}(G; H \otimes G)$. If E is a $\mathscr{B}$–$\mathscr{B}$-module, then $\mathscr{B}$ acts from the left via a *-representation ρ on $H \otimes G$. If E is centred, we have one of the extreme possibilities where $\mathscr{B}$ acts on G alone; see proof of Theorem 6.13.

The other extreme possibility is $\rho = \rho_H \otimes \mathbf{1}$ where ρ_H is a *-representation of $\mathscr{B}$ on H alone. For instance, suppose that $H = \mathbb{C}$ so that ρ_H is a character. Then $bx = 0$ for b in the kernel of ρ_H and all $x \in E$. (Such a module cannot be centred unless either $\mathscr{B} = \mathbb{C}$ or $E = \{0\}$.) A module with such a left multiplication indeed appears. We can use it to reduce the calculus for the third type of quantum independence (besides Bose and free independence), the so-called Boolean independence (see [14]), to the calculus on full Fock modules (see [17]).

We remark that the case where $\rho = \rho_H \otimes \mathbf{1}$ is nothing but the *exterior* tensor product of the Hilbert $\mathscr{B}$–$\mathbb{C}$-module H and the Hilbert $\mathbb{C}$–$\mathscr{B}$-module $\mathscr{B}$. In general, for a pre-Hilbert $\mathscr{A}_1$–$\mathscr{B}_1$-module E_1 and a pre-Hilbert $\mathscr{A}_2$–$\mathscr{B}_2$-module E_2 the exterior tensor product is the pre-Hilbert $\mathscr{A}_1 \otimes \mathscr{A}_2$–$\mathscr{B}_1 \otimes \mathscr{B}_2$-module $E_1 \otimes E_2$ with inner product defined by setting $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \otimes \langle x_2, y_2 \rangle$; see [6]. Again, to see positivity requires some knowledge of completely positive mappings, whereas *strict* positivity is seen easily from Cauchy–Schwartz inequality and a basic fact about linear independence in tensor products; see [16].

In between the two extreme cases $\rho = \mathbf{1} \otimes \mathsf{id}$ and $\rho = \rho_H \otimes \mathbf{1}$ we obtain other examples from a Hopf *-algebra $\mathscr{B}$ with a faithful Haar state $h: \mathscr{B} \to \mathbb{C}$. We identify $\mathscr{B}$ with the GNS-representation of h and denote by G the representation space. Then $E = G \otimes \mathscr{B}$ has a natural pre-Hilbert $\mathscr{B}$-module structure. We identify E as a subset of $\mathscr{B}(G, G \otimes G)$. The comultiplication $\Delta: \mathscr{B} \to \mathscr{B} \otimes \mathscr{B}$ gives rise to representation on $G \otimes G$ where we interpret elements of $\mathscr{B} \otimes \mathscr{B}$ as operators on $G \otimes G$. This is an example which is not among the types indicated before. (Otherwise, either $\mathscr{B} = \mathbb{C}$ or we obtain a contradiction to coassociativity.)

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Appendix A. General representations of Hilbert modules

In Section 3 we showed how for any Hilbert $\mathscr{B}$-module E a faithful representation of $\mathscr{B}$ can be extended to a faithful representation of $\mathscr{M}(E,\mathscr{B})$ on $G \oplus H$. Restricting this representation to E, we obtained an injective mapping η from E into $\mathscr{B}(G,H)$. Clearly, η fulfils

$$\eta(x\langle y,z\rangle) = \eta(x)\eta(y)^*\eta(z). \tag{A 1}$$

In this appendix we use (A 1) to define what we understand by a representation of a pre-Hilbert module E intrinsically, i.e. without embedding E into a matrix *-algebra. Then, using the notation from Remark 2.11, we show to what extent a representation η of E in $\mathscr{B}(G,H)$ extends to a representation Π of a matrix *-algebra $\mathscr{M}$ which contains E as a lower submodule in $\mathscr{B}(G \oplus H)$. We show that (A 1) is already sufficient to guarantee the existence of such an extension at least to what is generated algebraically by E. However, if we want to extend the representation to the C^*-completions of $\mathscr{B}$ and $\mathscr{A}$, then we have to pose the additional condition on η to be *completely bounded.*

Definition A.1. Let E be a pre-Hilbert $\mathscr{B}$-module. A *representation* of E on a pair (G,H) of Hilbert spaces is a mapping $\eta\colon E \to \mathscr{B}(G,H)$ fulfilling (A 1).

A representation has *non-degenerate range* if $H_\eta := \overline{\operatorname{span}}(\eta(E)G) = H$, and has *non-degenerate domain* if $G_\eta := \overline{\operatorname{span}}(\eta(E)^*H) = G$. We say that a representation is *non-degenerate* if it has both non-degenerate range and non-degenerate domain.

