A NOTE ON SINGULAR VOLTERRA FUNCTIONAL-DIFFERENTIAL EQUATIONS

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[Received 15 October 1998. Read 7 October 1999. Published 31 August 2000.]

Abstract

Functional-differential equations of the form $y'(t) = Vy(t)$ defined on either $(0, T]$ or $(0, T)$, and with initial condition $\lim_{t\to 0^+} h(t)y(t) = a$, are considered, and solutions $y \in C(0, T]$ and $y \in C(0, T)$ respectively are sought.

1. Introduction

Functional-differential equations of the form

$$\begin{cases} y'(t) = Vy(t), \text{ a.e. } t \in I \\ y(0) = y_0, \end{cases} \tag{1.1}$$

where V is an operator and $I = [0, T]$ or $[0, T)$, have received growing attention in the literature over the past thirty years [2; 3]. A study by the authors [7; 8, ch. 2] which details existence principles for (1.1) is just one of several in the area which examine this functional-differential equation and related equations. The cases when the operator V satisfies

$$V : C[0, T] \to L^p[0, T], \quad 1 \leq p \leq \infty \tag{1.2}$$

or

$$V : C[0, T) \to L^p_{loc}[0, T), \quad 1 \leq p \leq \infty, \quad 0 \leq T \leq \infty \tag{1.3}$$

and a solution $y \in AC[0, T]$ or $y \in AC_{loc}[0, T)$ respectively of (1.1) is sought are two examples of the results presented in [7] and [8].

Notation. Let I be an interval of $\mathbf{R}$. $L^p_{loc}(I)$ denotes the set of functions which are L^p-integrable on each compact interval of I, while analogously $AC_{loc}(I)$ denotes the set of functions which are absolutely continuous on each compact interval of I.

Indeed, in much of the literature the operator V in (1.1) is assumed to be an abstract Volterra operator.

Definition 1.1. An operator $V: E \to F$, where $E = E(I)$ and $F = F(I)$ are function spaces and $I = [0, T]$ or $[0, T)$, is an abstract Volterra operator if, for an arbitrary $T^\star \in I$, V satisfies the following condition:

$$x(t) = y(t), \quad t \in [0, T^\star] \text{ implies that } Vx(t) = Vy(t), \quad t \in [0, T^\star].$$

When considering spaces of measurable functions, the relevant equalities are replaced by equalities almost everywhere.

Two classic examples of Volterra operators are

$$Vy(t) = f(t, y(t)) \tag{1.4}$$

— a Niemytzki operator (a very special type of Volterra operator) — and

$$Vy(t) = h(t) + \int_0^t k(t,s) f(s, y(s))\, ds \tag{1.5}$$

— a Volterra integral operator. With V as described in (1.4), we see that (1.1) reduces to the Cauchy problem, while if V is as in (1.5), we have that (1.1) is an integrodifferential equation of the kind studied in [7] and [8].

The underlying spaces given in (1.2) and (1.3) are typical of those considered in the study of (1.1) in general, and indeed in the study of (1.1) when V is as described in (1.4) or (1.5). These spaces in turn encourage us to find solutions that lie in either $AC[0, T]$ or $AC_{loc}[0, T)$. In this note we wish to examine what happens if the operator V is not as 'well-behaved' as the V's described in the examples above, implying in turn that the underlying spaces are not 'as nice' as $C[0, T]$, $C[0, T)$, $L[0, T]$ or $L_{loc}[0, T)$.

Formally speaking, we wish to examine a variation of (1.1), namely

$$\begin{cases} y'(t) = Vy(t), \text{ a.e. } t \in I \\ \lim_{t \to 0^+} h(t) y(t) = a, \end{cases} \tag{1.6}$$

where $I = (0, T]$ or $(0, T)$ and V is a *singular* or *singular Volterra* operator.

Definition 1.2. Let $E[0, T]$ be a normed space with norm given by $|.|_E$. An operator $V: E[0, T] \to L[0, T]$ is a Carathéodory operator if it satisfies the following conditions:

(i) V is continuous,
(ii) for any $r > 0$, there exists $\mu_r \in L[0, T]$ such that for any $y \in E[0, T]$ with $|y|_E \le r$, we have $|Vy(t)| \le \mu_r(t)$, almost everywhere on $[0, T]$.

