# TOEPLITZ OPERATORS ON DISCRETE ABELIAN GROUPS

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Abstract

We study Toeplitz operators associated with quasi-partial ordered discrete abelian groups.

## 1. Introduction

The classical theory of Toeplitz operators and their associated $C^*$-algebras is an elegant and important area of modern mathematics. For this reason, many authors (e.g. R. Douglas, G. Murphy, A. Nica and E. Park) have sought to extend this theory to a more general setting. The study of Toeplitz operators in various contexts has been an important application of operator algebras, and there has been much progress lately in studying Toeplitz algebras defined on semigroups of discrete abelian groups. For related materials, see [1]–[6] and [9]. In this paper, we study Toeplitz algebras on discrete abelian groups under a general setting of quasi-partial ordered groups.

There are at least three streams of thought that lead us to study Toeplitz operators in such a general setting. Firstly, in recent years there has been much progress in studying Toeplitz operators on ordered groups. By comparison, less is known about Toeplitz operators on general partially ordered groups. To analyse such Toeplitz operators, a universal Toeplitz algebra is constructed in [1], and this plays a key role in the subsequent work of G. Murphy. It is proved in [1] that when $(G, G_+)$ is an ordered group, then the universal Toeplitz algebra is isomorphic to the Toeplitz algebra defined in the usual way. Naturally one might ask: is the analogous statement true for Toeplitz algebras on general partially ordered groups? In [9], a negative answer is given. In this paper, we obtain a further result (Proposition 2.5) in the case when $G$ is abelian. To study Toeplitz operators on general partially ordered groups, one approach is as follows: given a partially ordered group, find suitable ordered groups that contain the partially ordered group, and then use the Toeplitz algebras on the ordered groups to understand the Toeplitz algebra on

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the partially ordered group. To do this, a natural question arises. If $(G, G_1)$ is a partially ordered group and $(G, G_2)$ an ordered group with $G_1 \subseteq G_2$, is the natural morphism $\gamma^{G_2,G_1} : p^{G_1}u_g p^{G_1} \to p^{G_2}u_g p^{G_2}$ well defined and can it be extended as a $C^*$-algebra morphism from Toeplitz algebra $\mathscr{T}^{G_1}(G)$ to $\mathscr{T}^{G_2}(G)$? If $G = Z^2$, $G_1 = Z_+^2$ and $G_2 = \{(m,n) \in Z^2 \mid m > 0 \text{ or } m = 0 \text{ and } n \geq 0\}$, for example, we find that $\gamma^{G_2,G_1}$ is not well defined; see Example 3 below. But if we let $G_2$ be $Z_+ \times Z$ or $Z \times Z_+$, then $\gamma^{G_2,G_1}$ is well defined and can be extended; nevertheless in this case $(G, G_2)$ is only quasi-ordered, i.e. $G_2^0 = G_2 \cap (-G_2)$ is non-trivial. Secondly, just like $Z_+^2 = (Z_+ \times Z) \cap (Z \times Z_+)$, in many cases the positive part of a partially ordered group is the intersection of that of some finite quasi-ordered groups. Inspired by [1], [4], [5], [6], [9], under some suitable hypotheses one might be able to discuss the connection between these Toeplitz algebras. Thirdly, it seems that many properties of Toeplitz operators on ordered groups (Corollary 3.5 and Corollary 3.6) may turn out to be much more reasonable when considered in a more general setting.

## 2. The natural morphisms between Toeplitz algebras

Throughout this paper, $G$ denotes a discrete abelian group. Let $\{e_g | g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$, where

$$e_g(h) = \begin{cases} 1, & \text{if } g = h, \\ 0, & \text{otherwise} \end{cases}$$

for $g, h \in G$. For any $E \subseteq G$, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{e_g \mid g \in E\}$; its projection is denoted by $p^E$. For any $g \in G$, we define a unitary $u_g$ on $\ell^2(G)$ by $u_g(e_h) = e_{g+h}$ for $h \in G$.

**Definition.** The $C^*$-algebra generated by $\{p^E u_g p^E \mid g \in E\}$ is denoted by $\mathscr{T}^E(G)$, and is called the *Toeplitz algebra* with respect to $E$. Its *closed commutator ideal* is denoted by $C(\mathscr{T}^E(G))$.

