SOME COMMUTATIVITY THEOREMS FOR RINGS WITH POLYNOMIAL CONSTRAINTS

By Mohammad Ashraf
Department of Mathematics, King Abdulaziz University, Saudi Arabia

[Received 16 October 1998. Read 26 April 1999. Published 31 August 2000.]

Abstract

Let $m > 1, r, s$ be fixed non-negative integers, and let R be a ring with unity 1 in which for every x in R there exist polynomials $f(X), p(X), q(X) \in \mathbb{Z}[X]$, depending on x, such that $p(x)[f(x), y]q(x) = x^r[x, y^m]y^s$, for all y in R. The main result of the present paper asserts that R is commutative if R has the property $Q(m)$ (viz. for all x, y in R, $m[x, y] = 0$ implies that $[x, y] = 0$). Commutativity of R has also been obtained under different sets of constraints on integral exponents. Thus, many well-known commutativity theorems become corollaries of our results.

1. Introduction

Throughout the present paper, R will denote an associative ring with centre $Z(R)$, $N(R)$ the set of nilpotent elements of R and $C(R)$ the commutator ideal of R. R is said to be a *left* (resp. *right*) *s-unital* ring if for each $x \in R$, $x \in Rx$ (resp. $x \in xR$), and R is called *s-unital* if it is both left and right s-unital. As usual, $\mathbb{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbb{Z}$, the ring of integers. For any x, y in R, we set $[x, y] = xy - yx$. Consider the following ring properties:

- (P_1) For each x in R there exist polynomials $f(X), p(X), q(X) \in \mathbb{Z}[X]$ such that $p(x)[f(x), y]q(x) = y^s[x, y^m]x^r$, for all y in R, and fixed integers $m > 1$, $r \geq 0$, $s \geq 0$.
- (P_2) For each x in R there exist polynomials $f(X), p(X), q(X) \in \mathbb{Z}[X]$ such that $p(x)[f(x), y]q(x) = x^r[x, y^m]y^s$, for all y in R, and fixed integers $m > 1$, $r \geq 0$, $s \geq 0$.
- (P_3) For each $x \in R$ there exist polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ such that $p(x)[x^2 f(x), y]q(x) = y^r[x, y^m]$ and $\tilde{p}(x)[x^2\tilde{f}(x), y]\tilde{q}(x) = y^s[x, y^n]$, for all $y \in R$, where $r \geq 0$, $s \geq 0$, $m > 1, n > 1$ are fixed integers with $(m, n) = 1$.
- (P_4) For each $x \in R$ there exist polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ such that $p(x)[x^2 f(x), y]q(x) = [x, y^m]y^r$ and $\tilde{p}(x)[x^2\tilde{f}(x), y]\tilde{q}(x) = [x, y^n]y^s$, for all $y \in R$, where $r \geq 0$, $s \geq 0$, $m > 1$, $n > 1$ are fixed integers with $(m, n) = 1$.
- (P_{11}) For each $x \in R$, there exist polynomials $f(X), p(X), q(X) \in \mathbb{Z}[X]$ such that $p(x)[x^2 f(x), y]q(x) = y^s[x, y^m]$, for all $y \in R$, and fixed integers $m > 1$, $s \geq 0$.
- (P_{22}) For each $x \in R$, there exist polynomials $f(X), p(X), q(X) \in \mathbb{Z}[X]$ such that $p(x)[x^2 f(x), y]q(x) = [x, y^m]y^s$, for all y in R, and fixed integers $m > 1$, $s \geq 0$.
- $(P_3)^*$ For each $x, y \in R$ there exist polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ and integers $m = m(x, y) > 1$, $n = n(x, y) > 1$, $r = r(x, y) \geq 0$,

$s = s(x, y) \geq 0$ such that m and n are relatively prime and R satisfies $p(x)[x^2 f(x), y]q(x) = y^r[x, y^m]$ and $\tilde{p}(x)[x^2\tilde{f}(x), y]\tilde{q}(x) = y^s[x, y^n]$.