Definition 1.3. Let $I = [0, T]$ with one or both endpoints excluded. An operator $V: E[0, T] \to L_{loc}(I)$ is a local Carathéodory operator if it satisfies the following conditions:

(i) V is continuous,
(ii) for any $r > 0$, there exists $\mu_r \in L_{loc}(I)$ such that, for all $y \in E[0, T]$ with $|y|_E \le r$, we have $|Vy(t)| \le \mu_r(t)$, almost everywhere on I.

Definition 1.4. Let μ_r be as described in either of the two definitions above. If

$$\int_0^T \mu_r(t)\, dt < \infty,$$

then V is a regular operator. Otherwise we say that V is a singular operator.

(Note that the operator V described in Definition 1.2 is a regular operator.)

In the first half of the next section, we present an existence principle which guarantees the existence of a solution $y \in C(0, T]$ with $hy \in AC[0, T]$, of (1.6) with $I = (0, T]$. En route to this result, we examine the somewhat simpler functional-differential equation

$$\begin{cases} (h(t)y(t))' = Ny(t), \text{ a.e. } t \in [0, T] \\ \lim_{t\to 0^+} h(t)y(t) = a, \end{cases} \tag{1.7}$$

where $N: E_h[0, T] \to L[0, T]$ is a regular Carathéodory operator. (Here $E_h[0, T]$ is the space of functions $y \in C(0, T]$ for which $hy \in C[0, T]$, and $L[0, T] := L^1[0, T]$.) We prove the existence of a solution $y \in C(0, T]$, such that $hy \in AC[0, T]$, of (1.7) and then use this result to obtain the desired result for (1.6) with $I = (0, T]$.

In addition, we consider the functional equation

$$h(t)y(t) = a + N^\star y(t), \quad t \in [0, T], \tag{1.8}$$

where $N^\star: E_h[0, T] \to C[0, T]$, and present conditions which ensure that (1.8) has at least one solution $y \in C(0, T]$, such that $hy \in C[0, T]$. With this result established, further results for certain cases of (1.6) follow easily. For example, using the result obtained for (1.8), we will show that

$$\begin{cases} y'(t) = -\dfrac{1}{t^3} \displaystyle\int_0^t [sy(s) + As|sy(s)|^{\frac{1}{n}}]\, ds, \quad t \in (0, T] \\ \lim_{t\to 0^+} ty(t) = a \end{cases} \tag{1.9}$$

has a solution $y \in C(0, T]$ such that $hy \in C[0, T]$. We also discuss an existence result for the singular boundary value problem

$$\begin{cases} y''(t) + \dfrac{3y'(t)}{t} + \dfrac{y(t)}{t^2} = -\dfrac{A}{t^2}|ty(t)|^{\frac{1}{n}}, \quad t \in [0, 1] \\ \lim_{t\to 0^+} ty(t) \text{ exists}, \ y(1) = 1 \end{cases} \tag{1.10}$$

which is of a different form to that discussed in [4], but which also follows from the result established for (1.8).

The second half of Section 2 is dedicated to the more general version of (1.6) defined on the open interval $(0, T)$ and given by

$$\begin{cases} y'(t) = Vy(t), \text{ a.e. } t \in (0, T) \\ \lim_{t\to 0^+} h(t)y(t) = a. \end{cases} \tag{1.11}$$

We show that (1.11) has a solution $y \in C(0, T)$ with $hy \in AC_{loc}[0, T)$ by first presenting two existence principles for (1.7), where $[0, T]$ has been replaced by $[0, T)$, and then using these results to obtain the result for (1.11).

Finally we state the following theorem, which will be required later.

Theorem 1.1 (Nonlinear Alternative). *Let C be a convex subset of a normed linear space E, and let U be an open subset of C, with $p^\star \in U$. Then every compact, continuous map $G: \overline{U} \to C$ has at least one of the following two properties:*

(i) *G has a fixed point;*

(ii) *there is a $y \in \partial U$, with $y = (1-\lambda)p^\star + \lambda Gy$ for some $0 < \lambda < 1$.*

2. Existence principles

The main functional-differential equation of interest to us in this note is

$$\begin{cases} y'(t) = Vy(t), \ \text{a.e. } t \in (0, T] \\ \lim_{t \to 0^+} h(t)y(t) = a. \end{cases} \tag{2.1}$$