Let $G_+ \subseteq G$; we say that $(G, G_+)$ is a *quasi-partial ordered* (*qpo*) *group* if $0 \in G_+$, $G_+ + G_+ \subseteq G_+$ and $G = G_+ - G_+$. If furthermore $G = G_+ \cup (-G_+)$, then $(G, G_+)$ is referred to as a *quasi-ordered group*. Note that when $G_+^0 = G_+ \cap (-G_+) = \{0\}$, then qpo groups are just the usual *partially ordered groups*.

**Natural question.** Suppose that $(G, G_1)$ and $(G, G_2)$ are two qpo groups; can the morphism $\gamma^{G_2,G_1}$ defined by $\gamma^{G_2,G_1}(p^{G_1}u_g p^{G_1}) = p^{G_2}u_g p^{G_2}$ for any $g \in G$ be extended as a $C^*$-algebra morphism from $\mathscr{T}^{G_1}(G)$ to $\mathscr{T}^{G_2}(G)$? Note that if it is true, then $G_1$ must be contained in $G_2$. So in the following we always assume that $G_1 \subseteq G_2$.

**Definition** (cf. [9]). Let $(G, G_1)$ and $(G, G_2)$ be two qpo groups with $G_1 \subseteq G_2$. Let $G_2^C$ be the complement of $G_2$ in $G$. We say that $G_2$ *is finitely decomposed by* $G_1$ if for any finite subset $F$ of $G$ there exists an element $g \in G$ such that for all $s \in F$ the following hold:

(a) $s + g \in G_2$ if and only if $s \in G_2$;
(b) $s + g \in G_1 \cup G_2^C$.

*Remark.* When $G_2$ is finitely decomposed by $G_1$, then for any finite subset $F$ of $G$ we can further choose $g \in G_1$ satisfying the condition. In fact, replacing $F$ by $F \cup \{0\}$ will do. So for qpo groups $(G, G_1)$ and $(G, G_2)$ with $G_1 \subseteq G_2$, $G_2$ is finitely decomposed by $G_1$, if and only if for any finite subset $F = F_1 \cup F_2$ with $F_1 \subseteq G_2 \setminus G_1$ and $F_2 \subseteq G_2^C$ there exists $g \in G_1$ such that $g + F_1 \subseteq G_1$ and $g + F_2 \subseteq G_2^C$.

**Lemma 2.1** (cf. [9, theorem 1.1]). *Let $(G, G_1)$ and $(G, G_2)$ be two qpo groups with $G_1 \subseteq G_2$. If $G_2$ is finitely decomposed by $G_1$, then $\gamma^{G_2, G_1}$ is well defined and can be extended as a $C^*$-algebra morphism from $\mathscr{T}^{G_1}(G)$ to $\mathscr{T}^{G_2}(G)$.*

**Proposition 2.2.** *Let $(G, G_+)$ be a qpo group. For any $E \subseteq -G_+$, let $G_E$ be the semigroup of $G$ generated by $G_+$ and $E$. Then $G_E$ is finitely decomposed by $G_+$.*

PROOF. First we note that for any two qpo groups $(G, G_1)$ and $(G, G_2)$ with $G_1 \subseteq G_2$ it is always true that $G_2^0 + G_2^C = G_2^C$. So if for any finite subset $F$ of $G_2$ there exists $g \in G_2^0$ such that $F + g \subseteq G_1$, then $G_2$ is finitely decomposed by $G_1$.

Now let $G_1 = G_+$ and $G_2 = G_E = \{ g_0 + \sum_i n_i g_i \mid g_0 \in G_+, g_i \in E, n_i \in Z_+ \}$. Let $F = \{ e_1, e_2, \ldots, e_k \}$ be any finite subset of $G_E$ with

$$e_j = g_{0j} + \sum n_{ij} g_{ij},\ g_{0j} \in G_+,\ g_{ij} \in E \text{ and } n_{ij} \in Z_+.$$

Choose $g = -\sum n_{i1} g_{i1} - \sum n_{i2} g_{i2} \ldots - \sum n_{ik} g_{ik} \in G_E^0$, then $g + F \subseteq G_+$. ■

*Example 1.* Let $G = Z^2, G_1 = \{ (m,n) \in Z^2 \mid m \geq 0, m + n \geq 0 \}$ and $G_2 = \{ (m,n) \in Z^2 \mid m + n \geq 0 \}$. Then $G_2$ is finitely decomposed by $G_1$. In fact, $G_2 = G_E$, where $E = (-1, 1)$.

