COMMUTATIVITY THEOREMS THROUGH A STREB'S CLASSIFICATION

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Abstract

The aim of this paper is to investigate commutativity of rings with unity satisfying any one of the following conditions:

$$[x^n y x^m \pm x^r f(y)\, x^s, x] = 0,$$
$$\{1 - yx^m g(yx^m)\}[yx^m - yx^m f(yx^m), x]\{1 - yx^m h(yx^m)\} = 0,$$
$$y^t[x, y^n] = \pm g(x)[f(x), y]h(x) \quad \text{and} \quad [x, y^n]\, y^t = \pm g(x)[f(x), y]h(x),$$

for some $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$, where $m \geq 0, r \geq 0, s \geq 0, n > 0$ and $t > 0$ are integers. Also we extend these results to the case when integral exponents in the underlying conditions are no longer fixed but rather depend on the pair of ring elements x and y for their values. Finally, under different appropriate constraints on commutators, commutativity of rings has been studied.

1. Introduction

Throughout the paper, R will represent an associative ring (which may be without unity), $N = N(R)$ the set of nilpotent elements of R, $Z = Z(R)$ the centre of R, $C = C(R)$ the commutator ideal of R, and $U = U(R)$ the group of units of R. For any x, y in R, $[x, y]$ denotes the commutator $xy - yx$. As usual $\mathbf{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbf{Z}$, the ring of integers. By $GF(q)$ we mean the Galois field (finite field) with q elements and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. Set $e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $(GF(p))_2$ for p prime. Consider the following ring properties.

(I) For each x, y in R, there exists a polynomial $f(X)$ in $X^2\mathbf{Z}[X]$ such that either

$$[x^n y x^m - x^r f(y) x^s, x] = 0 \quad \text{or} \quad [x^n y x^m + x^r f(y) x^s, x] = 0,$$

where $n \geq 0, m \geq 0, r \geq 0, s \geq 0$ are fixed integers.

$(I)'$ For each x, y in R, there exist integers $n = n(x, y) \geq 0$, $m = m(x, y) \geq 0$, $r = r(x, y) \geq 0$, $s = s(x, y) \geq 0$ and a polynomial $f(X)$ in $X^2\mathbf{Z}[X]$ such that either

$$[x^n y x^m - x^r f(y) x^s, x] = 0 \quad \text{or} \quad [x^n y x^m + x^r f(y) x^s, x] = 0.$$

(II) For each x, y in R, there exist polynomials $f(X) \in X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $X\mathbf{Z}[X]$ such that

$$\{1 - yx^m g(yx^m)\}[yx^m - yx^m f(yx^m), x]\{1 - yx^m h(yx^m)\} = 0,$$

where $m \geq 0$ is a fixed integer.

(II)$'$ For each x, y in R, there exist an integer $m = m(x, y) \geq 0$ and polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $X\mathbf{Z}[X]$ such that

$$\{1 - yx^m g(yx^m)\}[yx^m - yx^m f(yx^m), x]\{1 - yx^m h(yx^m)\} = 0.$$

(III) For every x in R there exist polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$y^t[x, y^m] = \pm g(x)[f(x), y]h(x) \quad \text{and} \quad y^t[x, y^n] = \pm g(x)[f(x), y]h(x),$$

for all $y \in R$, where $t \geq 1, m \geq 1$ and $n \geq 1$ are fixed integers with $(m, n) = 1$.

(III)$'$ For every x, y in R there exist integers $t \geq 1, m \geq 1$ and $n \geq 1$ with $(m, n) = 1$ and polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$y^t[x, y^m] = \pm g(x)[f(x), y]\ h(x) \quad \text{and} \quad y^t[x, y^n] = \pm g(x)[f(x), y]h(x).$$

(IV) For every x in R there exist polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$[x, y^m]y^t = \pm g(x)[f(x), y]\ h(x) \quad \text{and} \quad [x, y^n]y^t = \pm g(x)[f(x), y]\ h(x),$$

for all y in R, where $t \geq 1, m \geq 1$ and $n \geq 1$ are fixed integers with $(m, n) = 1$.

