THE AFFINE WEYL GROUP OF TYPE $\tilde{C}_{n-1}$

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Abstract

We show that the group $\tilde{C}_{n-1}$ is the split extension of $n-1$ copies of the infinite dihedral group (D_∞) by the symmetric group of degree $n-1$ (S_{n-1}). We also consider some important subgroups of $\tilde{C}_{n-1}$.

1. Introduction

The affine Weyl group of type $\tilde{C}_{n-1}$ is one of the infinite irreducible Coxeter groups [11]. It has the following presentation:

$$\begin{aligned}\tilde{C}_{n-1} = \langle y_1, y_2, \ldots, y_n | y_i^2 &= 0, \quad 1 \le i \le n,\\ y_i y_j &= y_j y_i, \quad 1 \le i < j-1 \le n-1,\\ y_i y_{i+1} y_i &= y_{i+1} y_i y_{i+1}, \quad 2 \le i \le n-1,\\ (y_1 y_2)^4 &= (y_{n-1} y_n)^4 = e\rangle, \quad n \ge 3.\end{aligned}$$

The graph associated with this group is

$\tilde{C}_2$ is the triangle group $\triangle(2,4,4)$ which is one of the Euclidean triangle groups. It is an affine Weyl group [11].

In our paper [2], we showed that $\tilde{C}_3 \cong D_\infty^3 \rtimes S_3$, where D_∞ is the infinite dihedral group and S_3 is the symmetric group of degree 3. The proof contained difficult and tedious Tietze transformations. Some other properties of $\tilde{C}_3$ were also considered.

2. The structure of $\tilde{C}_{n-1}$

We show in this section that $\tilde{C}_{n-1}$ is the wreath product $D_\infty S_{n-1}$ (with the 'natural' action), which implies that $\tilde{C}_{n-1} \cong D_\infty^{n-1} \rtimes S_{n-1}$. We begin with some notation and a lemma. Let $\tilde{C}_{n-1}$ have the presentation given in §1. We consider the following

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presentation for S_{n-1}, the symmetric group of degree $n-1$:

$$\begin{aligned} S_{n-1} = \langle x_1, x_2, \ldots, x_{n-2} | x_i^2 = e, &\quad 1 \leq i \leq n-2, \\ x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}, &\quad 1 \leq i \leq n-3, \\ x_i x_j = x_j x_i, &\quad 1 \leq i < j-1 \leq n-3 \rangle, \quad n \geq 3. \end{aligned}$$

Let $\triangle_i = x_2 x_3 \cdots x_i$, where $2 \leq i \leq n-2$ and $\triangle_1 = e$. We consider the following presentation for $A = D_\infty^{n-1} = \langle a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} | a_i^2 = b_i^2 = e$, where $1 \leq i \leq n-1$:

$$a_i a_j = a_j a_i, \quad 1 \leq i < j \leq n-1, \quad \text{for all } i, j, \tag{1}$$

$$b_i b_j = b_j b_i, \quad 1 \leq i < j \leq n-1, \quad \text{for all } i, j, \tag{2}$$

$$a_i b_j = b_j a_i \text{ if } i \neq j \rangle . \tag{3}$$

To show that $\tilde{C}_{n-1} \cong D_\infty^{n-1} \rtimes S_{n-1}$, we use the method of presentations of group extensions explained in [1]. We find an epimorphism $\theta\colon \tilde{C}_{n-1} \to S_{n-1}$ such that the extension

$$1 \to \ker \theta \to \tilde{C}_{n-1} \to S_{n-1} \to 1 \tag{I}$$

splits. It will be required to find a presentation for ker θ. We guess that it will be isomorphic to $A = D_\infty^{n-1}$ (given by generators and relations). We then construct a new short exact sequence (II), where A is embedded as a normal subgroup of a group E (also given by generators and relations) in such a way that A is the kernel of an epimorphism $\theta'\colon E \to S_{n-1}$.

$$\begin{tikzcd} 1 \arrow[r] & \ker \theta \arrow[r] & \tilde{C}_{n-1} \arrow[r, "\theta"] & S_{n-1} \arrow[r] & 1 & \text{(I)} \\ 1 \arrow[r] & A \arrow[r] \arrow[u, dashed] & E \arrow[r, "\theta'"] \arrow[u, dashed, "\varphi"'] & S_{n-1} \arrow[r] \arrow[u, "\mathrm{id}"'] & 1. & \text{(II)} \end{tikzcd}$$

Then we use the Tietze transformations to identify E with $\tilde{C}_{n-1}$, i.e. to find an isomorphism $\phi\colon E \to \tilde{C}_{n-1}$ that makes the right square above commute. It then follows that θ induces an isomorphism $A \to$ ker θ. It also follows that (II) is isomorphic (as a short exact sequence) to (I), which means that we have a new version of our original presentation in which the kernel is displayed explicitly.