**Lemma 2.3** (cf. [9, proposition 1.2]). *Let $(G, G_+)$ be a qpo group. Then we have the following exact sequence:*

$$0 \longrightarrow C(\mathscr{T}^{G_+}(G)) \longrightarrow \mathscr{T}^{G_+}(G) \xrightarrow{\sigma^{G_+}} C(\widehat{G}) \longrightarrow 0$$

*where $\widehat{G}$ is the Pontryagin dual group of $G$, $\sigma^{G_+}(p^{G_+} u_g p^{G_+}) = \varepsilon_g$ for $g \in G$, and $\varepsilon_g(\gamma) = \gamma(g)$ for $\gamma \in \widehat{G}$.*

*Remark.* The special case of the preceding proposition, where $(G, G_+)$ is a partially ordered group, was obtained in [1].

**Proposition 2.4.**

(1) *Let $(G, G_1)$ and $(G, G_2)$ be two qpo groups with $G_1 \subseteq G_2$. If $(G, G_1)$ is quasi-ordered, then $G_2$ is finitely decomposed by $G_1$.*

(2) *Let $(G, G_+)$ be a qpo group. If $C(\mathscr{T}^{G_+}(G))$ is simple, then for any $g \in G_+ \setminus G_+^0$, $G_{-g} = G_+ - Z_+ g = G$.*

PROOF. (1) See [9, proposition 1.4].

(2) If $g \in G_+ \setminus G_+^0$, then $-g \notin G_+$, so $G_{-g} \neq G_+$. By Lemma 2.1 and Proposition 2.2 we know that $\gamma^{G,G_{-g}} \circ \gamma^{G_{-g},G_+} = \gamma^{G,G_+}$, which implies that $\mathrm{Ker}\,\gamma^{G_{-g},G_+} \subseteq \mathrm{Ker}\,\gamma^{G,G_+} = C(\mathscr{T}^{G_+}(G))$. Since $0 \neq 1 - p^{G_+}u_g p^{G_+}u_{-g}p^{G_+} \in \mathrm{Ker}\,\gamma^{G_{-g},G_+}$, by the simplicity of $C(\mathscr{T}^{G_+}(G))$ we know that $\mathscr{T}^{G_{-g}}(G) \cong \mathscr{T}^{G_+}(G)/_{C(\mathscr{T}^{G_+}(G))} \cong C(\widehat{G})$. It follows that $\mathscr{T}^{G_{-g}}(G)$ is commutative, but this can happen if and only if $G_{-g} = G$. ■

*Remark.* Let $(G, G_+)$ be an ordered group. If $C(\mathscr{T}^{G_+}(G))$ is simple, then for any $g_1 \in G_+, g_2 \in G_+ \setminus \{0\}$, there exist some $g \in G_+$ and $n \in N$ such that $-g_1 = g - ng_2$, i.e. $ng_2 - g_1 \in G_+$. It follows that $(G, G_+)$ is *archimedean* in the terminology of Rudin ([7]). The converse is also true; see [1].

*Example 2.* Let $G = Z^3$ and $G_+ = \{(m_1, m_2, m_3) \in Z^3 \mid m_1 + m_2 \geq 0, m_2 + m_3 \geq 0\}$. Take $g = (0,0,1) \in G_+ \setminus G_+^0$, then $G_{-g} \subseteq \{(m_1, m_2, m_3) \in Z^3 \mid m_1 + m_2 \geq 0\}$, so $C(\mathscr{T}^{G_+}(G))$ is not simple.

**Definition.** Let $(G, G_+)$ be a partially ordered group. $(G, G_+)$ is said to be *quasi-lattice ordered* if every finite subset of $G$ with an upper bound in $G_+$ has a least upper bound in $G_+$.