$(P_4)^*$ For each $x, y \in R$ there exist polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ and integers $m = m(x, y) > 1$, $n = n(x, y) > 1$, $r = r(x, y) \geq 0$, $s = s(x, y) \geq 0$ such that m and n are relatively prime and R satisfies $p(x)[x^2 f(x), y]q(x) = [x, y^m]y^r$ and $\tilde{p}(x)[x^2\tilde{f}(x), y]\tilde{q}(x) = [x, y^m]y^s$.

(CH) For every $x, y \in R$ there exist $f(X), g(X) \in X^2\mathbb{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$.

Q(d) For all $x, y \in R$, $d[x, y] = 0$ implies that $[x, y] = 0$, where d is some positive integer.

A well-known theorem due to Herstein [6] asserts that rings satisfying the polynomial identity $(x + y)^n = x^n + y^n$ for some $n > 1$ must have nil commutator ideal. Among other classes of rings in which $C(R)$ is known to be nil is the class of rings satisfying the polynomial identity $[x^n, y] = [x, y^n]$ for some $n > 1$ (cf. [6]). This class includes the rings satisfying the polynomial identity $(x + y)^n = x^n + y^n$. Motivated by this observation, Bell [7] proved that a ring R with unity 1 satisfying the polynomial identity $[x^n, y] = [x, y^n]$ is commutative if the additive group $(R, +)$ is n-torsion free. In attempts to generalise this result, several authors have considered various special cases of (P_1) and (P_2) (cf. [1], [2], [3], [4], [5], [7], [9], [14], [21] and [22]). In most of the cases the underlying polynomials are assumed to be monomials. In the present paper our objective is to prove commutativity of rings with unity 1 satisfying either (P_1) or (P_2) together with the property $Q(m)$. Further, if R satisfies (P_3) or (P_4), then $Q(m)$ is replaced by some other suitable constraints on the exponent m. Finally, commutativity of rings satisfying any one of the properties (P_{11})–$(P_4)^*$ has been investigated. Thus, we generalise many well-known commutativity theorems for rings.

2. Commutativity of rings with unity

The main result of the present section is stated as follows.

Theorem 1. *Let R be a ring with unity 1 satisfying the property (P_1). If R also satisfies $Q(m)$, then R is commutative (and conversely).*

We begin with the following known results, which are essentially proved in [12, p. 221], [13, theorem] and [8, theorem 1] respectively.

Lemma 1. *If x, y are elements of a ring R with $[x, [x, y]] = 0$, then for any positive integer k, $[x^k, y] = kx^{k-1}[x, y]$.*

Lemma 2. *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \cdots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(i) *For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.*
(ii) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*
(iii) *Every semiprime ring satisfying $f = 0$ is commutative.*

Lemma 3. *Let R be a ring in which for each x, y in R there exists a polynomial $f(X) \in X\mathbb{Z}[X]$ such that $[x, y] = [x, y]f(y)$. Then R is commutative.*

The following lemma is proved in [20] for a constant exponent k. However, with a slight modification in the proof, it can be proved for variable exponent k.

Lemma 4. *Let R be a ring with unity 1 and let f be a polynomial function of two variables on R with the property that $f(x+1, y) = f(x, y)$ for all x, y in R.*

(i) *If for all $x, y \in R$ there exists an integer $k = k(x, y) \geq 1$ such that $x^k f(x, y) = 0$, then necessarily $f(x, y) = 0$ for all $x, y \in R$.*
(ii) *If for all $x, y \in R$ there exists an integer $k = k(x, y) \geq 1$ such that $f(x, y)x^k = 0$, then necessarily $f(x, y) = 0$ for all $x, y \in R$.*

PROOF. We prove the result for case (i); the result for case (ii) follows similarly. Choose a positive integer $k_1 = k(1 + x, y)$ such that $(1 + x)^{k_1} f(x, y) = 0$. If $t = \max\{k, k_1\}$, then it can be easily seen that $x^t f(x, y) = 0$ and $(1 + x)^t f(x, y) = 0$. We have $f(x, y) = \{(1+x)-x\}^{2t+1} f(x, y)$. On expanding the expression on the right-hand side by the binomial theorem, we get the required result. ■