We assume that $h \in C[0, T]$, $a \in \mathbf{R}$ and that the operator V satisfies

$$V: E_h[0, T] \to L_{loc}(0, T], \tag{2.2}$$

where $E_h[0, T]$ will be described shortly. Equations of the form (2.1) are discussed in [6] and solutions $y \in C[0, T]$ are sought. However, prompted by the initial condition in (2.1), we define

$$E_h[0, T] = \{y \in C(0, T]: hy \in C[0, T]\} \tag{2.3}$$

and seek a solution y of (2.1) which resides in this weighted Banach space $E_h[0, T]$, with norm given by

$$|y|_{0,h} = |hy|_0 = \sup_{t \in [0,T]} |h(t)y(t)|.$$

If V is a Niemytzki operator as in (1.4), there is no difficulty in having V operate on $E_h[0, T]$ (or indeed $C(0, T]$), but if V is a Volterra integral operator as described in (1.5) and we insist that (2.2) is satisfied, then obviously a strong dependence of V on h is implied. In general, for an abstract operator V to satisfy (2.2), the reliance of V on h is inherent.

However, before considering (2.1) we will first examine the functional-differential equation

$$\begin{cases} (h(t)y(t))' = Ny(t), \ \text{a.e. } t \in [0, T] \\ \lim_{t \to 0^+} h(t)y(t) = a, \end{cases} \tag{2.4}$$

where $N: E_h[0, T] \to L[0, T]$ is a regular operator. An existence principle follows easily for (2.4) by incorporating the weighted space $E_h[0, T]$ with a result from [7] and [8].

Theorem 2.1. *Suppose that*

$$h \in C[0,T] \text{ with } h(t) \neq 0 \text{ for } t \in (0,T] \tag{2.5}$$

and

$$N : E_h[0,T] \to L[0,T] \text{ is a Carathéodory operator} \tag{2.6}$$

hold. In addition, suppose that there exists a constant $M > 0$, independent of λ, with

$$|a| < M \text{ and } |y|_{0,h} = |hy|_0 \neq M,$$

for any solution $y \in E_h[0,T]$ of

$$\begin{cases} (h(t)y(t))' = \lambda Ny(t), \text{ a.e. } t \in [0,T] \\ \lim_{t\to 0^+} h(t)y(t) = a, \end{cases} \tag{2.7}$$

for each $\lambda \in (0,1)$. Then (2.4) *has at least one solution $y \in C(0,T]$ with $hy \in AC[0,T]$.*

PROOF. By (2.6), solving (2.7) is equivalent to finding $y \in E_h[0,T]$ that satisfies

$$h(t)y(t) = a + \lambda \int_0^t Ny(s)\,ds, \quad t \in [0,T],$$

which on rewriting becomes

$$y(t) = (1-\lambda)\frac{a}{h(t)} + \lambda\left(\frac{a}{h(t)} + \frac{1}{h(t)}\int_0^t Ny(s)\,ds,\right), \quad t \in (0,T].$$

Defining

$$\tilde{N}y(t) = \frac{a}{h(t)} + \frac{1}{h(t)}\int_0^t Ny(s)\,ds, \quad t \in (0,T],$$

it is easy to check that $\tilde{N} : E_h[0,T] \to E_h[0,T]$ is well defined.

To apply the Nonlinear Alternative we must verify the continuity and complete continuity of $\tilde{N} : E_h[0,T] \to E_h[0,T]$. Since by (2.6) $N : E_h[0,T] \to L[0,T]$ is a continuous operator, it follows that if $y_n \to y$ in $E_h[0,T]$, then

$$|h(t)\tilde{N}y_n(t) - h(t)\tilde{N}y(t)| \leq \int_0^t |Ny_n(s) - Ny(s)|\,ds \leq \int_0^T |Ny_n(s) - Ny(s)|\,ds \to 0,$$

thus illustrating the continuity of $\tilde{N} : E_h[0,T] \to E_h[0,T]$.

We now show the complete continuity of $\tilde{N} : E_h[0,T] \to E_h[0,T]$. Let $\Omega \subseteq E_h[0,T]$ be bounded, that is, there exists $r > 0$ such that $|y|_{0,h} = |hy|_0 \leq r$ for each $y \in \Omega$. For $y \in \Omega$,

$$|\tilde{N}y|_{0,h} = |h\tilde{N}y|_0 \leq |a| + \int_0^T \mu_r(s)\,ds \equiv M_0 < \infty,$$

implying that $\tilde{N}\Omega$ is uniformly bounded in $E_h[0,T]$. Also for $y \in \Omega$ and $t, t^\star \in [0,T]$,

$t < t^\star$,

$$|h(t^\star)\tilde{N}y(t^\star) - h(t)\tilde{N}y(t)| \leq \int_t^{t^\star} |Ny(s)|\, ds \leq \int_t^{t^\star} \mu_r(s)\, ds \to 0 \text{ as } t \to t^\star.$$

Consequently, by the Arzéla–Ascoli Theorem, $\tilde{N}: E_h[0, T] \to E_h[0, T]$ is completely continuous.