Equivalently, $(G, G_+)$ is quasi-lattice ordered if and only if every element of $G$ having an upper bound in $G_+$ has a least such, and every two elements in $G_+$ having a common upper bound have a least common upper bound ([5, section 2.1]).

*Remark.* (1) The notation for quasi-lattice ordered groups was first introduced by Nica in [5] for discrete (not necessarily abelian) groups. In his definition, $G$ need not be equal to $G_+ \cdot (G_+)^{-1}$. But when $G$ is abelian, such a restriction will not result in any loss of generality since in this case $G_+ \cdot (G_+)^{-1}$ is a subgroup of $G$ ([5, section 2.4]).

(2) Let $(G, G_+)$ be a quasi-lattice ordered group. Since $G$ is partially ordered, it is easy to show that every finite subset of $G$ has an upper bound in $G_+$. For any $x$ and $y$ in $G_+$, their least common upper bound will be denoted by $\sigma(x, y)$.

(3) Ordered groups are quasi-lattice ordered. More generally, if $(G_1, (G_1)_+)$, $(G_2, (G_2)_+), \ldots, (G_n, (G_n)_+)$ are ordered groups, then the usual partially ordered group $(G_1 \times G_2 \ldots \times G_n, (G_1)_+ \times (G_2)_+ \ldots \times (G_n)_+)$ is also a quasi-lattice ordered group.

**Definition.** A *representation of a quasi-lattice ordered group* $(G, G_+)$ *by isometries on a Hilbert space* $H$ is a map $V : G_+ \to B(H)$ such that $V_0 = 1$, $V_x^* V_x = 1$ and $V_x V_y = V_{x+y}$ for any $x, y \in G_+$. A representation is said to be *covariant* if it satisfies

$$V_x V_x^* V_y V_y^* = V_{\sigma(x,y)} V_{\sigma(x,y)}^* \quad \text{for any} \quad x, y \in G_+.$$

Now let $(G, G_+)$ be a quasi-lattice ordered group. The *universal $C^*$-algebra for covariant isometric representations of* $G_+$, denoted by $C^*(G, G_+)$, is a $C^*$-algebra generated by a canonical covariant isometric representation $i : G_+ \to C^*(G, G_+)$ with the following property: if $V$ is any covariant isometric representation of $G_+$, then there is a $C^*$-morphism $\theta_V : C^*(G, G_+) \to C^*(\{V_x | x \in G_+\})$ such that $\theta_V(i(x)) = V_x$. The $C^*$-algebra $C^*(G, G_+)$ exists by [5, section 4.1], and since every discrete abelian group is amenable, the canonical covariant isometric representation $i(x) \to T_x = p^{G_+}u_x p^{G_+}$ induces an isomorphism between $C^*(G, G_+)$ and $\mathscr{T}^{G_+}(G)$ ([5, section 4]). So if $(G, G_+)$ is a quasi-lattice ordered group, then Toeplitz algebra $\mathscr{T}^{G_+}(G)$ has a universal property with respect to covariant isometric representations of $G_+$.

When $(G, G_+)$ is a partially ordered group, another *universal Toeplitz algebra for isometric (not necessarily covariant) representations of* $G_+$, denoted by $\mathscr{U}(\mathscr{T}^{G_+}(G))$, plays a fundamental role in [1]. But, to be honest, generally it does not equal the Toeplitz algebra $\mathscr{T}^{G_+}(G)$ discussed above (however, it does when $(G, G_+)$ is totally ordered; see [1, theorems 1.3 and 2.9]); in fact, we have the following proposition.

**Proposition 2.5.** *Let $(G, G_1)$ and $(G, G_2)$ be two partially ordered groups with $G_1 \subseteq G_2$. If $(G, G_1)$ is quasi-lattice ordered, then the natural morphism $\gamma^{G_2,G_1}$ exists and can be extended as a $C^*$-algebra morphism if and only if $G_1 = G_2$.*

