ON SOLUBILITY OF SOME GROUPS WITH A PERMUTATIONAL PROPERTY ON COMMUTATORS

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Abstract

Let G be a group. We say that $G \in C_3$ if and only if, for every $(a, x_1, x_2, x_3) \in G^4$, there exists a non-trivial permutation $\sigma \in S_3$ such that $[a, x_1, x_2, x_3] = [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$.

In this paper we prove that any finite group $G \in C_3$ is soluble. Moreover, we show that C_3 is not a variety and we prove that metabelian groups form the largest variety contained in C_3.

1. Introduction

Let G be a group and n be an integer greater than 1. We say that G has rewritable commutators of weight n if for each $(x_1, \ldots, x_n) \in G^n$ there exists a permutation $\sigma \in \mathbb{S}_n - \{1\}$ such that

$$[x_1, \ldots, x_n] = [x_{\sigma(1)}, \ldots, x_{\sigma(n)}]. \tag{1}$$

Notice that in this definition the permutation σ depends on the choice of the n-tuple $(x_1, \ldots, x_n)$ of elements of G. If σ is the same for any $(x_1, \ldots, x_n) \in G^n$, we have the variety V_σ defined by the identity (1).

If $n \geq 4$ and $\sigma = (1\ 2\ 3 \ldots\ n)$, a result of I. Macdonald [5] ensures that the variety V_σ is contained in the variety of nilpotent groups of class $\leq n-1$. If $n \geq 3$ and $\sigma = (n-1\ \ n)$, from a theorem of F. Levin [3] it follows that any group G satisfying the identity (1) is such that $[\Gamma_{n-2}(G), \Gamma_2(G)] = 1$. In 1970 N. Gupta and F. Levin [1] proved that if $\sigma \neq (1\ 2)$, then any group $G \in V_\sigma$ is nilpotent-by-nilpotent; if $n > 2$ and $\sigma(n) \neq n$, then any group $G \in V_\sigma$ is abelian-by-nilpotent, and if $\{1, 2\} \neq \{\sigma(1), \sigma(2)\}$, then V_σ is contained in the variety of nilpotent groups of class $\leq n+1$.

Groups with rewritable commutators of weight 2 constitute the variety defined by the law $[x, y]^2 = 1$ and described in [5].

In [4] P. Longobardi studied groups with rewritable commutators and, in particular, in the case $n = 3$. She proved that any finite group G of odd order having rewritable commutators of weight 3 is nilpotent of class ≤ 3, and it is nilpotent of class ≤ 2 if 3 does not divide its order. Moreover, each finite group with rewritable commutators of weight 3 is p-nilpotent for any prime $p \neq 2, 3$, hence soluble.

In this paper we consider particular classes of groups having rewritable commutators. Let n be an integer > 1; we say that a group G is in C_n if for any $(a, x_1, \ldots, x_n) \in G^{n+1}$ there exists a non-trivial permutation $\sigma \in \mathbb{S}_n$ such that

$$[a, x_1, \ldots, x_n] = [a, x_{\sigma(1)}, \ldots, x_{\sigma(n)}].$$

For $n = 2$ we get the class C_2 that clearly is the variety defined by the identity

$$[x_1, x_2, x_3] = [x_1, x_3, x_2]. \tag{2}$$

From lemma 2.1 of [3] it follows that C_2 is the variety of nilpotent groups of class ≤ 2.

We will now investigate the case $n = 3$.

First of all notice that C_3 contains each one of the varieties defined by the identity