Lemma 5. *Let R be a ring with unity 1 satisfying either (P_1) or (P_2). Then $C(R) \subseteq N(R)$.*

PROOF. Let R satisfy the property (P_1). If $s = 0$, then we have $p(x)[f(x), y]q(x) = [x, y^m]x^r$, and the replacing of y by $x + y$ yields that $[x, y^m]x^r = [x, (x + y)^m]x^r$ for all x, y in R. This is a polynomial identity and we see that $x = e_{11}$, $y = -e_{11} + e_{21}$ fail to satisfy this equality in $(GF(\wp))_2$, $\wp$ a prime, and hence by Lemma 2, $C(R) \subseteq N(R)$. On the other hand, if $s > 0$, then replace y by $1 + y$ in (P_1) to get $\{y^s[x, y^m] - (1 + y)^s[x, (1 + y)^m]\}x^r = 0$. Now application of Lemma 4 yields that $y^s[x, y^m] - (1 + y)^s[x, (1 + y)^m] = 0$ for all x, y in R. This is again a polynomial identity and we see that the same choice of x and y fail to satisfy this equality in $(GF(\wp))_2$, $\wp$ a prime; hence by Lemma 2, $C(R) \subseteq N(R)$.

Further, if R satisfies the property (P_2), then by using similar techniques as used above, with the choice of $x = e_{11}$, $y = -e_{11} + e_{12}$, we get the required result. ■

PROOF OF THEOREM 1. First we shall show that nilpotents are central. Let $u \in N(R)$. Then there exists a minimal positive integer t such that

$$u^k \in Z(R) \text{ for all integers } k \geq t. \tag{1}$$

If $t = 1$, each such u is central. Therefore, now assume that $t > 1$. Replace y by u^{t-1} in (P_1) to get $p(x)[f(x), u^{t-1}]q(x) = u^{s(t-1)}[x, u^{m(t-1)}]x^r$ for all x in R. Now in view of (1) and the fact that $m(t-1) \geq t$ for $m > 1$, we find that

$$p(x)[f(x),\ u^{t-1}]q(x) = 0. \tag{2}$$

Further, replace y by $1+u^{t-1}$ in (P_1) to get

$$p(x)[f(x), 1+u^{t-1}]q(x) = (1+u^{t-1})^s[x, (1+u^{t-1})^m)]x^r.$$

This, in view of (2), yields that $(1+u^{t-1})^s[x,(1+u^{t-1})^m]x^r = 0$. However, since $1+u^{t-1}$ is invertible, the last equation reduces to $[x,(1+u^{t-1})^m]x^r = 0$. Now apply Lemma 4 to get $[x,(1+u^{t-1})^m] = 0$ for all x in R. Combining this with (1), we get

$$0 = [x, 1+mu^{t-1}] = [x,(1+u^{t-1})^m].$$

This yields that $m[x,u^{t-1}] = 0$ for all x in R, and application of $Q(m)$ gives that $u^{t-1} \in Z(R)$. This is a contradiction, and hence $t = 1$. Thus we find that $N(R) \subseteq Z(R)$. Combining this fact with Lemma 5, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{3}$$

Notice that the left-hand side of the property involved in (P_1) remains unchanged if y is replaced by $1+y$; therefore $\{(1+y)^s[x,(1+y)^m] - y^s[x,y^m]\}x^r = 0$. By applying Lemma 4, we find that $(1+y)^s[x,(1+y)^m] - y^s[x,y^m] = 0$ for all x, y in R. However, in view of (3), Lemma 1 is applicable in the present case, and the last identity implies that $m[x,y]\{(1+y)^{m+s-1} - y^{m+s-1}\} = 0$, that is

$$m[x\{(1+y)^{m+s-1} - y^{m+s-1}\}, y] = 0, \text{ for all } x, y \in R. \tag{4}$$

Now, apply the property $Q(m)$ to get $[x\{(1+y)^{m+s-1} - y^{m+s-1}\}, y] = 0$. This is a polynomial identity and can be rewritten in the form $[x,y] = [x,y]yh(y)$ for some $h(X) \in \mathbb{Z}[X]$. Hence, by Lemma 3, R is commutative.