We define the map $\theta\colon \tilde{C}_{n-1} \to S_{n-1}$ by $\theta(y_i) = x_{i-1}$, where $2 \leq i \leq n-1$, $\theta(y_1) = \theta(y_n) = e$. θ is an epimorphism and the short exact sequence

$$1 \to \ker \theta \to \tilde{C}_{n-1} \to S_{n-1} \to 1$$

splits because θ is a retract. We define a natural action of the group S_{n-1} on A as follows:

$$(a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{n-1})^{x_i} = (a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_{n-1}),$$

$$(b_1, b_2, \ldots, b_i, b_{i+1}, \ldots, b_{n-1})^{x_i} = (b_1, b_2, \ldots, b_{i+1}, b_i, \ldots, b_{n-1}).$$

We change the action to the following form:

$$a_k^{x_i} = \begin{cases} a_{k+1} & \text{if } i = k, \quad 1 \le i \le n-2 \quad (4) \\ a_{k-1} & \text{if } i = k-1, \quad 1 \le i \le n-2 \quad (5) \\ a_k & \text{otherwise}, \quad 1 \le i \le n-2. \quad (6) \end{cases}$$

$$b_k^{x_i} = \begin{cases} b_{k+1} & \text{if } i = k, \quad 1 \le i \le n-2 \quad (4') \\ b_{k-1} & \text{if } i = k-1, \quad 1 \le i \le n-2 \quad (5') \\ b_k & \text{otherwise}, \quad 1 \le i \le n-2. \quad (6') \end{cases}$$

We now consider the group E, which is the split extension of A by S_{n-1} with the above action. A presentation for E is:

$$E = \langle a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1}, x_1, x_2, \ldots, x_{n-2} | RA, RS_{n-1}, A^{S_{n-1}} \rangle,$$

where RA are the relations of A, RS_{n-1} are the relations of S_{n-1}, and $A^{S_{n-1}}$ is the action of S_{n-1} on A. Our aim will be to show that E is isomorphic to $\tilde{C}_{n-1}$. We begin by simplifying the relations of E using the following lemma.

Lemma 1. *In* S_{n-1} *the following identities hold:*
- (i) $\triangle_k x_i = x_{i+1} \triangle_k$ *if* $2 \le i < k$,
- (ii) $\triangle_k x_i = \triangle_{k-1}$ *if* $i = k$,
- (iii) $\triangle_k x_i = \triangle_{k+1}$ *if* $i = k+1$,
- (iv) $\triangle_k x_i = x_i \triangle_k$ *if* $i > k+1$,
- (v) $\triangle_k \triangle_i = (x_3 x_4 \cdots x_{i+1}) \triangle_k$ *if* $2 \le i < k$,
- (vi) $\triangle_k \triangle_i = (x_3 x_4 \cdots x_i) \triangle_{k-1}$ *if* $i \ge k$.

PROOF. We prove this lemma using the relations of the group S_{n-1}.

(i)
$$\begin{aligned} \triangle_k x_i &= (x_2 x_3 \cdots x_i x_{i+1} \cdots x_k) x_i \\ &= x_2 x_3 \cdots x_i x_{i+1} x_i \cdots x_k \\ &= x_2 x_3 \cdots x_{i+1} x_i x_{i+1} \cdots x_k \ \cdot \\ &= x_{i+1} x_2 x_3 \cdots x_i x_{i+1} \cdots x_k \\ &= x_{i+1} \triangle_k. \end{aligned}$$

(ii), (iii) and (iv) are clear. (v) and (vi) are applications of (i) to (iv).

We observe that relation (4) implies that $a_i^{x_i} = a_{i+1}$, where $2 \le i \le n-2$, which easily implies that $a_i = a_1^{\triangle_i}$, where $1 \leq i \leq n-1$. Similarly relation (4′) implies that $b_i = b_1^{\triangle_i}$, where $2 \leq i \leq n-1$. Using these two relations we simplify the presentation of E as in the following proposition.

Proposition 2. *In the group E the following holds:*

(i) *Relation* (1) *is equivalent to* $(a_1x_1)^4 = e$.
(ii) *Relation* (2) *is equivalent to* $(b_1x_1)^4 = e$.
(iii) *Relation* (3) *is equivalent to* $(a_1x_1b_1x_1)^2 = e$.
(iv) *Relations* (5) *and* (5′) *become redundant.*
(v) *Relation* (6) *becomes* $(a_1x_j)^2 = e, \quad 2 \leq j \leq n-2$.
(vi) *Relation* (6′) *becomes* $(b_1x_j)^2 = e, \quad 2 \leq j \leq n-2$.

