ON A QUESTION OF NIELS GRØNBAEK

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Abstract

Let $F(X)$ denote the norm closure of the space of all finite-rank operators on a Banach space X. We show that there are Banach spaces X for which the product map $a \otimes b \to ab$ does not define a surjective map from the projective tensor product $F(X)\widehat{\otimes}F(X)$ onto $F(X)$.

Let X be a Banach space and let $B(X)$ be the space of all bounded operators on X equipped with the usual norm. Let $F(X)$ denote the closure in $B(X)$ of the set of all finite-rank maps on X.

Niels Grønbaek asked me the following question: is it true that every Banach space has the following property?

(P) The product map $F(X)\widehat{\otimes}F(X) \to F(X)$ is onto, or equivalently there exists a constant C such that for any finite-rank map $u: X \to X$, there are finite-rank maps $v_n: X \to X$ and $w_n: X \to X$ such that $u = \sum_1^\infty v_n w_n$ and $\sum_1^\infty \|v_n\| \, \|w_n\| \le C\|u\|$.

As observed by Grønbaek, any space with the bounded approximation property has this property, while, at the other end of the spectrum, any Banach space X such that $X\widehat{\otimes}X = X\check{\otimes}X$ (as constructed in [4]) also satisfies (P) but fails the approximation property. However, as shown by the next statement, which is the main result of this note, there are intermediate cases for which (P) fails. This seems to be the first application of the variant of the construction in [4] for the case of cotype q with $q \neq 2$.

Theorem 1. *There is a* (*separable*) *Banach space failing the above property* (P). *More precisely, for any $q > 2$ any Banach space X_q which is of cotype q and such that the dual $(X_q)^*$ is a GT space of cotype 2* (*see below*), *but which contains ℓ_q isomorphically, must fail* (P).

Let us briefly recall some terminology. Let $u: X \to Y$ be an operator between Banach spaces. Let $0 < p \le q < \infty$. Then u is called (q,p)-absolutely summing (resp. p-absolutely summing in case $q = p$), if there is a constant C such that, for any finite

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sequence (x_i) in X, we have

$$\left(\sum \|u(x_i)\|^q\right)^{1/q} \leq \sup\left\{\left(\sum |\xi(x_i)|^p\right)^{1/p} \mid \xi \in X^*, \|\xi\| \leq 1\right\}.$$

We denote by $\pi_{q,p}(u)$ (resp. $\pi_p(u)$ in case $q = p$) the smallest constant C for which this holds. Actually, for short we will use the terms (q, p)-summing or p-summing.

A Banach space X is called of cotype q ($q \geq 2$) if there is a constant C such that, for any finite sequence (x_i) in X, we have

$$\left(\sum \|x_i\|^q\right)^{1/q} \leq \left(\int \|\sum r_i(t)x_i\|^2 dt\right)^{1/2},$$

where we have denoted by (r_i) the sequence of the Rademacher functions on the Lebesgue interval.

As Grothendieck's fundamental theorem shows (see [5]), there are Banach spaces X (for instance $X = L_1$) such that every bounded operator $u\colon X \to \ell_2$ is automatically 1-summing. As in [4], we call these GT spaces. We refer the reader to [6] and [5] for background on p-summing or (q, p)-summing operators, and cotype of Banach spaces.

Remark 2. Let X be a Banach space. It is proved in [4, proposition 1.11] that X^* is a GT space of cotype 2 if and only if X satisfies the following: there is a constant C such that, if we denote by $R \subset L_1$ the closed span of the Rademacher functions in L_1 over the Lebesgue interval, every $v\colon R \to X$ admits an extension $\tilde{v}\colon L_1 \to X$ such that $\|\tilde{v}\| \leq C\|v\|$.

Lemma 3. *Let X be of cotype $q \geq 2$, and such that X^* is a GT space of cotype 2. Then there is a constant K' such that any finite-rank map $u\colon X \to X$ satisfies*

$$\pi_{q,2}(u) \leq K'\|u\|. \tag{1}$$

Proof. We refer to [3] for the definitions and the main properties of the K-convexity constant of an operator u, which we denote, as in [3], by $K(u)$ (precisely, this is defined as the best constant in (3) below). It is proved in [3] that if X^* has cotype 2 and X has cotype q then there is a constant K'' such that every finite-rank map $u\colon X \to X$ satisfies

$$K(u) \leq K''\|u\|. \tag{2}$$

Moreover, since X^* is a GT space of cotype 2, by Remark 2 we have the following property: there is a constant C' such that for any n and any $x_1, \ldots, x_n$ in X there is a function $\Phi \in L^\infty([0, 1]; X)$ such that for all $i = 1, \ldots, n$ we have

$$\int r_i(t)\Phi\, dt = x_i \quad \text{and} \quad \|\Phi\|_{L^\infty(X)} \leq C' \max\left\{\left(\sum |\xi(x_i)|^2\right)^{1/2} \mid \xi \in B_{X^*}\right\}.$$

