FINITE NON-NILPOTENT GROUPS ALL OF WHOSE SECOND MAXIMAL SUBGROUPS ARE TI-GROUPS

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Abstract

A subgroup H of a finite group G is called a TI-subgroup if $H \cap H^g = 1$ or H for each $g \in G$. This paper classifies the finite non-nilpotent groups all of whose second maximal subgroups are TI-groups.

1. Introduction

Given a finite group G, a subgroup K of G is called second maximal if there exists a maximal subgroup M of G such that $K \leq M$ and K is maximal in M. A long series of papers deals with the finite groups with given second maximal subgroups. First, one may assume that second maximal subgroups possess some structure. For example, Janko [3] showed that a finite non-solvable group G in which every second maximal subgroup is nilpotent is isomorphic to $PSL(2, 5)$ or $SL(2, 5)$. Janko's result has been generalised by many scholars. Another direction is to consider conditions on the normality. A well-known theorem says that if every second maximal subgroup of a finite group G is S-quasinormal in G, then G is supersolvable [6, I, theorem 6.5].

The aim of the present paper is to classify the finite non-nilpotent groups in which every second maximal subgroup is a TI-subgroup. Recall that a subgroup H of G is called a TI-subgroup if $H \cap H^g = 1$ or H for each $g \in G$. The concept of TI-subgroups is a generalisation of the normality since a normal subgroup obviously is a TI-subgroup. In [5], Walls classified the finite groups all of whose subgroups are TI-subgroups. In particular, such groups are solvable and possess simple structure.

Throughout this paper, G denotes a finite group, Z_n denotes the cyclic group of order n. All unexplained notation and terminology are standard.

2. Preliminaries

A minimal non-abelian group G is a non-abelian group in which every proper subgroup is abelian. If, in addition, G is non-nilpotent, then $G = PQ$ where P is a minimal normal p-subgroup of G and Q a cyclic q-group, p, q are distinct primes [2, III, Satz 5.2].

Proposition 2.1. *If M is a TI-subgroup and $core_G(M) > 1$, then M is normal in G.*

Proof. If $N = core_G(M) > 1$, then $1 < N \leq M \cap M^g$ for each $g \in G$, and so $M^g = M$ because M is a TI-subgroup. Thus the conclusion follows. ■

Proposition 2.2. *If every second maximal subgroup of G is normal in G, then G is either nilpotent or $G = PQ$ is a minimal non-abelian group of order $p^n q$ with $Q \lhd G$ where $P = \langle x \rangle \in Syl_p(G)$ and $Q \in Syl_q(G)$, p and q are distinct primes.*

Proof. Assume that G is non-nilpotent. Then G has at least one abnormal maximal subgroup, say P. We claim that P is a cyclic p-group for some prime p and hence G is solvable. Assume that P contains two distinct maximal subgroups, K_1 and K_2. Then both K_1 and K_2 are normal in G by the hypothesis. It follows that $P = K_1K_2$ is normal in G, a contradiction. Thus P has a unique maximal subgroup, which proves our claim. Since every maximal subgroup of a finite solvable group is of prime power index, P has prime power index in G. So we have $G = PQ$ where Q is a Sylow q-subgroup and $p \neq q$. By a theorem of Burnside [2, IV, Hauptsatz 2.6], Q is normal in G. Let $P = \langle x \rangle$ and let F be a maximal subgroup of Q. Then $\langle x^p \rangle$ is a second maximal subgroup of G, so normal in G by the hypothesis. It is clear that $\langle x^p \rangle F$ is also second maximal in G and nilpotent. Again apply the hypothesis to see that $\langle x^p \rangle F$ is normal in G and hence $F \lhd G$. Thus $P \leq PF < G$ and then we have $F = 1$ because P is maximal in G. Thus Q is of order q. Now we see easily that G is a minimal non-abelian group, thus completing the proof. ■

Proposition 2.3. *By Q_{2^n} denote the generalised quaternion group of order 2^n. If each second maximal subgroup of Q_{2^n} is normal, then $n = 3$.*