Proof. In all parts we use the relations $a_i = a_1^{\triangle_i}$, and $b_i = b_1^{\triangle_i}$, where $2 \leq i \leq n-1$ together with Lemma 1.

(i) Relation (1) is $a_ia_j = a_ja_i$, where $1 \leq i \leq j \leq n-1$. $\Rightarrow$ $a_1^{\triangle_i}a_1^{\triangle_j} = a_1^{\triangle_j}a_1^{\triangle_i}$. Using (6), (6′) and Lemma 1 (v), we get $(a_1x_1)^4 = e$. Parts (ii) to (vi) are proved similarly. ■

We observe that $a_2 = x_1a_1x_1$. Thus $a_1 = x_1a_2x_1$ and $a_2^2 = e$. We change the relations in the proposition as follows:

$$(a_1x_1)^4 = e \quad \text{to} \quad (x_1a_2)^4 = e,$$

$$(b_1x_1a_1x_1)^2 = e \quad \text{to} \quad (b_1a_2)^2 = e,$$

$$(a_1x_i)^2 = e \quad \text{to} \quad (x_1a_2x_1x_i)^2 = e.$$

Hence we get the following presentation for E:

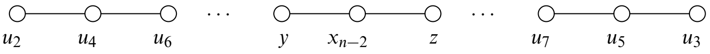
$$\begin{aligned} E = \langle a_2, b_1, x_1, x_2, \ldots, x_{n-2} | a_2^2 = b_1^2 = x_1^2 = x_2^2 = \cdots = x_{n-2}^2 = e, \\ (x_i + x_{i+1})^3 = e \text{ if } 1 \leq i \leq n-3, \\ (x_ix_j)^2 = e \text{ if } 1 \leq i < j-1 \leq n-3, \\ (x_1a_2)^4 = (b_1x_1)^4 = (b_1a_2)^2 = e, \\ (x_1a_2x_1x_i)^2 = (b_1x_i)^2 = e, \ 2 \leq i \leq n-2 \rangle. \end{aligned}$$

We let $u_i = x_ix_{i-1}x_i$ in the group S_n, where $2 \leq i \leq n-1$.

Lemma 3. *The group S_{n-1} has the presentation*

$$\begin{aligned} G = \langle u_2, u_3, \ldots, u_{n-2}, x_{n-2} | u_2^2 = u_3^2 = \cdots = u_{n-2}^2 = x_{n-2}^2 = e, \\ (u_iu_{i+2})^3 = e, \ 2 \leq i \leq n-4, \\ (u_iu_j)^2 = e, \ j \neq i+2, \\ (u_{n-2}x_{n-2})^3 = (u_{n-3}x_{n-2})^3 = e, \\ (u_ix_{n-2})^2 = e, \ i \neq n-2 \text{ or } n-3 \rangle. \end{aligned}$$

The Coxeter graph of S_{n-1} associated with this presentation is



where y or z is u_{n-2} and the other is u_{n-3}.

PROOF. Using the relations of S_{n-1}, it is easy to see that $u_i^2 = e$, where $2 \leq i \leq n-2$. We show that $(u_i u_{i+2})^3 = e$, where $2 \leq i \leq n-4$.

$$\begin{aligned}(u_i u_{i+2})^3 &= (x_i x_{i-1} x_i x_{i+2} x_{i+1} x_{i+2})^3\\ &= x_i x_{i-1} x_i x_{i+2} x_{i+1} x_{i+2} x_i x_{i-1} x_i x_{i+2} x_{i+1} x_{i+2} x_i x_{i-1} x_i x_{i+2} x_{i+1} x_{i+2}\\ &= x_{i-1} x_i x_{i-1} x_{i+2} x_{i+1} x_{i+2} x_{i-1} x_i x_{i-1} x_{i+2} x_{i+1} x_{i+2} x_{i-1} x_i x_{i-1} x_{i+2} x_{i+1} x_{i+2}\\ &= x_{i-1} x_{i+2} x_i x_{i+1} x_i x_{i+1} x_i x_{i+1} x_{i-1} x_{i+2}\\ &\quad\; x_{i-1} x_{i+2} x_{i-1} x_{i+2} = e.\end{aligned}$$

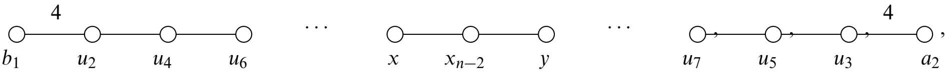
The other relations of G can be shown similarly. The graph is obvious from the relations. ■

Now we use the substitution $u_i = x_i x_{i-1} x_i$ in the presentation of E and replace the generators and relations of S_{n-1} by those of G. We also do the following Tietze transformations in E.