Applying the Nonlinear Alternative with $U = \{y \in E_h[0, T]: |y|_{0,h} < M\}$, $C = E = E_h[0, T]$, $G = \tilde{N}$ and $p^\star = a/h$, we notice that (ii) cannot occur. Consequently, $\tilde{N}$ has a fixed point in $E_h[0, T]$ or equivalently (2.4) has a solution $y \in E_h[0, T]$ with $hy \in AC[0, T]$. ∎

With Theorem 2.1 established we are now in a better position to examine (2.1). We note that finding a solution $y \in E_h[0, T]$ of (2.1) is equivalent (under appropriate conditions) to finding a solution $y \in E_h[0, T]$ of

$$\begin{cases} (h(t)y(t))' = h'(t)y(t) + h(t)Vy(t), \text{ a.e. } t \in (0, T] \\ \lim_{t\to 0^+} h(t)y(t) = a. \end{cases} \tag{2.8}$$

Therefore, defining

$$Ny(t) := h'(t)y(t) + h(t)Vy(t), \tag{2.9}$$

we see that we are able, in certain circumstances, to write (2.1) in the form of (2.4). In particular, if h and V are such that

$$N: E_h[0, T] \to L[0, T], \tag{2.10}$$

and the hypotheses of Theorem 2.1 are satisfied, then it will follow that (2.1) has a solution $y \in E_h[0, T]$ with $hy \in AC[0, T]$. We now state this formally and illustrate with an example.

Theorem 2.2. *Suppose that*

$$h \in AC[0, T] \text{ and } h(t) \neq 0 \text{ for } t \in (0, T] \tag{2.11}$$

and

$$\begin{cases} \text{the operator } V: E_h[0, T] \to L_{loc}(0, T] \text{ is such that the operator} \\ N, \text{ defined by } Ny(t) = h'(t)y(t) + h(t)Vy(t), \text{ satisfies} \\ N: E_h[0, T] \to L[0, T], \text{ and is a Carathéodory operator} \end{cases} \tag{2.12}$$

hold. In addition, suppose that there exists a constant $M > 0$, independent of λ, with

$$|a| < M \text{ and } |y|_{0,h} = |hy|_0 \neq M,$$

for any solution $y \in E_h[0,T]$ *of*

$$\begin{cases} (h(t)y(t))' = \lambda\left(h'(t)y(t) + h(t)Vy(t)\right), & a.e.\ t \in [0,T] \\ \lim_{t\to 0^+} h(t)y(t) = a, \end{cases} \tag{2.13}$$

for each $\lambda \in (0,1)$. *Then* (2.1) *has at least one solution* $y \in C(0,T]$ *with* $hy \in AC[0,T]$.

PROOF. The proof follows directly from Theorem 2.1. ■

In many of the examples that we wish to consider, the weight function is given by $h(t) = t^n$, $n > 0$. We now examine one such example which satisfies the hypotheses of Theorem 2.2 and which has also been discussed in [5].

Example 2.1. Consider

$$\begin{cases} y'(t) = Vy(t) := -\frac{n}{t}y(t) + mt^{m-1-n}, & t \in (0,T] \\ \lim_{t\to 0^+} t^n y(t) = 0, \end{cases} \tag{2.14}$$

where $n > 0$, $m \in (0,n)$ and V is a singular operator. This problem has a unique solution $y(t) = t^{m-n}$, which of course lies in $E_h[0,T]$, where $h(t) = t^n$. Here

$$Ny(t) = h'(t)y(t) + h(t)Vy(t) = nt^{n-1}y(t) + t^n\left(-\frac{n}{t}y(t) + mt^{m-1-n}\right) = mt^{m-1},$$

which clearly satisfies (2.12).