(IV)$'$ For all x, y in R there exist integers $t \geq 1, m \geq 1$ and $n \geq 1$ with $(m, n) = 1$ and polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$[x, y^m]y^t = \pm g(x)[f(x), y]h(x) \quad \text{and} \quad [x, y^n]y^t = \pm g(x)[f(x), y]h(x).$$

(V) For every x, y in R there exist $f(t), g(t)$ in $t^2\mathbf{Z}[t]$ such that

$$[x - f(x), y - g(y)] = 0.$$

Recently, in an attempt to extend the famous Jacobson's theorem '$x^{n(x)} = x$', Ó Searcóid and MacHale [14] proved the commutativity of rings satisfying the condition $(xy)^{n(x,y)} = xy$ with $n(x, y) > 1$. Tominaga and Yaqub [18, theorem 2] established that if R is a ring such that either $xy = p(xy)$ or $xy = p(yx)$, where $p(X)$ is in $X^2\mathbf{Z}[X]$, then R is commutative. Also the author, jointly with Bell and Quadri [5, theorem 2], obtained the commutativity of the rings with unity 1 satisfying polynomial identities of the form $[xy - p(xy), x] = 0$ and $[xy - q(xy), x] = 0$, where $p(X), q(X)$ are in $X^2\mathbf{Z}[X]$. Our first aim is to investigate commutativity of rings with unity 1 satisfying either of the properties (I) or (II). Further, we shall consider the properties (I)$'$ and (II)$'$, where integral exponents are allowed to vary with the pair of ring elements x, y and where the ring also satisfies Chacron's condition (V). Our second goal is to establish commutativity of rings with unity 1 satisfying any one of the properties $(III), (IV), (III)'$ and $(IV)'$. There are several results in the existing literature concerning commutativity of rings with unity 1 satisfying certain special cases of these conditions (cf. [2, theorem 2]; [4, theorems 5 and 6]; [9, theorem 1]; [15, theorems 1 and 2]).

In this paper we will confine our attention mainly to the case when polynomials in the underlying conditions vary with the pair of ring elements x, y which offer simultaneous extensions of these results for rings with unity 1. Finally, some related cases of conditions (III) and (IV) have been considered and the commutativity of rings has been investigated under appropriate torsion restrictions on commutators. The idea of the proofs presented in the last is based on some iteration techniques developed by Tong [19].

2. Preliminary results

In order to develop the proofs of our theorem, let us first consider the following types of rings.

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p prime.

(ii) $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \mid a, b \in F \right\}$, where F is a finite field with a non-trivial automorphism σ.

(iii) A non-commutative division ring.

(iv) $S = \langle 1 \rangle + T$, T is a non-commutative radical subring of S.

(v) $S = \langle 1 \rangle + T$, T is a non-commutative subring of S such that $T[T,T] = [T,T]T = 0$.

In [17], Streb classified non-commutative rings which have been used effectively as a tool by several authors to prove a number of commutativity theorems (cf. [1]; [2]; [3]; [10]; [11]; [12]; [13]; [16]; [18]). From the proof of [17, corollary 1], it is easy to see that if R is a non-commutative ring with unity 1, then there exists a factorsubring of R which is of type (i), (ii), (iii), (iv) or (v). This gives us the following lemma, which plays a vital role in our subsequent discussion (cf. [11, Meta theorem]).

Lemma 2.1. *Let P be a ring property which is inherited by factorsubrings. If no ring of type (i), (ii), (iii), (iv) or (v) satisfies P, then every ring with unity 1 and satisfying P is commutative.*

The proofs of the following lemmas can be found in [8], [11, corollary 1], [7] and [19, lemma 1].