Proof. We have

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle.$$

So $[a, b] = a^{-1}b^{-1}ab = a^{-2}$ and the derived subgroup $Q'_{2^n} = \langle a^{-2} \rangle$ is of order 2^{n-2}. Let H be a second maximal subgroup of Q_{2^n}. Then $|H| = 2^{n-2}$ and H is normal in Q_{2^n} by the hypothesis. Hence Q_{2^n}/H is abelian of order 4, so $H = Q'_{2^n}$. We thus know that Q_{2^n} has a unique subgroup of order 2^{n-2}. By [2, III, Hauptsatz 8.3], this situation occurs only when $n = 3$, as desired. ■

3. Theorems

Theorem 1. *If each maximal subgroup of G is a TI-subgroup, then G is either nilpotent or a Frobenius group with a nilpotent complement, and the Frobenius kernel is the unique minimal normal subgroup of G. In particular, G is solvable and $C_G(x)$ is nilpotent for all $1 \neq x \in G$.*

Proof. If every maximal subgroup of G is normal, then G is nilpotent. Hence assume that G has a non-normal maximal subgroup H. Then H is a self-normalising TI-subgroup, and hence G is a Frobenius group with complement H [2, V, Hauptsatz 7.6]. Let K be the kernel of G and let V be a minimal normal subgroup of G. Then V is contained in K and is nilpotent [2, V, Satz 8.7]. So V is elementary abelian of order p^n for some prime p. Moreover, $H \cong G/V$, it follows by Proposition 2.1 that H is nilpotent. The remaining conclusions of the theorem follow. ■

Theorem 2. *If G is a non-nilpotent group, all of whose second maximal subgroups are TI-subgroups, then one of the following holds:*

(a) *$G = PQ$ is a minimal non-abelian group, where $P = \langle x \rangle$ is a Sylow p-subgroup and $Q \lhd G$ is a Sylow q-subgroup of order q, p and q are distinct primes.*

(b) *G is a non-abelian group of order pq^2 or pqr, where p, q and r are distinct primes.*

(c) *$G = PH$ is a Frobenius group with kernel P elementary abelian and with complement H, each maximal subgroup of H acts irreducibly on P, and H is either cyclic or the direct product of a cyclic group of odd order with the quaternion group Q_8 of order 8.*

(d) *$G \cong S_4$, the symmetric group of degree 4.*

(e) *$G = PSL(2,5)$. Conversely, each of the groups listed in the above satisfies the condition of the theorem.*

PROOF. It is easy to check the fact that every group (a)–(e) has only second maximal subgroups, which are TI-subgroups, and the proof is omitted here. Conversely, assume that G satisfies the hypothesis of the theorem. We first consider the case when G is non-solvable. By Theorem 1, each maximal subgroup of G is solvable and so $G/\Phi(G)$ is a non-abelian simple group, where $\Phi(G)$ is the Frattini subgroup of G. Further, $\Phi(G) = 1$ by Proposition 2.1. Hence G is a non-abelian simple group and $C_G(x)$ is nilpotent for all $1 \neq x \in G$ by applying the last conclusion of Theorem 1. Such groups have been classified by M. Suzuki in [4]. The groups concerned are $PSL(3,4), PSL(2,9) \cong A_6, S_Z(2^n)$ and $PSL(2,q)$ where either $q = 2^m$ with $m > 1$ or $q > 3$ is a Fermat or Mersenne prime. ■

Since both $PSL(3,4)$ and A_6 have $PSL(2,4) \cong A_5$ as a proper subgroup, and since every proper subgroup G is solvable, we see that G cannot be one of them.

$S_Z(2^n)$ is a Zassenhaus group of type (H,S) where H is cyclic of order $2^n - 1$ and S is a non-abelian special 2-group of order 2^{2n} [1, p. 466]. So $F = HS$ is a Frobenius group with kernel S and complement H. Furthermore, F is maximal in $S_Z(2^n)$. Since S is not abelian, $S_Z(2^n)$ can be ruled out by applying Theorem 1.