$$[a, x_1, \ldots, x_n] = [a, x_{\sigma(1)}, \ldots, x_{\sigma(n)}],$$

where σ is a fixed permutation in $\mathbb{S}_3 - \{1\}$. All these varieties consist of metabelian groups. In fact, by using the results of Levin [3], Macdonald [6], and Kikodze [2], we have that if $\sigma = (2, 3)$, the prescribed variety is the one of metabelian groups and in each other case we obtain that this is the variety of nilpotent groups of class ≤ 3. So any metabelian group is in C_3. In [8] we observed that there exists a group $G \in C_3$, which is not metabelian. Such a group G is 2-metabelian, that is for any $a, x \in G$ the subgroup $\langle a, x \rangle$ is metabelian. This agrees with the result, also obtained in [8], which ensures that any nilpotent group in C_3 is 2-metabelian. Moreover, in [7] we proved that if $G \in C_3$ and it is periodic, with no involution, then G is 2-metabelian. It is still an open question whether every C_3-group is 2-metabelian. Notice that if a finite group G is 2-metabelian, then it is soluble, as all finite non-abelian simple groups are 2-generator. In the first section we will show that any finite group $G \in C_3$ is soluble. In the second section we point out that C_3 is not a variety and we prove that metabelian groups form the largest variety contained in C_3.

2. Finite groups in C_3

The main result of this section shows that each finite group $G \in C_3$ is soluble. The proof is a consequence of a solubility condition holding for some (possibly infinite) 2-generator C_3-groups (Corollary 3) and the well-known theorem of Malle *et al.* [6], which ensures that every finite non-abelian simple group is generated by two elements, one of which is an involution.

First we need the following easy remark.

Proposition 1. *Let $G \in C_3$, $a, x \in G$ and H be an abelian subgroup of G containing a. Then the subgroup $K = \langle\, [a, x, a_1] \mid a_1 \in H \rangle$ is abelian.*

Proof. Since H is abelian, for any $a_1, a_2 \in H$ we have $[a, x, a_1 a_2] = [a, x, a_2 a_1]$, that is

$$[a, x, a_2][a, x, a_1][a, x, a_1, a_2] = [a, x, a_1][a, x, a_2][a, x, a_2, a_1]. \tag{3}$$

Consider the commutator $[a, x, a_1, a_2]$.

From $G \in C_3$ it follows that either $[a, x, a_1, a_2] = 1$ or $[a, x, a_1, a_2] = [a, x, a_2, a_1]$. Suppose $[a, x, a_1, a_2] = 1$ and consider $[a, x, a_2, a_1]$. Again from $G \in C_3$ it follows that either $[a, x, a_2, a_1] = 1$ or $[a, x, a_2, a_1] = [a, x, a_1, a_2]$. So in any case $[a, x, a_1, a_2] = [a, x, a_2, a_1]$ and from (3) it follows that

$$[a, x, a_1][a, x, a_2] = [a, x, a_2][a, x, a_1].$$

Hence K is abelian. ∎

In particular, from Proposition 1 it follows that if $G \in C_3$ then for any $a, x \in G$ the subgroup $A = \langle [a, x, a^n] \mid n \in \mathbb{Z} \rangle$ is abelian.

Now we are able to prove the key lemma.

Lemma 2. *Let* $G = \langle a, x \rangle \in C_3$. *If* $o(x) = 2$, *then the abelian subgroup* $A = \langle [a, x, a^n] \mid n \in \mathbb{Z} \rangle$ *is normal in* G.

PROOF. Obviously $a \in N_G(A)$, so we have to show that $x \in N_G(A)$.

Notice that from $o(x) = 2$ it follows that

$$[a, x, x] = [x, a][a, x]^x = [x, a]^2 \quad \text{and} \quad [x, a, x] = [a, x][x, a]^x = [a, x]^2.$$

Moreover, for every $n \in \mathbb{Z}$,

$$[x, a, a^n] = [[a, x]^{-1}, a^n] = [a^n, [a, x]]^{[a,x]^{-1}} = ([a, x, a^n]^{-1})^{[a,x]^{-1}}. \tag{4}$$

In particular, $[x, a, a]^{[a,x]} = [a, x, a]^{-1}$ and

$$[a, x, a]^x = [x, a, a^x] = [x, a, a[a, x]] = [x, a, a]^{[a,x]} = [a, x, a]^{-1}. \tag{5}$$

We claim that $[a, x] \in N_G(A)$. To start with, we show that $A^{[a,x]} \subseteq A$, that is $[a, x, a^n]^{[a,x]} \in A$ for any $n \in \mathbb{Z}$.