Corollary 1. *Let $m > 1$, r, s be fixed non-negative integers and R be a ring with unity 1 in which for every x in R there exist integers $p = p(x) \geq 0$, $k = k(x) \geq 0$, $t = t(x) \geq 0$, depending on x, such that $x^p[x^k, y]x^t = y^s[x, y^m]x^r$ for all y in R. If R satisfies $Q(m)$, then R is commutative (and conversely).*

Using similar arguments with necessary variations, one can prove the following.

Theorem 2. *Let R be a ring with unity 1 satisfying the property (P_2). If R satisfies $Q(m)$, then R is commutative (and conversely).*

Remark 1. The ring of 3×3 strictly upper triangular matrices over a field provides an example showing that the above theorems are not valid for arbitrary rings. Moreover, the following ring shows that the property $Q(m)$ in the hypotheses of the above theorems cannot be deleted.

Example 1. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in GF(2) \right\}$. For $m = 4$, R satisfies the property $[x^m, y] = [x, y^m]$ for all x, y in R. However, R is a non-commutative ring with unity.

3. Commutativity theorems through Streb's classification

In view of the above example, it seems natural to ask under what additional assumption R turns out to be commutative if the property $Q(m)$ is dropped from the hypotheses of the above theorems. The theorem below gives an answer to this question.

Theorem 3. *Let R be a left s-unital ring satisfying the property (P_3). Then R is commutative (and conversely).*

In order to develop the proof of the above theorem, we begin with the following types of rings:

(a) $\begin{pmatrix} GF(\wp) & GF(\wp) \\ 0 & GF(\wp) \end{pmatrix}$, $\wp$ a prime.

$(a)_\ell$ $\begin{pmatrix} GF(\wp) & GF(\wp) \\ 0 & 0 \end{pmatrix}$, $\wp$ a prime.

$(a)_r$ $\begin{pmatrix} 0 & GF(\wp) \\ 0 & GF(\wp) \end{pmatrix}$, $\wp$ a prime.

(b) $M_\sigma(K) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| a, b \in K \right\}$, where K is a finite field with a non-trivial automorphism σ.

(c) A non-commutative ring with no non-zero divisors of zero.

(d) $S = < 1 > +T, T$ is a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

In a recent paper, Streb [23] gave a classification of non-commutative rings that has been used effectively as a tool to obtain several commutativity theorems (cf. [17], [18] and [19]). From the proof of [23, corollary (1)], it is easy to see that if R is a non-commutative left s-unital ring, then there exists a factor subring S of R which is of the type $(a)_\ell$, (b), (c) or (d). This gives us the following lemma, which plays a vital role in our subsequent discussion (cf. [18, meta theorem]).

Lemma 6. *Let P be a ring property which is inherited by factor subrings. If no rings of type $(a)_\ell$, (b), (c) or (d) satisfy P, then every left s-unital ring satisfying P is commutative.*

We pause to remark that the dual of the above lemma holds, that is, if P is a ring property which is inherited by factor subrings, and if no rings of type $(a)_r$, (b), (c) or (d) satisfy P, then every right s-unital ring satisfying P is commutative.

The first two of the following lemmas can be found in [19] and [17] respectively.

Lemma 7. *If R is a left (resp. right) s-unital ring which is not right (resp. left) s-unital, then R has a factor subring of type $(a)_\ell$ (resp. $(a)_r$).*

Lemma 8. *Suppose that a ring with unity 1 satisfies (CH). If R is non-commutative, then there exists a factor subring of R which is of type (a) or (b).*

Lemma 9. *If a ring S is of type $(a)_\ell$ or (b), then S does not satisfy $(P_3)^*$.*

Proof. If S is of type $(a)_\ell$, then in $(GF(\wp))_2$, $\wp$ a prime,

$$p(e_{12})[e_{12}^2 f(e_{12}), e_{11}+e_{12}]q(e_{12}) - (e_{11}+e_{12})^r[e_{12}, (e_{11}+e_{12})^m] = e_{12} \neq 0$$

for some integers $m > 1, r \geq 0$, and polynomials $p(X), f(X), q(X) \in \mathbb{Z}[X]$.