Consider $PSL(2,q)$. By Dickson's Theorem [5, III, Hauptsatz 8.27], $PSL(2,q)$ contains two dihedral subgroups D_1 and D_2, both of which are maximal in $PSL(2,q)$. For the case when $q = 2^n$, D_1 and D_2 are both non-nilpotent with order $2(2^n - 1)$ and $2(2^n + 1)$, respectively. If $G \cong PSL(2,2^n)$, each of D_1 and D_2 must be a Frobenius group, and hence both $2^n - 1 = p$ and $2^n + 1 = r$ are primes, which implies that $p = 3$ and $r = 5$. We conclude that $G \cong PSL(2,4) \cong PSL(2,5)$, as desired. If $q = 2^n - 1$ or $2^n + 1$, then one of D_1 and D_2 is nilpotent. However, when $n \geq 3$, $q^2 \equiv 1 \pmod{16}$, G has S_4 as a subgroup by Dickson's Theorem [5, III, Hauptsatz 8.27], and the only possible case is when $q = 2^2 + 1$, and $G \cong PSL(2,5) \cong A_5$, as desired again.

From now on we assume that G is solvable. Let V be an arbitrary minimal normal subgroup of G with order r^m where r is a prime and $m \geq 1$. By Proposition 2.1, each second maximal subgroup of G/V is normal in G/V and it follows by Proposition 2.2 that G/V is either nilpotent or a minimal non-abelian group of order $p^n q$ with normal Sylow q-subgroup of order q and with cyclic Sylow p-subgroup of order p^n, where p, q are distinct primes and $p \mid q - 1$.

First let G/V be non-nilpotent for some V.

Let P be a Sylow p-subgroup of G and let Q be a Sylow q-subgroup. Then PV/V is cyclic of order p^n and QV/V is normal of order q.

Case 1. $r = p$.

For this case, $|Q| = q$ and $V \leq P$ and P is abnormal maximal in G. We need discuss only two cases.

Claim 3.1. *If P is cyclic, then conclusion (a) in Theorem 2 holds.*

To show this, let $P = \langle x \rangle$ and let M be any maximal subgroup of G. We need to claim that M is abelian. The claim is true obviously if $P \leq M$, so we may assume that $p \mid |G : M|$ and $Q \leq M$. Then $M = \langle x^p \rangle Q = \langle x^p \rangle \times Q$ because $\langle x^p \rangle$ is second maximal in G and hence normal in G by hypothesis. This shows our claim and the result follows.

Claim 3.2. *If P is non-cyclic, then one of conclusions (b) and (d) in Theorem 2 holds.*

Since P is an abnormal maximal subgroup of G, P possesses at least one maximal subgroup H which is not normal in G. But H is second maximal in G, so H is a TI-subgroup by the hypothesis. Therefore

$$H^x \cap H = 1 \quad \forall x \in G - N_G(H) = G - P.$$

Also, $V \nleq H$ by Proposition 2.1, and so $P^x = (HV)^x = H^x V$ holds for any $x \in G$, thus

$$|V : H^x \cap V| = |P^x : H^x| = |P : H| = p \ \forall x \in G.$$

Recall that $|V| = p^m$. We now claim that if $m \geq 2$ then $G \cong S_4$. As $(H^x \cap V) \cap (H \cap V) \leq H^x \cap H = 1$ for all $x \in G - P$ and $H \cap V$ and $H^x \cap V$ both have index p in V, we get $|H \cap V| = p$ and $|V| = p^2$. Furthermore, $V \cong Z_p \times Z_p$ since V is minimal normal in G. Consider the subgroup VQ. If VQ is nilpotent, choosing $1 \neq x \in Q$, we have $(H \cap V)^x = H \cap V$ and so $H^x \cap H \geq H \cap V > 1$. This is a contradiction. Hence VQ cannot be nilpotent and it follows that $C_G(V)$ is a p-group. Assume that $VQ \leq M < G$ for some maximal subgroup M of G. Since VQ is non-nilpotent, Theorem 1 implies that M must be a Frobenius group with kernel V. Hence $VQ = M$, namely, VQ is a maximal subgroup of G. In particular, V is maximal in P and hence $|P| = p^3$. On the other hand, $G/C_G(V)$ is isomorphic to some subgroup of $Aut(V) \cong GL(2, p)$, so

$$q \mid |GL(2, p)| = p(p-1)^2(p+1),$$

which forces $q = p + 1$ since $q > p$. Thus $p = 2$ and $q = 3$. It follows that $|V| = 4$ and $|P| = 2^3$. Then since VQ is normal but Q is not normal in G, by the Frattini argument, $G = VN_G(Q) = V \times N_G(Q)$. Hence $|G : N_G(Q)| = |V| = 4$ and $|N_G(Q)| = 6$, which show that G has a faithful permutation representation of degree 4. Thus we conclude that $G \cong S_4$. This proves conclusion (d) of Theorem 2.