By C_3 and (5)

$$[a, x, a^n, [a, x]] = \begin{cases} 1; \text{ or} \\ [a, [a, x], a^n, x]; \text{ or} \\ [a, [a, x], x, a^n] = [[a, x, a]^{-1}, x, a^n] = [[a, x, a][a, x, a]^{-x}, a^n] \\ \qquad = [[a, x, a]^2, a^n] \in A. \end{cases}$$

If $[a, x, a^n, [a, x]] \notin A$ then $[a, x, a^n, [a, x]] = [a, [a, x], a^n, x]$ and $[[a, x, a]^2, a^n] = 1$, that is $[a, x, a, a^n]^2 = 1$. Thus

$$\begin{aligned} [a, x, a^n, [a, x]] &= [a, [a, x], a^n, x] = [[a, x, a]^{-1}, a^n, x] \\ &= [[a, x, a, a^n]^{-1}, x] = [a, x, a, a^n, x] = [a, x, a^n, a, x]. \end{aligned} \tag{6}$$

(The last equality follows from Proposition 1.) Consider the commutator

$$[a, x, a^n, a[a, x]] = [a, x, a^n, [a, x]][a, x, a^n, a]^{[a,x]}.$$

$G \in C_3$ yields that one of the following holds:

$$[a, x, a^n, a[a, x]] = \begin{cases} 1; \text{ or} \\ [a, a[a, x], x, a^n] = [a, [a, x], x, a^n] = 1; \text{ or} \\ [a, x, a[a, x], a^n] = [[a, x, a]^{[a,x]}, a^n]; \text{ or} \\ [a, a[a, x], a^n, x] = [a, [a, x], a^n, x] = [a, x, a^n, [a, x]]. \end{cases}$$

If $[a, x, a^n, a[a, x]] = [a, x, a^n, a^x] = 1$, then $[a, x, a^n]^{a^x} = [a, x, a^n] \in A$ and so $[a, x, a^n]^{[x,a^{-1}]} = [a, x, a^n]^{[a,x]^{a^{-1}}} \in A$. This yields $[a, x, a^n, [a, x]^{a^{-1}}] \in A$, that is $[a, x, a^n, [a, x][a, x, a^{-1}]] \in A$, so $[a, x, a^n, [a, x]]^{[a,x,a^{-1}]} \in A$ and $[a, x, a^n]^{[a,x]} \in A$.

If $[a, x, a^n, a[a, x]] = [a, x, a^n, [a, x]]$, then $[a, x, a^n, a]^{[a,x]} = 1$ and $[a, x, a^n, a] = 1$. From the last equality and from (6) we get $[a, x, a^n, [a, x]] = [a, x, a^n, a, x] = 1$, a contradiction.

Finally, if $[a, x, a^n, a[a, x]] \neq 1$ and $[a, x, a^n, a[a, x]] \neq [a, x, a^n, [a, x]]$, then $[a, x, a^n, [a, x]] = 1$ and so $[a, x, a^n]^{[a,x]} = [a, x, a^n] \in A$, the final contradiction.

Thus $[a, x, a^n]^{[a,x]} \in A$ for all $n \in \mathbb{Z}$, that is $A^{[a,x]} \subseteq A$; equivalently $A \subseteq A^{[x,a]}$.

Now, consider a^{-1}, x and the abelian subgroup $A_1 := \langle [a^{-1}, x, a^n] \mid n \in \mathbb{Z} \rangle$. The same argument shows that $A_1 \subseteq A_1^{[x,a^{-1}]}$.

Since $A_1 = A_1^a$ and $[a^{-1}, x, a^n]^a = [x, a, a^n] = ([a, x, a^n]^{-1})^{[x,a]}$, by (4), for every $n \in \mathbb{Z}$, we have

$$A_1 = A_1^a = \langle [a, x, a^n]^{[x,a]} \mid n \in \mathbb{Z} \rangle = A^{[x,a]}.$$

So from $A_1 \subseteq A_1^{[x,a^{-1}]}$ it follows that $A^{[x,a]} \subseteq A^{[x,a][x,a^{-1}]} = A^{[a,x]^{-1}[a,x]^{a^{-1}}} = A^{[a,x,a^{-1}]} = A$. Then we get $A^{[x,a]} = A$ and $[a, x] \in N_G(A)$, as claimed.