Further examples will be discussed after we make the following observation. We note that in the proof of Theorem 2.1 we were able to integrate the functional-differential equation (2.4), and hence write it as a functional equation of the form

$$h(t)y(t) = a + N^\star y(t), \quad t \in [0,T], \tag{2.15}$$

where $N^\star : E_h[0,T] \to C[0,T]$, or, alternatively,

$$y(t) = \tilde{N}y(t) := \frac{a}{h(t)} + \frac{1}{h(t)}N^\star y(t), \quad t \in (0,T], \tag{2.16}$$

where $\tilde{N} : E_h[0,T] \to E_h[0,T]$. It transpires that, for certain Volterra operators V, it is more beneficial to rewrite (2.1) directly in the form (2.15), and then use an existence principle for (2.15) to establish the existence of a solution of (2.1). We will better illustrate this point in the next example. However, we first state the following existence principle for (2.15).

Theorem 2.3. *Suppose that* (2.5) *and*

$$N^\star : E_h[0,T] \to C[0,T] \text{ is a continuous and completely continuous operator} \tag{2.17}$$

hold. In addition, suppose that there exists a constant $M > 0$, *independent of* λ, *with*

$|a| < M$ and $|y|_{0,h} = |hy|_0 \neq M$, for any solution $y \in E_h[0,T]$ of

$$h(t)y(t) = a + \lambda N^\star y(t), \quad t \in [0,T],$$

for each $\lambda \in (0,1)$. Then (2.15) *has at least one solution $y \in C(0,T]$ with $hy \in C[0,T]$.*

Proof. It is easy to see that $\tilde{N} := \frac{a}{h(t)} + \frac{1}{h(t)} N^\star : E_h[0,T] \to E_h[0,T]$ is a continuous and completely continuous operator. The proof now follows by using the Nonlinear Alternative in a similar fashion to that described in the proof of Theorem 2.1. ■

Remark 2.1. The solution $y \in C(0,T]$ of (2.15) guaranteed by Theorem 2.3 satisfies $hy \in C[0,T]$. However, in this case it is not necessarily true that $(hy)' \in L[0,T]$, that is, unlike the solution obtained for (2.4) in Theorem 2.1, here hy may not belong to $AC[0,T]$.

We now present two applications of Theorem 2.3.

Example 2.2. Consider

$$\begin{cases} y'(t) = Vy(t) := -\dfrac{1}{t^3}\displaystyle\int_0^t [sy(s) + As|sy(s)|^{\frac{1}{n}}]\, ds, \quad t \in (0,T] \\ \lim_{t\to 0^+} ty(t) = a, \end{cases} \tag{2.18}$$

where $1 < n \in \mathbf{N}$, $A, a \in \mathbf{R}$ and V is a singular Volterra operator. If y is a solution of (2.18), then by differentiating we see that

$$y''(t) + \frac{3y'(t)}{t} + \frac{y(t)}{t^2} = -\frac{A}{t^2}|ty(t)|^{\frac{1}{n}}.$$

Using the method of variation of parameters, we obtain

$$y(t) = \frac{C_1}{t} + \frac{C_2 \ln t}{t} + \frac{A}{t}\int_0^t \ln s\, |sy(s)|^{\frac{1}{n}}\, ds - \frac{A \ln t}{t}\int_0^t |sy(s)|^{\frac{1}{n}}\, ds, \quad t \in (0,T],$$

that is,

$$ty(t) = C_1 + C_2 \ln t + A\int_0^t \ln s\, |sy(s)|^{\frac{1}{n}}\, ds - A \ln t \int_0^t |sy(s)|^{\frac{1}{n}}\, ds, \quad t \in (0,T]. \tag{2.19}$$

We seek a solution $y \in E_t[0,T] = \{y \in C(0,T] : ty \in C[0,T]\}$ of (2.19), and therefore we let $C_2 = 0$. Also, since the initial condition in (2.18) must be satisfied and since for $y \in E_t[0,T]$, $\lim\limits_{t\to 0^+}\displaystyle\int_0^t \ln s |sy(s)|^{\frac{1}{n}}\, ds = \lim\limits_{t\to 0^+} \ln t \int_0^t |sy(s)|^{\frac{1}{n}}\, ds = 0$, we choose $C_1 = a$ and can rewrite (2.19) as

$$ty(t) = a - A\int_0^t \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\, dz\, ds, \quad t \in [0,T]. \tag{2.20}$$

Defining

$$N^{\star}y(t) := -A\int_0^t \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds, \quad t \in [0, T],$$

we obtain

$$ty(t) = a + N^{\star}y(t), \quad t \in [0, T],$$

and consequently (2.18) is written in the form (2.15).