PROOF. Suppose that there exists some partially ordered group $(G, G_2)$ with $G_1 \subseteq G_2$ and $G_1 \neq G_2$, such that $\gamma^{G_2,G_1}$ is well defined and can be extended as a $C^*$-algebra morphism. Then choose any $x \in G_2 \setminus G_1$ with $x = g_1 - g_2$ for some $g_1, g_2 \in G_1$. Let

$$T = T^{G_1}_{g_1} T^{G_1}_{-g_1} T^{G_1}_{g_2} T^{G_1}_{-g_2} = T^{G_1}_{\sigma(g_1,g_2)} T^{G_1}_{-\sigma(g_1,g_2)},$$

where $T^{G_i}_{g_j} = p^{G_i} u_{g_j} p^{G_i}$ for $i, j = 1, 2$. Then

$$T^{G_2}_{g_1} T^{G_2}_{-g_1} T^{G_2}_{g_2} T^{G_2}_{-g_2} = T^{G_2}_{\sigma(g_1,g_2)} T^{G_2}_{-\sigma(g_1,g_2)}.$$

In particular,

$$\left(T^{G_2}_{g_1} T^{G_2}_{-g_1} T^{G_2}_{g_2} T^{G_2}_{-g_2}\right) e_{g_1} = \left(T^{G_2}_{\sigma(g_1,g_2)} T^{G_2}_{-\sigma(g_1,g_2)}\right) e_{g_1}.$$

It follows that $g_1 - \sigma(g_1, g_2) \in G_2$.

On the other hand, $\sigma(g_1, g_2) - g_1 \in G_1 \subseteq G_2$. So $g_1 = \sigma(g_1, g_2)$, which is a contradiction since $x = g_1 - g_2 \notin G_1$. ■

*Remark.* Let $G$ be a discrete abelian group. For any subset $M$ of $G$, $M$ is said to be a *cone* if $(G, M)$ is a partially ordered group. By Zorn's Lemma, every cone is contained in some maximal one. When $G$ is torsion-free, it is proved in [1] that $M$ is maximal if and only if $(G, M)$ is an ordered group. Applying the above proposition, we know that when $G$ is torsion-free, the following two conditions are equivalent:

(1) $(G, G_+)$ is an ordered group;

(2) $(G, G_+)$ is quasi-lattice ordered and $\mathscr{U}(\mathscr{T}^{G_+}(G)) \simeq \mathscr{T}^{G_+}(G)$.

*Example 3.* For the usual partially ordered group $(Z^2, Z^2_+)$ and the *ℓexico-ordered group* $(Z^2, Z^2_{\ell ex})$, $\gamma^{Z^2_{\ell ex}, Z^2_+}$ cannot be extended as a $C^*$-algebra morphism.

*Example 4.* Let $G = Z^2$, and let $\alpha$ and $\beta$ be any two real numbers with $\alpha < \beta$. Let $G_1 = \{(m, n) \in Z^2 \mid -\alpha m + n \geq 0\}, G_2 = \{(m, n) \in Z^2 \mid -\beta m + n \leq 0\}$, and $G_0 = G_1 \cap G_2$. By [6, lemma 1.1 and proposition 1.2], we know that $(G, G_0)$ is a qpo group, $\gamma^{G_1,G_0}$ and $\gamma^{G_2,G_0}$ both are well defined and can be extended as $C^*$-algebra morphisms. So if either $\alpha$ or $\beta$ is irrational (i.e. $(G,G_0)$ is partially ordered), then $(G,G_0)$ is not quasi-lattice ordered.

**Proposition 2.6.** *Let $(G,G_1)$ be a quasi-lattice ordered group, and $(G,G_2)$ a qpo group with $G_1 \subseteq G_2$. Then $\gamma^{G_2,G_1}$ is well defined and can be extended as a $C^*$-algebra morphism if and only if $G_2$ is finitely decomposed by $G_1$.*

Proof. It suffices to prove '$\Longrightarrow$'. Suppose that $\gamma^{G_2,G_1}$ is well defined. Then

$$T^{G_2}_{x_1} T^{G_2}_{-x_1} T^{G_2}_{x_2} T^{G_2}_{-x_2} = T^{G_2}_{\sigma(x_1,x_2)} T^{G_2}_{-\sigma(x_1,x_2)}$$

for any $x_1, x_2 \in G_1$. So for any $x_1, x_2 \in G_1$ and $x_3 \in G_2$,

$$x_3 - \sigma(x_1,x_2) \in G_2 \text{ whenever } x_3 - x_1 \in G_2 \text{ and } x_3 - x_2 \in G_2. \qquad (*)$$