Using the very definition of $K(u)$, this implies

$$\begin{aligned}\left\|\sum r_i(t)u(x_i)\right\|_{L^2(X)} &\le K(u)\|\Phi\|_{L^2(X)}\\ &\le K(u)C' \max_{\xi\in B_{X^*}}\left(\sum|\xi(x_i)|^2\right)^{1/2}, \qquad (3)\end{aligned}$$

and since X is of cotype q we conclude that $\pi_{q,2}(u) \le C'C''K(u)$ for some constant C'' (equal to the cotype q constant of X). Thus by (2) we obtain (1). ∎

Lemma 4. *Let $q > 2$. Any space X of cotype q such that X^* is a GT space of cotype 2, but which contains a subspace isomorphic to ℓ_q (or which contains ℓ_q^n's uniformly, in the sense of* e.g. [5, p. 39]), *fails the property* (P) *introduced above.*

Proof. By a result due to König–Retherford–Tomczak (see [6, §22] or [2]), if $p < 2$, $k \ge 2$ and $1/p = 1/q_1 + \cdots + 1/q_k$ where $2 \le q_i < \infty$ for each $i = 1, \dots, k$, there is a constant B such that for any k-tuple $u_1, \dots, u_k$ of operators in $B(X)$ we have

$$\pi_2(u_1u_2\dots u_k) \le B\prod_{i=1}^k \pi_{2,q_i}(u_i).$$

Now let us assume for simplicity that $2 < q < 4$. Then if we take $k = 2$ and $q_1 = q_2 = q$, we have $p < 2$, so that this implies, using (1),

$$\pi_2(vw) \le BK'^2\|v\|\,\|w\|.$$

So if the property (P) appearing in Theorem 1 held, we would have, if $u = \sum v_nw_n$,

$$\begin{aligned}\pi_2\left(\sum v_nw_n\right) &\le BK'^2\sum\|v_n\|\,\|w_n\|\\ &\le CBK'^2\|u\|,\end{aligned}$$

hence, for all $u\colon X \to X$ with finite rank, $\pi_2(u) \le CBK'^2\|u\|$. Now applying the property again we obtain by a classical composition result of Pietsch (see [6, p. 55])

$$\begin{aligned}\pi_1\left(\sum v_nw_n\right) &\le \sum\pi_2(v_n)\pi_2(w_n)\\ &\le (CBK'^2)^2\sum\|v_n\|\,\|w_n\|\\ &\le C(CBK'^2)^2\|u\|,\end{aligned}$$

hence we find a constant K'' such that any finite-rank $u\colon X \to X$ satisfies

$$\pi_1(u) \le K''\|u\|,$$

and *a fortiori* satisfies

$$\pi_1(u) \le K''\pi_2(u).$$

But now this fails when $X \supset \ell_q$, $q > 2$, because, by a standard argument (for details see the equivalence of $(iv)_a$ and $(iv)_b$ in [1, proposition 2.1]), this implies that

$$\Pi_2(\ell_\infty, X) = B(\ell_\infty, X)$$

and this obviously fails if the space ℓ_q, $q > 2$, embeds into X. Indeed, it would immediately follow that $\Pi_2(\ell_\infty, \ell_q) = B(\ell_\infty, \ell_q)$, but it is easy to check that the diagonal multiplication operators from ℓ_∞ into ℓ_q are in general *not* 2-summing when $q > 2$.

Similarly, if $q > 4$ we choose an even integer m such that $\frac{m}{q} < 1$. By property (P), any finite-rank $u: X \to X$ can be written as

$$\sum_i v_1(i) v_2(i) \ldots v_m(i)$$

with finite-rank maps $v_j(i)$ such that

$$\sum_i \|v_1(i)\| \ldots \|v_m(i)\| \leq C^{m-1} \|u\|.$$

Then the preceding argument leads again to $\pi_1(u) \leq K'' \|u\|$ for some constant K'', which is impossible. ■

Proof of Theorem 1. By [4, theorem 2.4], any space Y of cotype q (resp. which is separable) can be embedded in a space X of cotype q (resp. separable) such that X^* is a GT space of cotype 2. Then, by Lemma 4, it suffices to take $Y = \ell_q$ to conclude. ■

Remark. Let $K(X)$ denote the space of all compact operators on X. It is apparently not known whether there is a Banach space X such that the natural product map

$$K(X) \widehat{\otimes} K(X) \to K(X)$$

is not onto.

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