The relation $(x_1 a_2 x_2 x_1)^2 = e$ becomes $(a_2 u_2)^2 = e$. The relations $(x_1 a_2 x_1 x_i)^2 = e$ become $(a_2 x_i)^2 = e$, which become $(a_2 x_{n-2})^2 = e$ and $(a_2 u_i)^2 = e$, where $4 \leq i \leq n-2$. The relation $(x_1\, a_2)^4 = e$ becomes $(u_3 a_2)^4 = e$ by using also $(a_2 u_2)^2 = e$. The relation $(b_1 x_1)^4 = e$ becomes $(b_1 u_2)^4 = e$. A presentation for E will be

$$\begin{aligned}E = \langle b_1, u_2, u_3, \ldots, u_{n-2}, x_{n-2}, a_2 \mid b_1^2 = u_i^2 = x_{n-2}^2 = a_2^2 = e, \quad 2 \leq i \leq n-2,&\\ (u_i u_{i+2})^3 = e, \quad 2 \leq i \leq n-4,&\\ (u_i u_j)^2 = e, \quad |i-j| \neq 2,&\\ (u_{n-2} x_{n-2})^3 = (u_{n-3} x_{n-2})^3 = e,&\\ (u_i x_{n-2})^2 = e, \quad i \neq n-2 \text{ or } n-3,&\\ (b_1 u_2)^4 = (u_3 a_2)^4 = (b_1 a_2)^2 = (b_1 x_{n-2})^2 = (a_2 x_{n-2})^2,&\\ (a_2 u_2)^2 = (a_2 u_i)^2 = e, \quad 4 \leq i \leq n-2,&\\ (b_1 u_i)^2 = e, \quad 3 \leq i \leq n-2\rangle.&\end{aligned}$$

The graph of E associated with this presentation is:



where x or y is u_{n-2} and the other is u_{n-3}.

This shows that E is isomorphic to $\tilde{C}_{n-1}$, and hence $\tilde{C}_{n-1}$ is the split extension of D_∞^{n-1} by S_{n-1}.

3. Important subgroups of $\tilde{C}_{n-1}$

We let $\tilde{C}_{n-1}$ have the presentation given in section 1 and Z_2 has the presentation $\langle y | y^2 = e\rangle$. We define the epimorphism $\theta\colon \tilde{C}_{n-1} \to Z_2$ by $\theta(x_1) = y$ and $\theta(x_i) = e$, for $2 \le i \le n$. Then $\frac{\tilde{C}_{n-1}}{\ker\theta} \cong Z_2$. We use $\{e, x_1\}$ as a Schreier transversal for ker θ in $\tilde{C}_{n-1}$. Using the Reidemeister–Schreier process, we find that ker θ is isomorphic to T_n [6], the Coxeter group whose graph is

Thus we have shown that T_n is a normal subgroup of $\tilde{C}_{n-1}$ of index 2.

We now consider an epimorphism from $\tilde{C}_{n-1}$, $n \ge 5$ to $Z_2 \times Z_2 = \langle y_1, y_2 | y_1^2 = y_2^2 = (y_1y_2)^2 = e\rangle$ defined by $\theta(x_1) = y_1$, $\theta(x_n) = y_2$, $\theta(x_i) = e$, for $2 \le i \le n-1$. Thus $\frac{\tilde{C}_{n-1}}{\ker\theta} \cong Z_2 \times Z_2$ and $\{e, x_1, x_n, x_1x_n\}$ is a Schreier transversal for ker θ in $\tilde{C}_{n-1}$. The Reidemeister–Schreier process gives a presentation for ker θ isomorphic to the Coxeter group Q_n whose graph is

Thus the group Q_n is a normal subgroup of $\tilde{C}_{n-1}$ of index 4. Using similar arguments it is easy to see that Q_n is a normal subgroup of T_n of index 2 and that P_4 (the Coxeter group whose graph is square) is a normal subgroup of T_4 of index 2.

To find the centre of $\tilde{C}_{n-1}$ we use the fact $\tilde{C}_{n-1} = D_\infty^{n-1} \rtimes S_{n-1}$, which gives us $Z(\tilde{C}_{n-1}) \subseteq Z(D_\infty^{n-1})$ since $Z(S_{n-1}) = \{e\}$. It is easy to see that $Z(D_\infty) = \{e\}$ and so $Z(\tilde{C}_{n-1}) \subseteq Z(D_\infty^{n-1}) \subseteq (Z(D_\infty))^{n-1} = \{e\}$. Hence $\tilde{C}_{n-1}$ has a trivial centre.

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