For any $g_1, g_2, \ldots, g_n \in G_2 \setminus G_1$ and $f_1, f_2, \ldots, f_m \in G_2^C$, let $g_i = x_i - y_i$ with $x_i, y_i \in G_1$ for $i = 1, 2, \ldots, n$. By $(*)$, we know that $x_i - \sigma(x_i, y_i) \in G_2$. Let

$$g = \sum_{i=1}^{n} \big(\sigma(x_i,y_i) - x_i\big) \in G_2^0 \cap G_1.$$

Then

$$g + g_i = \sum_{j \neq i} \big(\sigma(x_j,y_j) - x_j\big) + \big(\sigma(x_i,y_i) - y_i\big) \in G_1$$

for $i = 1, 2, \ldots, n$. Since $G_2^0 + G_2^C = G_2^C$, $G_2$ is finitely decomposed by $G_1$. ■

## 3. Index theory for Toeplitz operators

Let $G$ be a discrete abelian group and $\widehat{G}$ its dual group. $\widehat{G}$ is compact and is connected if and only if $G$ is torsion-free. When given the normal Haar measure, it is easy to show that $\{\varepsilon_x | x \in G\}$ is an orthonormal basis for $L^2(\widehat{G})$. Let $(G, G_+)$ be a qpo group. From another point of view, we may also regard $\mathscr{T}^{G_+}(G)$ as a subalgebra of $B(L^2(\widehat{G}))$ as follows.

Let $H^{G_+}(\widehat{G})$ be the closed subspace of $L^2(\widehat{G})$ generated by $\{\varepsilon_x | x \in G_+\}$; its projection is denoted by $p^{G_+}(\widehat{G})$. For any $\varphi \in C(\widehat{G})$, define $T^{G_+}_\varphi$ on $H^{G_+}(\widehat{G})$ by $T^{G_+}_\varphi(f) = p^{G_+}(\varphi f)$ for $f \in H^{G_+}(\widehat{G})$. The $C^*$-algebra generated by $\{ T^{G_+}_\varphi \mid \varphi \in C(\widehat{G}) \}$ is denoted by $\mathscr{T}^{G_+}_r(\widehat{G})$ and is also called the *Toeplitz algebra* with respect to $G_+$. Its *closed commutator ideal* is denoted by $C(\mathscr{T}^{G_+}_r(\widehat{G}))$. By the Stone–Weierstrass Theorem, $\mathscr{T}^{G_+}_r(\widehat{G})$ is also generated by $\{T^{G_+}_{\varepsilon_x} \mid x \in G\}$. Define $\rho : \mathscr{T}^{G_+}_r(\widehat{G}) \to B(L^2(\widehat{G}))$ by $\rho(T)(\xi \oplus \eta) = T(\xi)$ for any $T \in \mathscr{T}^{G_+}_r(\widehat{G})$, $\xi \in p^{G_+}(L^2(\widehat{G}))$, and $\eta \in (I - p^{G_+})(L^2(\widehat{G}))$. Clearly $\rho$ is a $C^*$-algebra morphism, therefore $\mathscr{T}^{G_+}_r(\widehat{G})$ may be regarded as a $C^*$-subalgebra of $B(L^2(\widehat{G}))$.

Define a unitary $u : \ell^2(G) \to L^2(\widehat{G})$ by $ue_g = \varepsilon_g$ for $g \in G$, then $u^* \circ T^{G_+}_{\varepsilon_g} \circ u = p^{G_+} u_g p^{G_+}$. So the two Toeplitz algebras are unitarily equivalent.

The following result belongs to E. Van Kampen, as mentioned in [4], which extends a result of Bohr, to get an analogue of the winding number.