It is easy to check that $N^{\star}: E_t[0, T] \to C[0, T]$ is a continuous operator. To check the complete continuity of $N^{\star}$, let Ω be a bounded set in $E_t[0, T]$, that is, there exists $r > 0$ such that $|y|_{0,t} = |ty|_0 \le r$, for all $y \in \Omega$. Clearly $N^{\star}\Omega$ is uniformly bounded since, for $y \in \Omega$,

$$|N^{\star}y|_0 \le |A|\int_0^t \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds \le |A|Tr^{\frac{1}{n}}.$$

Letting $t_1, t_2 \in [0, T]$, with $t_2 < t_1$, we have that

$$|N^{\star}y(t_1) - N^{\star}y(t_2)| \le |A|\int_{t_2}^{t_1} \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds \le |A|(t_1 - t_2)r^{\frac{1}{n}} \to 0 \text{ as } t_1 \to t_2,$$

thus illustrating the equicontinuity of $N^{\star}\Omega$. The complete continuity of $N^{\star}: E_t[0, T] \to C[0, T]$ now follows from the Arzéla–Ascoli Theorem. In summary, we have that $N^{\star}: E_t[0, T] \to C[0, T]$ is a continuous and completely continuous operator and thus satisfies (2.17) of Theorem 2.3.

The existence of a solution $y \in E_t[0, T]$ of (2.18) will now follow from Theorem 2.3 if we show that there exists an $M > 0$, such that any solution $y \in E_t[0, T]$ of

$$ty(t) = a + \lambda N^{\star}y(t) = a - \lambda A\int_0^t \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds, \quad t \in [0, T] \tag{2.21}$$

satisfies $|ty|_0 \ne M$. Since

$$|ty|_0 \le |a| + |A|T|ty|_0^{\frac{1}{n}},$$

the existence of such an M is clear (that is, one can find $M > 0$ such that $M/(|a| + |A|TM^{\frac{1}{n}}) > 1$).

Note that if $A = 0$ in (2.18), then $y = a/t$ is a unique solution of the problem and clearly $y \in E_t[0, T]$.

Another result, this time concerning a singular boundary value problem, different to the type discussed in [4] but related to Example 2.2, is given in the following example.

Example 2.3. Consider

$$\begin{cases} y''(t) + \dfrac{3y'(t)}{t} + \dfrac{y(t)}{t^2} = -\dfrac{A}{t^2}|ty(t)|^{\frac{1}{n}}, \quad t \in [0,1], \\ \lim_{t\to 0^+} ty(t) \text{ exists, } y(1) = 1, \end{cases} \tag{2.22}$$

where $1 < n \in \mathbf{N}$, and $A \in \mathbf{R}$. Notice that

$$ty(t) = 1 - A\int_t^1 \ln s|sy(s)|^{\frac{1}{n}}\,ds - A\ln t\int_0^t |sy(s)|^{\frac{1}{n}}\,ds = 1$$

$$+A\int_t^1 \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds, \quad t \in [0,1],$$

or if we let

$$N^\star y(t) := A\int_t^1 \frac{1}{s}\int_0^s |zy(z)|^{\frac{1}{n}}\,dz\,ds, \quad t \in [0,1],$$

we see that

$$ty(t) = 1 + N^\star y(t), \quad t \in [0,1]. \tag{2.23}$$

By essentially the same reasoning as in Example 2.2, we obtain that (2.23) has a solution $y \in C(0,1]$ with $ty \in C[0,1]$. Hence (2.22) has a solution $y \in E_t[0,1]$.

(Unlike Example 2.2, the $N^\star : E_t[0,1] \to C[0,1]$ described above is not a Volterra operator. However, Example 2.2 and the example above follow from Theorem 2.3, which does not discriminate against operators which are not Volterra.)