Consider the case of $m = 1$. We claim that $|G| = p^2q$. Indeed, V is a normal subgroup of G of order p for this case, so $V \leq Z(P)$. It follows that $V \leq Z(G)$ since P is an abnormal maximal subgroup of G. If G has only nilpotent maximal subgroups, then G is a minimal non-nilpotent group. Applying [2, III, Satz 5.2], P is cyclic, contrary to the assumption. Hence there exists a maximal subgroup M in G such that M is non-nilpotent. By Theorem 1, M must be a Frobenius group. Thus, as $V \leq Z(G)$, we have $V \nleq M$, and so $G = VM$ and $G/V \cong M/V \cap M = M$. Since G/V is minimal non-nilpotent by hypothesis and a Frobenius group since M is, we can conclude that $|M| = |G/V| = pq$ and $|G| = p^2q$, and thus conclusion (b) of Theorem 2 holds.

Case 2. $r \neq p$.

In this case, $P = \langle x \rangle$ and PV is an abnormal maximal subgroup of G and QV is normal in G. We first claim the following.

Claim 3.3. *If PV is nilpotent, then $|G| = pqr$ or pq^2.*

We have $PV \leq C_G(V) \lhd G$. If $C_G(V) < G$, then $PV = C_G(V) \lhd G$, which is a contradiction since PV is abnormal in G. Thus $C_G(V) = G$, which means $V \leq Z(G)$, so $|V| = r$. We claim that there exists a maximal subgroup M of G such that $G = V \times M$. In fact, the claim is clear if $r \neq q$. Otherwise, $r = q$ and $V \leq Q$ and $|Q| = q^2$. If Q is cyclic, then V is a unique subgroup of Q of order q and contained in $Z(G)$, by Ito's lemma [2, IV, Satz 5.5], P is normal in G, a contradiction. Hence $Q \cong Z_q \times Z_q$. Now, as V is a subgroup of Q of order q and contained in $Z(G)$, applying Mashcke's Theorem [2, I, Satz 17.6], we see that V has a complement subgroup $M = PZ_q$, which is a maximal subgroup of G. Hence $G = V \times M$ as desired. Now $M \cong G/V$ is a minimal non-abelian group of order p^nq with a normal Sylow q-subgroup. On the other hand, Theorem 1 asserts that M is a Frobenius group. Thus we can conclude that M must be a non-abelian group of order pq, completing the proof of Claim 3.3.

Claim 3.4. *If PV is non-nilpotent, then the conclusion in Claim 3.3 holds as well.*

Since PV is maximal in G and non-nilpotent, apply Theorem 1 to see that PV is a Frobenius group with kernel V. Let P_1 be a maximal subgroup of P. Then P_1V is a second maximal subgroup of G, so P_1V is a TI-subgroup by hypothesis. By Proposition 2.1, P_1V is normal in G. It follows that $G = N_G(P_1)V$ by the Frattini argument. Suppose $P_1 \neq 1$. Since P is a Frobenius complement of PV, we see that $N_G(P_1) \leq P$, so $N_G(P_1) \cap V = 1$ and $G = V \times N_G(P_1)$. It is clear that $N_G(P_1)$ is maximal in G. As in the proof of Claim 3.3, $N_G(P_1) \cong G/V$ is a minimal non-abelian group of order p^nq and a Frobenius group. We obtain $|N_G(P_1)| = pq$, which is absurd since $1 < P_1 < P \leq N_G(P_1)$. Thus we have shown that P is of order p. Set $K = QV$. Then $G = PK$. We claim that G is a Frobenius group with kernel K. Assume that the claim is not true. Then $C_P(x) > 1$ for some $1 \neq x \in K$. But PV is a Frobenius group with kernel V, we see that $x \notin V$. So xV is a generator of $QV/V = K/V$ since QV/V is of order q. Also, from $|P| = p$ and $C_P(x) > 1$ it follows that xV