As in Remark 2.11, let $\mathscr{M}$ be a matrix *-algebra which contains E as a lower submodule. By

$$\mathscr{M}_0 = \begin{pmatrix} \mathscr{B}_0 & E^* \\ E & \mathscr{A}_0 \end{pmatrix}$$

we denote the *-subalgebra of $\mathscr{M}$ generated by E. Obviously, $\mathscr{B}_0$ is the linear span of all x^*y $(x,y \in E)$ and $\mathscr{A}_0$ is the linear span of all xy^* $(x,y \in E)$. We conclude that $\mathscr{B}_0$ is a *-ideal in $\mathscr{B}$, that $\mathscr{A}_0$ is a *-ideal in $\mathscr{A}$ and, consequently, that $\mathscr{M}_0$ is a *-ideal in $\mathscr{M}$.

Proposition A.2. *Any representation η of E extends to a *-representation*

$$\Pi_0 = \begin{pmatrix} \pi_0 & \eta^* \\ \eta & \rho_0 \end{pmatrix}$$

of $\mathscr{M}_0$. If η has non-degenerate domain and non-degenerate range, then π_0 and ρ_0, respectively, are unique.

PROOF. We put $\eta^*(x^*) = \eta(x)^*$, we put $\pi_0(x^*y) = \eta(x)^*\eta(y)$ and we put $\rho_0(xy^*) = \eta(x)\eta(y)^*$. From (A 1) we see that if Π_0 is well defined, then it is a *-representation of $\mathscr{M}_0$. So we have to show that π_0 and ρ_0 are well defined.

Suppose that $b = \sum_i x_i^* y_i = 0$. Therefore, by (A 1) we find that $\sum_i \eta(z)\eta(x_i)^*\eta(y_i) = \eta(zb) = 0$ for all $z \in E$. Since all operators $\eta(x)^*\eta(y)$ map into G_η, this implies that $\sum_i \eta(x_i)^*\eta(y_i) = 0$. Similarly, for $a = \sum_i x_i y_i^* = 0$ we find that $\sum_i \eta(x_i)\eta(y_i)^*\eta(z) = 0$ for all $z \in E$, which implies that $\sum_i \eta(x_i)\eta(y_i)^* = 0$. This shows well-definedness.

Clearly, the represention Π_0 is determined uniquely on the invariant subspace $G_\eta \oplus H_\eta$. ■

An arbitrary element in H_η may be approximated by sums of elements of the form $\eta(x)g$ where $x \in E$ and $g \in G_\eta$. One might try to extend the representation ρ_0 to an element $a \in \mathscr{A}$ by defining an operator on H_η which sends $\eta(x)g$ to $\eta(ax)g$. However, this need not always be possible, as there are representations of a pre-C^*-algebra which do not extend to the C^*-completion. The following definition will exclude such representations.

Definition A.3. A representation η of E is called *completely bounded for $\mathscr{B}$* if for any $b \in \mathscr{B}$ there exists a constant $M_b > 0$, such that

$$\left\|\sum_i \eta(x_i b^*)^*\eta(y_i)\right\| \le M_b \left\|\sum_i \eta(x_i)^*\eta(y_i)\right\|$$

for all $x_i, y_i \in E; i = 1, \ldots, n; n \in \mathbb{N}$.

η is called *completely bounded for $\mathscr{A}$* if for any $a \in \mathscr{A}$ there exists a constant $M_a > 0$, such that

$$\left\|\sum_i \eta(ax_i)\eta(y_i)^*\right\| \le M_a \left\|\sum_i \eta(x_i)\eta(y_i)^*\right\|$$

for all $x_i, y_i \in E; i = 1, \ldots, n; n \in \mathbb{N}$.

η is called *completely bounded* if it is both completely bounded for $\mathscr{B}$ and completely bounded for $\mathscr{A}$.

Theorem A.4. *A representation η of E extends (uniquely on $G_\eta \oplus H_\eta$) to a *-representation*

$$\Pi = \begin{pmatrix} \pi & \eta^* \\ \eta & \rho \end{pmatrix}$$

of $\mathscr{M}$ on $G \oplus H$ if and only if η is completely bounded.