Lemma 2.2. *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \cdots, x_n$ with relatively prime integral coefficients. Then the following are equivalent.*

(a) *For any ring R satisfying the polynomial identity $f = 0$, C is a nil ideal.*

(b) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

(c) *Every semiprime ring satisfying $f = 0$ is commutative.*

Lemma 2.3. *If R is a non-commutative ring satisfying (V), then there exists a factor-subring of R which is of type (i) or (ii).*

Lemma 2.4. *Let R be a ring in which for all x, y in R there exists some polynomial $f(X)$ in $X^2\mathbf{Z}[X]$ such that $[x - f(x), y] = 0$. Then R is commutative.*

Lemma 2.5. *Let R be a ring with unity 1. For all $x \in R$ define I_p^k recursively as*

$$I_0^k(x) = x^k, \quad and$$

$$I_p^k(x) = I_{p-1}^k(x+1) - I_{p-1}^k(x), \quad if \quad p \geq 1.$$

Then $I_{k-1}^k(x) = \frac{1}{2}(k-1)k! + k!x$; $I_k^k = k!$ and $I_j^k(x) = 0$ for $j > k$.

3. Main results

Theorem 3.1. *Let R be a ring with unity 1 satisfying any one of the conditions (I) and (II). Then R is commutative.*

Theorem 3.2. *Let R be a ring with unity 1 satisfying any one of the conditions (III) and (IV). Then R is commutative.*

We prove the assertion by a step-by-step reduction from division rings to the considered rings.

Step 3.1. *Let R be a division ring satisfying any one of the properties (I) and (II). Then R is commutative.*

Before proving Step 3.1, we begin with the following lemma.

Lemma 3.1. *Let R be a ring with unity 1 satisfying the property (I). If $x \in U$, then for each $y \in R$ there exists $q(X)$ in $X^2\mathbf{Z}[X]$ such that $[x, y - q(y)] = 0$.*

Proof. Choose $f(X)$ in $X^2\mathbf{Z}[X]$ such that

$$[x^{-n}yx^{-m} - x^{-r}f(y)x^{-s}, x^{-1}] = 0.$$

This implies that $[x^{-n}yx^{-m} - x^{-r}f(y)x^{-s}, x] = 0$, that is,

$$x^{-n}[x, y]x^{-m} = x^{-r}[x, f(y)]x^{-s},$$

or

$$x^r[x, y]x^s = x^n[x, f(y)]x^m. \tag{1}$$

Choose some polynomial $p(X) \in X^2\mathbf{Z}[X]$ such that

$$[x^n f(y)x^m - x^r p(f(y))x^s, x] = 0.$$

This yields that

$$x^n[x, f(y)]x^m = x^r[x, p(f(y))]x^s. \tag{2}$$

From (1) and (2) we obtain

$$x^r[x, y]x^s = x^r[x, q(y)]x^s, \tag{3}$$

where $q(X) = p(f(X)) \in X^2\mathbf{Z}[X]$. Since x is a unit, (3) yields $[x, y - q(y)] = 0$. ■

PROOF OF STEP 3.1. For each $x, y \in R$ there exists $f(X)$ in $X^2\mathbf{Z}[X]$ such that $[x, y - q(y)] = 0$, by Lemma 3.1. Hence R is commutative by Lemma 2.4. ■

Remark 3.1. By making use of remark 12 of [6] one can prove that 'If a ring R with unity satisfies the property (I), then U is commutative'.

Using similar arguments, we can get the required result if R satisfies $[x^n yx^m + x^r f(y)x^s, x] = 0$.

Suppose that R satisfies the property (II). Let u be a unit in R, that is $u \in U$, and for arbitrary element y in R we obtain polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $X\mathbf{Z}[X]$ such that

$$\{1 - (yu^{-m}u^m)\ g(yu^{-m}u^m)\}\ [yu^{-m}u^m - yu^{-m}u^m f(yu^{-m}u^m), u] \times$$
$$\{1 - (yu^{-m}u^m)h(yu^{-m}u^m)\} = 0,$$

or

$$\{1 - yg(y)\}[y - yf(y), u]\{1 - yh(y)\} = 0.$$

This shows that either $1 - yg(y) = 0$, or $1 - yh(y) = 0$, or $[y - yf(y), u] = 0$. In all cases R is commutative by Lemma 2.4.