centralises PV/V and hence G/V is nilpotent, a contradiction, which shows our claim. Now since the kernel of a Frobenius group is nilpotent [2, V, Hauptsatz 8.7], $K = QV$ is nilpotent. If $r \neq q$, then $K = Q \times V$, we get $Q \lhd G$ since K is normal in G. Consequently, $G/Q \cong PV$ is non-nilpotent. Since PV is a Frobenius group with kernel V, applying Proposition 2.2, G/Q is a non-abelian group of order pr. Thus we conclude $|G| = pqr$ as desired. If $r = q$, then $V \leq Q$ and $G = PQ$ with $Q \lhd G$. So G is supersolvable because $|P| = p$ and $p \mid q - 1$, it follows that $|V| = q$ and $|G| = pq^2$ as desired again.

Next let G/V be nilpotent for each V.

Since G is non-nilpotent, we have $V \nsubseteq \Phi(G)$ in this case. Thus G possesses a maximal subgroup M such that

$$G = VM, \quad V \cap M = 1$$

and $M \cong G/V$ is nilpotent. Suppose that $core_G(M) > 1$. Then we may choose a minimal normal subgroup U of G such that U is contained in $core_G(M)$. Then G/U is nilpotent by the assumption of this section. Hence G is isomorphic to a subgroup of $G/V \times G/U$ and thus G is nilpotent, a contradiction. Therefore $core_G(M) = 1$. Recall that $|V| = r^m$. If r is a divisor of $|M|$, let R be a Sylow r-subgroup of M. Then $M \leq N_G(R)$ since M is nilpotent. Also, as VR is an r-group and $V \cap R \leq V \cap M = 1$, we have $N_V(R) > 1$. But then, R is normal in G, contrary to $core_G(M) = 1$. Hence we conclude that M is an r'-group.

We now show that M is a TI-subgroup of G. Assume the contrary. Then there exists an element $g \in M$ such that $M^g \neq M$ while $M^g \cap M = D > 1$. Choose g such that D has the largest possible order. Since M^g is nilpotent, we have $D < N_{M^g}(D)$, thus $N_{M^g}(D) \nsubseteq M$ by the maximality of D. On the other hand, since $core_G(M) = 1$, we have $N_G(D) < G$. Consequently, $N_M(D) < M$. Therefore M has a maximal subgroup L which contains $N_M(D)$. It is clear that VL is a maximal subgroup of G, and $VL \lhd G$ as G/V is nilpotent. In view of $core_G(M) = 1$ and $L \lhd M$, we have $N_G(L) = M$, so VL connot be nilpotent. It follows from Theorem 1 that VL is a Frobenius group with kernel V. In particular, L is an r'-group. Hence $N_G(D) \leq L \leq M^x$ for some $x \in G$. But then, $D < N_M(D) \leq N_G(D) \leq M^x$, contrary to the choice of D. Thus M is certainly a TI-subgroup of G, and it follows that $G = MV$ is a Frobenius group with kernel V.

By the structure theorem on Frobenius groups [2, V, Hauptsatz 8.7], every Sylow subgroup of M of odd order is cyclic and every Sylow 2-subgroup of M is either cyclic or a generalised quaternion group Q_{2^n} for some $n \geq 3$. Moreover, M is nilpotent, so we can conclude that $M \cong Z_m$ for some integer $m \geq 2$ or $M \cong Z_d \times Q_{2^n}$ where $d \geq 1$ is odd and $n \geq 3$. Suppose that $M \cong Z_d \times Q_{2^n}$. Since every second maximal subgroup of G/V is normal in G/V, it follows that every second maximal subgroup of Q_{2^n} is normal in Q_{2^n} as well. Applying Proposition 2.3, we obtain $M \cong Z_d \times Q_8$.

Finally, let F be an arbitrary maximal subgroup of M. If V has a proper subgroup $B > 1$ such that B is F-invariant, then we may assume that FB is a second maximal subgroup of G, so FB is a TI-subgroup of G; this is impossible because any $x \in V - B$ satisfies $(FB)^x \cap FB = B$. Thus F acts irreducibly on V. The proof of the theorem is now complete. ■

From Theorem 1, we have the following.

Corollary 3. *Let G be a finite group without any normal non-trivial Sylow subgroups. Then each second maximal subgroup of G is a TI-subgroup if and only if $G \cong S_4$ or A_5.*

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