Proof. The 'only if' direction is clear from the remark before Definition A.3. So let us assume that η is completely bounded. Let b be in $\mathscr{B}$. The first condition on a completely bounded representation guarantees that $||\pi_0(bb_0)|| \le M_b||\pi_0(b_0)||$ and $||\pi_0(b_0 b)|| \le M_{b^*}||\pi_0(b_0)||$ for all $b_0 \in \mathscr{B}_0$. Moreover, $b_0\pi_0(bb_0') = \pi_0(b_0 b)b_0'$ so that b acts as double centraliser on $\pi_0(\mathscr{B}_0)$. Hence the representation π_0 of $\mathscr{B}_0$ on G extends to a representation π of $\mathscr{B}$ on G. A similar statement is true for the representation ρ_0 of $\mathscr{A}_0$ on H. Clearly, these are the ingredients of the claimed representation Π. ■

Remark A.5. Let π be an isometric *-representation of $\mathscr{B}$ on G. In the construction of the mapping $\eta : E \to \mathscr{B}(G, H)$ in Section 3 it is not necessary to know that (3.3a) is valid. In particular, it is not necessary to know that (3.3b) defines a norm on E. We easily find that η is completely bounded for $\mathscr{B}$ so that η extends to a *-representation of $\mathscr{M}(E, \mathscr{B})$ by Theorem A.4.

Now it is clear that (3.3a) holds with all implications. In particular, $\mathscr{B}^a(E)$ is known to be a C^*-algebra. Clearly, η is completely bounded for $\mathscr{B}^a(E)$ so that, again by Theorem A.4, η extends to a *-representation of $\mathscr{M}^0(E, \mathscr{B})$. Obviously, this *-representation must be faithful. This may serve as an alternative proof of Theorem 3.10 without proving (3.3a) first.

By the algebraic methods developed in the appendix of [16] we can further extend the *-representation ρ of $\mathscr{B}^a(E)$ to a representation of $\mathscr{B}^r(E)$ on H. (These methods are purely algebraic and do not use the existence of an inner product.) Clearly, this extension must be an isometry so that we also have an alternative proof for Theorem 3.11.

Finally, we remark that the construction in Section 3 does not depend on the fact that the representation π of $\mathscr{B}$ is faithful. (This only guarantees that the extension of π to $\mathscr{M}^0(E, \mathscr{B})$ is faithful.) Also, when π is no longer faithful, we still obtain an extension of π to $\mathscr{M}^0(E, \mathscr{B})$ in the same manner. Similarly, here the representation ρ also extends from $\mathscr{B}^a(E)$ to $\mathscr{B}^r(E)$. Of course, ρ will no longer be an isometry. But we still have $\|\rho(a)\| \leq \|a\|$, which was the more difficult part in the proof of Lemma 3.9.

Appendix B. A lemma on semi-normed spaces

Lemma B.1. *Let E denote a vector space with a (non-trivial) semi-norm $\|\bullet\|$. Denote by $\mathcal{N}$ the subspace of elements $x \in E$ with $\|x\| = 0$. We define $\mathscr{B}(E)$ as the space consisting of all linear operators T on E such that*

$$\sup_{\|x\| \leq 1} \|Tx\| < \infty.$$

Then the elements of $\mathscr{B}(E)$ leave invariant $\mathcal{N}$ so that $T \in \mathscr{B}(E)$ gives rise to a bounded operator $x + \mathcal{N} \mapsto Tx + \mathcal{N}$ on the normed space $E/\mathcal{N}$.

PROOF. We conclude indirectly. Suppose that $T \in \mathscr{L}(E)$ is a linear operator on E and x is an element in E such that $\|x\| = 0$, but $\|Tx\| = 1$. Then $\|\lambda x + Tx\| \leq \lambda\|x\| + \|Tx\| = 1$, but $\|T(\lambda x + Tx)\| \geq \big|\lambda\|Tx\| - \|T^2x\|\big|$ so that $\sup_{\|y\| \leq 1} \|Ty\| \geq \sup_{\lambda} \|T(\lambda x + Tx)\| = \infty$. Therefore T is not an element of $\mathscr{B}(E)$. ■

Note added: We would like to mention some interesting results on von Neumann modules found in [3]. The GNS-module of a *normal* completely positive mapping between two von Neumann algebras is a two-sided von Neumann module, automatically. Each von Neumann module E over $\mathscr{B}(G)$ is necessarily all of $\mathscr{B}(G, H)$. If, additionally, E is a two-sided von Neumann $\mathscr{B}(G)$–$\mathscr{B}(G)$-module, then it has the form $\mathscr{B}(G, H' \otimes G)$ where H' is a suitable Hilbert space. In particular, it is centred automatically. This shows that there are many natural examples of centred Hilbert modules. For instance, the theory of E_0-semigroups on $\mathscr{B}(G)$ as developed by Arveson [2] has a plentiful supply of centred Hilbert modules. In [3] it is shown that Arveson's tensor product systems of Hilbert spaces form precisely the $\mathscr{B}(G)$-centres of an associated tensor product system of Hilbert modules.