Step 3.2. *Suppose that $k \geq 1, t \geq 1$ are fixed integers and R is a ring with unity 1 in which for every x in R there exist polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that either $y^t[x, y^k] = \pm\ g(x)\ [f(x), y]\ h(x)$ or $[x, y^k]y^t = \pm g(x)[f(x), y]h(x)$ for all y in R. Then $C \subseteq N$.*

PROOF. Suppose that R satisfies the condition

$$y^t[x, y^k] = \pm g(x)\ [f(x), y]h(x).$$

Set $1 + y$ for y in the given condition to obtain

$$(1 + y)^t[x, (1 + y)^k] = y^t[x, y^k].$$

As $x = e_{22}$ and $y = e_{12}$ fail to satisfy the above polynomial identity in $(GF(p))_2, p$ prime, by Lemma 2.2, R has nil commutator ideal, that is $C \subseteq N$.

A similar argument can be used to obtain the result if R satisfies the condition $[x, y^k]y^t = \pm\ g(x)\ [f(x), y]\ h(x)$. ■

We are now well equipped to prove our theorems.

Proof of Theorem 3.1. Suppose that R is a ring of the type (i).

Let R satisfy the property (I). Taking $x = e_{22}$ and $y = e_{12}$, we get

$$[e_{22}^n e_{12} e_{22}^m - e_{22}^r f(e_{12}) e_{22}^s, e_{22}] = e_{12} \neq 0.$$

Suppose that R satisfies the property (II). Taking $x = e_{11}$ and $y = e_{21}$, we have

$$\{1 - e_{21}e_{11}^m g(e_{21}e_{11}^m)\}[e_{11}, e_{12}e_{11}^m - e_{12}e_{11}^m f(e_{12}e_{11}^m)] \times \{1 - e_{21}e_{11}^m h(e_{12}e_{11}^m)\} = e_{21} \neq 0.$$

Hence in all cases we get a contradiction, and so no ring of type (i) satisfies (I) and (II).

Further, consider the ring $M_\sigma(F)$ of type (ii).

Suppose that R satisfies the property (I). Then take

$$x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, (\alpha \neq \sigma(\alpha)), \quad \text{and} \quad y = e_{12},$$

such that

$$[x^n y x^m - x^r f(y) x^s, x] = -\alpha^n(\alpha - \sigma(\alpha))(\sigma(\alpha))^m e_{12} \neq 0,$$

for all $f(X) \in X^2\mathbf{Z}[X]$.

If R satisfies the property (II), then choose $x = \begin{pmatrix} \alpha & 0 \\ 0 & \sigma(\alpha) \end{pmatrix}, (\alpha \neq \sigma(\alpha))$ and $y = e_{21}$. We get

$$\{1 - yx^m g(yx^m)\}[x, yx^m - yx^m f(yx^m)]\{1 - yx^m h(yx^m)\} = e_{21}\alpha^m(\alpha - \sigma(\alpha)) \neq 0.$$

Thus in all cases R cannot be of type (ii).

Let R be of type (iii). Then by Step 3.1 we get a contradiction.

Let R be of type (iv). If R satisfies either property (I) or (II), then a careful scrutiny of the proof of Step 3.1 gives that there exist x and u in U and arbitrary y in R such that either $[x, y - q(y)] = 0$ or $y - y^2 g(y) = 0$ or $y - y^2 h(y) = 0$ for all $q(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $X\mathbf{Z}[X]$. But in the present case if $t_1, t_2 \in T$, then $u = 1 + t_1$ is a unit and there exist $q(X) \in X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $X\mathbf{Z}[X]$ such that either $[t_2 - q(t_2), 1 + t_1] = 0$ or $t_2 - t_2^2 g(t_2) = 0$, or $t_2 - t_2^2 h(t_2) = 0$. Thus in every case T is commutative by Lemma 2.4, which gives a contradiction.

Further, let R be of type (v). Let $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$.

Suppose that R satisfies (I). Then there exist polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ such that

$$[(1 + t_1)^n t_2 (1 + t_1)^m - (1 + t_1)^r f(t_2)(1 + t_1)^s, 1 + t_1] = 0.$$

This implies that $[t_2, t_1] = 0$, which is a contradiction.

In the same way, if R satisfies (II) then again we get a contradiction.