Let $S = \{\begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} | a, b \in K\}$, where K is a finite field with a non-trivial automorphism σ. Then notice that $N(S) = Ke_{12}$. Suppose, on the contrary, that S satisfies $(P_3)^*$. Then for any $a \in N(S)$ and arbitrary unit u there exist integers $m = m(a,u) > 1, n = n(a,u) > 1, r = r(a,u) \geq 0, s = s(a,u) \geq 0$, with the property that m and n are relatively prime, and polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ such that $p(a)[a^2 f(a), u]q(a) = u^r[a, u^m]$ and $\tilde{p}(a)[a^2\tilde{f}(a), u]\tilde{q}(a) = u^s[a, u^n]$. Since $a^2 = 0$ and u is a unit, the last two equations imply that $[a, u^m] = 0$ and $[a, u^n] = 0$. Now, the relative primeness of m and n shows that $[a, u] = 0$. Thus, for the non-central element $a = e_{12}$, we have $[e_{12}, u] = 0$, which implies that e_{12} is central, a contradiction. ■

Using similar arguments with the choice of $x = e_{12}, y = e_{12} + e_{22}$, one can prove the following.

Lemma 10. *If a ring S is of type $(a)_r$ or (b), then S does not satisfy $(P_4)^*$.*

Proof of Theorem 3. We shall show that no rings of type $(a)_\ell$, (b), (c) or (d) satisfy (P_3).

In view of Lemma 9, we see that no rings of type $(a)_\ell$ and (b) satisfy (P_3), so by Lemma 7 R is also right s-unital and hence s-unital. Thus by Proposition 1 of [11] we can assume that R has unity 1. Since arguments given in the proof of Lemma 5 are still valid in the present situation, Lemma 2 shows that no rings of type (c) satisfy (P_3). Finally, let R be a ring of type (d). Suppose that $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$. Then there exist polynomials $f(X), p(X), q(X), \tilde{f}(X), \tilde{p}(X), \tilde{q}(X) \in \mathbb{Z}[X]$ such that

$$m[t_1, t_2] = (1+t_2)^r[t_1, (1+t_2)^m] = p(t_1)[t_1^2 f(t_1), 1+t_2]q(t_1) = 0$$

and

$$n[t_1, t_2] = (1+t_2)^s[t_1, (1+t_2)^n] = \tilde{p}(t_1)[t_1^2\tilde{f}(t_1), 1+t_2]\tilde{q}(t_1) = 0.$$

This implies that $[t_1, t_2] = 0$, a contradiction.

Thus no rings of type $(a)_\ell$, (b), (c) or (d) satisfy (P_3) and hence by Lemma 6 R is commutative. ■

The proof of the following theorem proceeds exactly as above except at the point where Lemma 9 is used. Instead, Lemma 10 should be used, and it can be easily shown that no rings of type $(a)_r$, (b), (c) or (d) satisfy (P_4). We omit the details of the proof to avoid repetition.

Theorem 4. *Let R be a right s-unital ring satisfying the property* (P_4). *Then R is commutative* (*and conversely*).

Corollary 2 ([7, theorem 6]). *Let $m > 1$, $n > 1$ be relatively prime positive integers and let R be a ring with unity* 1 *satisfying the identities* $[x^m, y] = [x, y^m]$ *and* $[x^n, y] = [x, y^n]$ *for all $x, y \in R$. Then R is commutative.*

A careful scrutiny of the proof of Lemmas 9 and 10 shows that if R is a left (resp. right) s-unital ring satisfying the property (P_{11}) (resp. (P_{22})), then no rings of type $(a)_\ell$ (resp. $(a)_r$) satisfy (P_{11}) (resp. (P_{22})). Hence by Lemma 7, R is right (resp. left) s-unital, and hence s-unital. Thus, by proposition 1 of [11], we can assume that R has unity 1. Now, application of Theorems 1 and 2 yields the following.