For any $n \in \mathbb{Z}$, we have $[a^{n+1}, x] = [a, x][a, x, a^n][a^n, x]$. Then $[a^{n+1}, x] \in N_G(A)$ if and only if $[a^n, x] \in N_G(A)$. Therefore $[a^n, x] \in N_G(A)$ for any $n \in \mathbb{Z}$, since $[a, x] \in N_G(A)$.

Now notice that for any $n \in \mathbb{Z}$ we have $[a, x, a^{n+1}]^x = [a, x, a^n]^x [a, x, a]^{a^n x} = [a, x, a^n]^x [a, x, a]^{x a^n [a^n, x]} = [a, x, a^n]^x [a, x, a]^{-a^n [a^n, x]}$ by (5). Thus $[a, x, a^{n+1}]^x \in A$ if and only if $[a, x, a^n]^x \in A$. So from $[a, x, a]^x = [a, x, a]^{-1} \in A$ it follows that $[a, x, a^n]^x \in A$ for any $n \in \mathbb{Z}$. Therefore $A^x \subseteq A$ and $A^x = A$, because $x^2 = 1$, that is $x \in N_G(A)$. ■

Clearly the class C_3 is closed under taking homomorphic images, so we have the following.

Corollary 3. *Let $G = \langle a, x \rangle \in C_3$. If $o(x) = 2$, then G is abelian-by-cyclic-by-abelian.*

Proof. From Lemma 2 it follows that the abelian subgroup $A = \langle [a, x, a^n] \mid n \in \mathbb{Z} \rangle$ is normal in G. If $A = \{1\}$, then $[a, x, a^n] = 1$ for any $n \in \mathbb{Z}$. Moreover, $[a, x]^x = [x, a] = [a, x]^{-1}$. So we get that $G' = \langle [a, x] \rangle^G = \langle [a, x] \rangle$ is cyclic and G is cyclic-by-abelian. If $A \neq \{1\}$, we may consider the factor group G/A, which again satisfies the hypothesis in the same situation as the case $A = \{1\}$, so G/A is cyclic-by-abelian, and G is abelian-by-cyclic-by-abelian. ■

Now, by using Corollary 3 and the theorem in [6] we get the following.

Proposition 4. *A finite simple group is in C_3 if and only if it is abelian.*

Obviously the class C_3 is also closed under taking subgroups. Therefore if we consider a finite group $G \in C_3$, its composition factors are abelian and we get the following.

Theorem 5. *Every finite group in C_3 is soluble.*

3. The class C_3 and the variety of metabelian groups

In [8] we observed that the standard wreath product of the dihedral group of order 8 and of the group of order 2, $G = D_8 \,\mathrm{wr}\, \mathbb{Z}_2$, which is not metabelian, is in C_3. Now the direct product $G \times G$ does not belong to C_3, and therefore C_3 is not a variety. This can be checked directly, but it is also an immediate consequence of the following result.

Theorem 6. *Let G be a group. If the direct product $G \times G$ is in C_3, then G is metabelian.*

PROOF. First notice that for any $(a, x_1, x_2, x_3), (b, y_1, y_2, y_3) \in G^4$ there exists a permutation $\sigma \in \mathbb{S}_3 - \{1\}$ such that

$$[a, x_1, x_2, x_3] = [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] \quad \text{and} \quad [b, y_1, y_2, y_3] = [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}].$$