Hence we observe that no ring of type (i), (ii), (iii), (iv) or (v) satisfies (I) and (II), and by Lemma 2.1 R is commutative. ∎

PROOF OF THEOREM 3.2. Using Lemma 2.2 and Step 3.2, R cannot be of type (iii) or (iv). Next, if R is assumed to be of type (i), then choosing $x = e_{12}$ and $y = e_{11}$ in $(GF(p))_2, p$ prime we get

$$\begin{aligned} e_{11}^t[e_{12}, e_{11}^n] &= \pm g(e_{12})[f(e_{12}), e_{11}]\ h(e_{12}) \\ &= -e_{12} \neq 0, \end{aligned}$$

for all $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$. Hence in both cases we get a contradiction.

Now, consider the ring $M_\sigma(F)$, a ring of type (ii).

If R satisfies property (III), then note that $N(M_\sigma(F)) = Fe_{12}$. Hence for any a in $N(M_\sigma(F))$ and arbitrary unit u in U there exist polynomials $f(X)$ in $X^2\mathbf{Z}[X]$ and $g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$u^t[a, u^n] = \pm g(a)[f(a), u]h(a) = 0.$$

Since $a^2 = 0$ and u is a unit, $[a, u^n] = 0$. Similarly, we get $[a, u^m] = 0$. But $(m, n) = 1$, so we get $[a, u] = 0$. Thus for non-central element $a = e_{12}$ and arbitrary unit u one gets $[e_{12}, u] = 0$, which leads to a contradiction that e_{12} is central.

By a similar argument we obtain a contradiction if R satisfies (IV).

Finally, let R be a ring of type (v). Let $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$.

If R satisfies property (III), then there exist polynomials $f(X) \in X^2\mathbf{Z}[X]$ and $g(X), h(X) \in \mathbf{Z}[X]$ such that

$$n[t_2, t_1] = (1 + t_1)^t\ [t_2, (1 + t_1)^n] = \pm\ g(t_2)\ [f(t_2), 1 + t_1]h(t_2) = 0.$$

Similarly one can show that $m[t_2, t_1] = 0$. This shows that $[t_2, t_1] = 0$, which yields a contradiction.

Similarly, we can obtain a contradiction if R satisfies (IV).

Hence no ring of type (i), (ii), (iii), (iv) or (v) satisfies (III) and (IV), and thus by Lemma 2.1 R is commutative. ■

The proofs of Theorems 3.1 and 3.2 reveal that if R satisfies any one of the conditions $(I)'$, $(II)'$, $(III)'$ and $(IV)'$, then in each case R has no factorsubring of type (i) or (ii). Combining this fact with Lemma 2.3, we obtain the following theorem.

Theorem 3.3. *Suppose that R is a ring with unity 1 satisfying (V). Moreover, if R satisfies any one of the properties $(I)'$ and $(II)'$, then R is commutative (and conversely).*

Theorem 3.4. *Let R be a ring with unity 1 satisfying (V). Suppose further that R satisfies any one of the conditions $(III)'$ and $(IV)'$. Then R is commutative (and conversely).*

The following example demonstrates that in the hypothesis of Theorem 3.2 the existence of both conditions in properties (III) and (IV) is not superfluous (even if the ring R has unity 1).

Example 3.1. Consider $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}$. Then R is a non-commutative ring with unity satisfying the condition $y^t[x, y^4] = x^r[x^4, y]x^s$ where r, s and t may be any non-negative integers.

4. Commutativity of torsion-free rings

In view of Example 3.1, it is natural to ask: what additional conditions are required to establish the commutativity of a ring R if we simply assume $y^t[x, y^m] = \pm\, g(x)[f(x), y]h(x)$ and $[x, y^m]\ y^t = \pm\, g(x)[f(x), y]h(x)$ in properties (III) and (IV) respectively? Finally, it is tempting to conjecture that an m-torsion-free ring with unity 1 satisfying any one of the above properties must be commutative. Thus, under certain appropriate constraints on the commutators involved in the underlying conditions, we may prove some interesting cases of the conjecture. In fact, we shall consider the following ring properties.

(VI) For every y in R, there exist polynomials $f(X), g(X), h(X)$ in $\mathbf{Z}[X]$ such that

$$[x^n, y^r] = \pm\, g(x)[f(x), y]h(x),$$

for all x in R where $r > 1$ and $n \geq 1$ are fixed integers.