Theorem 5. *Let R be a left* (*resp. right*) *s-unital ring satisfying the property* (P_{11}) (*resp.* (P_{22})). *If R satisfies $Q(m)$, then R is commutative.*

If R satisfies Chacron's criterion together with the properties $(P_3)^*$ or $(P_4)^*$, then by using similar arguments to those above and combining Lemmas 7–10, we get the following.

Theorem 6. *Let R be a left* (*resp. right*) *s-unital ring satisfying the property* $(P_3)^*$ (*resp.* $(P_4)^*$). *If R satisfies (CH), then R is commutative* (*and conversely*).

Acknowledgement

The author is grateful to the referee for his valuable suggestions.

References

[1] H.A.S. Abujabal, On commutativity of left s-unital rings, *Acta Scientiarum Mathematicarum* (*Szeged*) **56** (1992), 51–62.
[2] H.A.S. Abujabal and M.A. Obaid, Some commutativity theorems for right s-unital rings, *Mathematica Japonica* **37** (1992), 591–600.
[3] H.A.S. Abujabal, M. Ashraf and M.A. Obaid, Commutativity with certain constraints, *Mathematica Japonica* **37** (1992), 965–72.
[4] M. Ashraf, A commutativity theorem for associative rings, *Archivum Mathematicum* (*Brno*) **31** (1995), 201–4.
[5] M. Ashraf, Commutativity of rings with certain constraints, *Proceedings of the Royal Irish Academy* **96**A (1996), 177–83.
[6] H.E. Bell, On some commutativity theorems of Herstein, *Archiv der Mathematik* **24** (1973), 34–8.
[7] H.E. Bell, On the power map and ring commutativity, *Canadian Mathematical Bulletin* **21** (1978), 399–404.
[8] H.E. Bell, Commutativity of rings with constraints on commutators, *Results in Mathematics* **8** (1985), 123–31.
[9] A. Harmanci, Two elementary commutativity theorems for rings, *Acta Mathematica* (*Scientiarum*) *Hungarica* **29** (1977), 23–9.
[10] I.N. Herstein, Power maps in rings, *Michigan Mathematical Journal* **8** (1961), 29–32.
[11] Y. Hirano, Y. Kobayashi and H. Tominaga, Some polynomial identities and commutativity of s-unital rings, *Mathematical Journal of Okayama University* **24** (1982), 7–13.

[12] N. Jacobson, *Structure of rings*, American Mathematical Society Colloquium Publication, Providence, RI, 1956.
[13] T.P. Kezlan, A note on commutativity of semi-prime PI-rings, *Mathematica Japonica* **27** (1982), 267–8.
[14] T.P. Kezlan, On commutativity theorems for PI-rings with unity, *Tamkang Journal of Mathematics* **24** (1993), 29–36.
[15] H. Komatsu, A commutativity theorem for rings, *Mathematical Journal of Okayama University* **26** (1984), 135–9.
[16] H. Komatsu, A commutativity theorem for rings–II, *Osaka Journal of Mathematics* **22** (1985), 811–14.
[17] H. Komatsu and H. Tominaga, Chacron's condition and commutativity theorems, *Mathematical Journal of Okayama University* **31** (1989), 101–20.
[18] H. Komatsu and H. Tominaga, Some commutativity theorems for left s-unital rings, *Results in Mathematics* **15** (1989), 335–42.
[19] H. Komatsu, T. Nishinaka and H. Tominaga, On commutativity of rings, *Radovi Matematiki* **6** (1990), 303–11.
[20] W.K. Nicholson and A. Yaqub, A commutativity theorem for rings and groups, *Canadian Mathematical Bulletin* **22** (1979), 419–23.
[21] E. Psomopoulos, A commutativity theorem for rings involving a subset of the ring, *Glasnik Matematiki* **18** (1983), 231–6.
[22] E. Psomopoulos, Commutativity theorems for rings and groups with constraints on commutators, *International Journal of Mathematics and Mathematical Sciences* **7** (1984), 513–17.
[23] W. Streb, Zur Struktur nichtkommutativer Ringe, *Mathematical Journal of Okayama University* **31** (1989), 135–40.