Indeed, from $G \times G \in C_3$ it follows that

$$\begin{aligned}([a, x_1, x_2, x_3], [b, y_1, y_2, y_3]) &= [(a, b), (x_1, y_1), (x_2, y_2), (x_3, y_3)] \\ &= [(a, b), (x_{\sigma(1)}, y_{\sigma(1)}), (x_{\sigma(2)}, y_{\sigma(2)}), (x_{\sigma(3)}, y_{\sigma(3)})] \\ &= ([a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}])\end{aligned}$$

for some $\sigma \in \mathbb{S}_3 - \{1\}$. Thus, if there exists $(a, x_1, x_2, x_3) \in G^4$ such that $[a, x_1, x_2, x_3] = [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for $\sigma \in \mathbb{S}_3 - \{1\}$ and $[a, x_1, x_2, x_3] \neq [a, x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}]$ for any $\tau \in \mathbb{S}_3 - \{1, \sigma\}$, then for each $(b, y_1, y_2, y_3) \in G^4$ we have $[b, y_1, y_2, y_3] = [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}]$. This implies that G is in the variety defined by the identity

$$[a, x_1, x_2, x_3] = [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$$

with fixed $\sigma \in \mathbb{S}_3 - \{1\}$. So G is metabelian.

Therefore we may suppose that for any $(a, x_1, x_2, x_3) \in G^4$ there exist at least two different permutations $\sigma, \tau \in \mathbb{S}_3 - \{1\}$ such that

$$[a, x_1, x_2, x_3] = [a, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = [a, x_{\tau(1)}, x_{\tau(2)}, x_{\tau(3)}]. \tag{7}$$

Fix such (a, x_1, x_2, x_3), σ and τ. If $\mathbb{S}_3 = \{1, \sigma, \tau, \eta, \rho, \lambda\}$ then we have that $[a, x_{\eta(1)}, x_{\eta(2)}, x_{\eta(3)}] = [a, x_{\rho(1)}, x_{\rho(2)}, x_{\rho(3)}] = [a, x_{\lambda(1)}, x_{\lambda(2)}, x_{\lambda(3)}]$.

Moreover, for each $(b, y_1, y_2, y_3) \in G^4$ either $[b, y_1, y_2, y_3] = [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}]$ or $[b, y_1, y_2, y_3] = [b, y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}]$.

Suppose that for some $(b, y_1, y_2, y_3) \in G^4$ we have:

$$[b, y_1, y_2, y_3] = [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}] \neq [b, y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}]. \tag{8}$$

Then the previous argument ensures that we may assume that

$$[b, y_1, y_2, y_3] = [b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}] = [b, y_{\lambda(1)}, y_{\lambda(2)}, y_{\lambda(3)}]$$

and

$$[b, y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}] = [b, y_{\eta(1)}, y_{\eta(2)}, y_{\eta(3)}] = [b, y_{\rho(1)}, y_{\rho(2)}, y_{\rho(3)}].$$

We will prove that for any $(c, z_1, z_2, z_3) \in G^4$ the equality

$$[c, z_1, z_2, z_3] = [c, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}] \tag{9}$$

holds.

Let $[c, z_1, z_2, z_3] \neq [c, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}]$. Then from (7) and (8) we get

$$[c, z_1, z_2, z_3] = [c, z_{\tau(1)}, z_{\tau(2)}, z_{\tau(3)}] = [c, z_{\lambda(1)}, z_{\lambda(2)}, z_{\lambda(3)}]$$

and so

$$[c, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}] = [c, z_{\rho(1)}, z_{\rho(2)}, z_{\rho(3)}] = [c, z_{\eta(1)}, z_{\eta(2)}, z_{\eta(3)}].$$

Consider $(c, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ and $(b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)})$. There exists a permutation $\epsilon \in \mathbb{S}_3 - \{1\}$ such that

$$[c, z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}] = [c, z_{\epsilon\sigma(1)}, z_{\epsilon\sigma(2)}, z_{\epsilon\sigma(3)}] \tag{10}$$

and

$$[b, y_{\sigma(1)}, y_{\sigma(2)}, y_{\sigma(3)}] = [b, y_{\epsilon\sigma(1)}, y_{\epsilon\sigma(2)}, y_{\epsilon\sigma(3)}]. \tag{11}$$

From (10) it follows that either $\epsilon\sigma = \rho$ or $\epsilon\sigma = \eta$ and from (11) it follows that either $\epsilon\sigma = \lambda$ or $\epsilon\sigma = 1$, a contradiction.

So (9) holds and G is metabelian, since it belongs to the variety defined by this equation. ■

Corollary 7. *The variety of metabelian groups is the largest variety contained in* C_3.

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