(VII) For every x in R, there exist polynomials $f(X), g(X), h(X)$ in $\mathbf{Z}[X]$ such that either

$$y^r[x^n, y] = \pm g(x)[f(x), y]h(x) \quad \text{or} \quad [x^n, y]y^r = \pm g(x)[f(x), y]h(x),$$

for all x in R, where $r \geq 1$ and $n \geq 1$ are fixed integers.

$(VIII)$ For all y in R, there exist polynomials $f(X), g(X), h(X)$ in $\mathbf{Z}[X]$ such that either

$$x^n[x, y^r] = \pm g(y)[x, f(y)]h(y) \quad \text{or} \quad [x, y^r]x^n = \pm g(y)[x, f(y)]h(y),$$

for all x in R, where $r \geq 1$ and $n \geq 1$ are fixed integers.

To prove the commutativity of rings R with the above properties, we need some extra conditions on commutators in R, such as the condition

$$Q(m) : m[x, y] = 0 \quad \text{implies} \quad [x, y] = 0,$$

for all x, y in R and m a positive integer.

Our method of proof uses some iteration techniques which are based on Lemma 2.5 due to Tong [19].

Theorem 4.1. *Let R be a ring with unity 1 satisfying any one of the properties (VI), (VII), $(VIII)$. Also if R satisfies $Q((max\{r, n\})!)$, then R is commutative.*

Proof. Suppose that R satisfies property (VI). We shall first use induction on y^r. From Lemma 2.5, we have

$$I_k(x) = I_k^r(x),$$

for $k = 0, 1, 2, 3, 4, 5, \cdots$. Then condition (VI) can be written as

$$[x^n, I_0(y)] = \pm g(x)[f(x), y]h(x). \tag{4}$$

Replacing y by $y+1$ in (4) and using Lemma 2.5, we get

$$[x^n, I_0(y) + I_1(y)] = \pm\, g(x)[f(x), y]h(x).$$

Again using (4), we get

$$[x^n, I_1(y)] = 0. \tag{5}$$

Replacing y by $y+1$ and using Lemma 2.5, we get

$$[x^n, I_1(y+1)] = [x^n, I_1(y) + I_2(y)] = 0.$$

Again by (5) we get $[x^n, I_2(y)] = 0$. Hence one can observe that replacing y by $y+1$ and iterating $(r-1)$ times, we get

$$[x^n, I_{r-1}(y)] = 0,$$

that is,

$$r![x^n, y] = 0.$$

Finally, replacing x by $x+1$ and using techniques similar to those above, we obtain $r!n![x, y] = 0$, and hence by the property Q $((max\{r, n\})!)$ we get the commutativity of R.

Suppose that R satisfies property (VII). Then using the same techniques we get either

$$I_0(y)[x^n, y] = \pm g(x)[f(x), y]h(x)$$

or

$$[x^n, y]I_0(y) = \pm g(x)[f(x), y]h(x).$$

Replacing y by $y+1$ and using Lemma 2.5, we obtain either $I_1(y)[x^n, y] = 0$, or $[x^n, y]\ I_1(y) = 0$; proceeding in the same way, we finally obtain $I_r(y)[x^n, y] = 0$, or $[x^n, y]\ I_r(y) = 0$. Thus in both cases we get $r![x^n, y] = 0$. Next, using the same method of replacing x by $x+1$ and iterating $(n-1)$ times, we get that $r!n![x, y] = 0$ and by the property Q $((max\{r, n\})!)$ we obtain commutativity of R.

Similarly we can prove commutativity of R, if R satisfies the property $(VIII)$. ■

We conclude our discussion with the following conjecture.

Conjecture 4.2. Let R be a ring with unity 1 in which for every y in R there exists a polynomial $f(\lambda)$ in $\lambda^2\mathbf{Z}[\lambda]$ such that $y^r[x^m, y] = \pm x^t[f(x), y]y^s$, where $m \geq 1$ and r, s, t are non-negative integers. Moreover, if commutators in R are m-torsion-free, then R is commutative.

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