To complete the note, we remark that one can also consider a variation of (2.1) defined on the open interval $(0,T)$, $0 \leq T \leq \infty$, namely

$$\begin{cases} y'(t) = Vy(t), \text{ a.e. } t \in (0,T) \\ \lim_{t\to 0^+} h(t)y(t) = a \end{cases} \tag{2.24}$$

where

$$V : E_h[0,T) \to L_{loc}(0,T).$$

Here $E_h[0,T)$ is the Fréchet space

$$E_h[0,T) = \{y \in C(0,T) : hy \in C[0,T)\},$$

where, for every $m \in \{1,2,\ldots\}$, the seminorm $\rho_{m,h}(y)$ is defined by

$$\rho_{m,h}(y) = \sup_{t\in[0,t_m]} |h(t)y(t)|, \text{ where } t_m \uparrow T.$$

As in tackling (2.1), when we examined (2.4) first, here it helps to first consider the somewhat simpler problem

$$\begin{cases} (h(t)y(t))' = Ny(t), \text{ a.e. } t \in [0, T) \\ \lim_{t\to 0^+} h(t)y(t) = a, \end{cases} \tag{2.25}$$

where $N: E_h[0, T) \to L_{loc}[0, T)$.

We just state the following two existence principles for (2.25), since the proofs follow by modifying the ideas presented in [7] and [8] to take into consideration the space $E_h[0, T)$.

Theorem 2.4. *Let K be a closed, convex subset of $E_h[0, T)$, with*

$$h \in C[0, T) \text{ with } h(t) \neq 0 \text{ for } t \in (0, T), \tag{2.26}$$

$$N: E_h[0, T) \to L_{loc}[0, T) \text{ a continuous operator,} \tag{2.27}$$

$$\begin{cases} \text{for all } y \in K, \text{ there exists } a \in L_{loc}[0, T) \\ \text{with } |Ny(t)| \leq a(t) \text{ almost everywhere on } [0, T) \end{cases} \tag{2.28}$$

and

$$\begin{cases} \tilde{N}: K \to K \text{ where } \tilde{N}: E_h[0, T) \to E_h[0, T) \text{ is the operator} \\ \text{defined by } \tilde{N}y(t) = \dfrac{a}{h(t)} + \dfrac{1}{h(t)} \displaystyle\int_0^t Ny(s)\, ds, \quad t \in (0, T). \end{cases} \tag{2.29}$$

Then (2.25) has at least one solution $y \in C(0, T)$ with $hy \in AC_{loc}[0, T)$.

Theorem 2.5. *Let $0 < t_1 < t_2 < \ldots < t_n < \ldots$, with $t_n \uparrow T$. Suppose that (2.26),*

$$N: E_h[0, T) \to L_{loc}[0, T) \text{ is a continuous Volterra operator,} \tag{2.30}$$

$$\begin{cases} \text{for any constant } A_k > 0, \text{ there exists } a_{A_k} \in L[0, t_k] \text{ such that for any} \\ y \in E_h[0, t_k], \text{ with } \rho_{k,h}(y) \leq A_k, \text{ we have } |Ny(t)| \leq a_{A_k}(t), \text{ a.e. } t \in [0, t_k] \end{cases} \tag{2.31}$$

and

$$\begin{cases} \text{suppose that for each } n = 1, 2, \ldots, \text{ there exists } y_n \in E_h[0, t_n] \text{ that solves} \\ \quad \begin{cases} y_n'(t) = Ny_n(t), \text{ a.e. } t \in [0, t_n] \\ \lim_{t\to 0^+} h(t)y_n(t) = a \end{cases} \\ \text{and that there are bounded sets } B_k \subseteq \mathbf{R} \text{ for } k = 1, 2, \ldots, \\ \text{such that } n \geq k \text{ implies } h(t)y_n(t) \in B_k \text{ for } t \in [0, t_k] \end{cases} \tag{2.32}$$

hold. Then (2.25) has at least one solution $y \in C(0, T)$ with $hy \in AC_{loc}[0, T)$ and $h(t)y(t) \in \overline{B_k}$ on $[0, t_k]$, for each k.

Remark 2.2. Theorem 2.5 is the first theorem in this note where we have specifically requested that the operator V be of Volterra type.

It is now clear that if one can rewrite (2.24) in the form of (2.25), that is,

$$\begin{cases} (h(t)y(t))' = Ny(t) := h'(t)y(t) + h(t)Vy(t), \text{ a.e. } t \in [0, T) \\ \lim_{t\to 0^+} h(t)y(t) = a, \end{cases}$$

where h and N satisfy the hypotheses of either Theorem 2.4 or Theorem 2.5, then (2.24) has at least one solution $y \in C(0, T)$ with $hy \in AC_{loc}[0, T)$. We omit the detail.

Remark 2.3. As in the first half of this section, existence principles can also be presented for

$$h(t)y(t) = a + N^{\star}y(t), \quad t \in [0, T), \tag{2.33}$$

and additional results then discussed for (2.24).

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