**Lemma 3.1** (cf. [8]). *Let $E$ be a connected compact group and suppose that $\varphi$ is a continuous complex-valued function on $E$ which does not vanish anywhere. Then there exists a continuous homomorphism $\chi$ from $E$ to $T$ and a continuous complex-valued function $\psi$ on $E$, such that $\varphi = \chi e^{\psi}$.*

Now let $(G, G_+)$ be an ordered group. In this case $G$ is torsion-free, so $\widehat{G}$ is connected. For any $\varphi \in C(\widehat{G})$, if it does not vanish anywhere, then by Lemma 3.1 there exist some $x$ in $G$ and $\psi \in C(\widehat{G})$ such that $\varphi = \varepsilon_x e^{\psi}$. As observed in [2], such $x$ is unique; we denote it by $\mathrm{ind}_t(\varphi)$ and call it the *topological index* of $\varphi$. The following lemma belongs to G. Murphy. In our terms, it can be stated as follows.

**Lemma 3.2** (cf. [2]). *Let $(G, G_+)$ be an ordered group and $\phi \in C(\widehat{G})$. Then $T_\phi^{G_+}$ is invertible if and only if $\phi$ does not vanish anywhere and* $\mathrm{ind}_t(\phi) = 0$.

**Theorem 3.3.** *Let $(G, G_+)$ be a quasi-ordered group. If $G$ is torsion-free, then for any $\phi \in C(\widehat{G})$, $T_\phi^{G_+}$ is invertible if and only if $\phi$ does not vanish anywhere and* $\mathrm{ind}_t(\phi) \in G_+^0$.

PROOF. Suppose that $G$ is torsion-free and $(G, G_+)$ is a quasi-ordered group. By definition, $G = G_+ \cup (-G_+)$ and $0 \in G_+^0 = G_+ \cap (-G_+)$. Let $G_+^* = G_+ \setminus G_+^0$. Then it is easy to show that $G_+^* + G_+ = G_+^*$ and $G = G_+^* \cup G_+^0 \cup (-G_+^*)$. Let $G_1 = G_+^* \cup \{0\}$. Since for any $x \in G_+^0$ and $y \in G_+^*$ we have $x = (x + y) - y \in G_1 - G_1$, $G_1$ is in fact a cone. Let $E$ be a maximal cone containing $G_1$, then $(G, E)$ is an ordered group. Note that $G_+^* \subseteq G_1 \subseteq E$ and $E \cap (-E) = \{0\}$, and we know that $(-G_+^*) \cap E = \emptyset$; in other words, $E \subseteq G \setminus (-G_+^*) = G_+$. By Proposition 2.4, $\gamma^{G_+, E}$ is well defined. For any $\psi \in C(\widehat{G})$, $T_{e^\psi}^{G_+} = \gamma^{G_+, E}(T_{e^\psi}^E)$, which is invertible by Lemma 3.2.

Now let $\phi \in C(\widehat{G})$. If $T_\phi^{G_+}$ is invertible in $\mathscr{T}_r^{G_+}(\widehat{G})$, then by Lemma 2.3 we know that $\phi$ does not vanish anywhere. Let $\phi = \varepsilon_x e^\psi$ with $x = \mathrm{ind}_t(\phi)$ and $\psi \in C(\widehat{G})$. Then

$$T_\phi^{G_+} = \begin{cases} T_{\varepsilon_x}^{G_+} T_{e^\psi}^{G_+}, & \text{if } x \in -G_+, \\ T_{e^\psi}^{G_+} T_{\varepsilon_x}^{G_+}, & \text{if } x \in G_+. \end{cases}$$

Since $T_\phi^{G_+}$ and $T_{e^\psi}^{G_+}$ both are invertible, in any case $T_{\varepsilon_x}^{G_+}$ is invertible. But this can happen only if $x \in G_+^0$. ■

Now suppose that $(G, G_+)$ is a qpo group. Let $\mathfrak{A} = \{ E \mid E \subseteq -G_+, G_E^0 \neq 0 \}$. Clearly, $-G_+ \in \mathfrak{A}$. Let

$$G_F = \bigcap_{E \in \mathfrak{A}} G_E \quad \text{and} \quad G_F^0 = G_F \bigcap (-G_F) = \bigcap_{E \in \mathfrak{A}} G_E^0.$$

For any $E$ in $\mathfrak{A}$, $G_E$ is also generated by $G_F$ and $E$. Since $E \subseteq -G_+ \subseteq -G_F$, by

Proposition 2.2 we know that $\gamma^{G_E,G_F}$ and $\gamma^{G_E,G_+}$ are both well defined. Let

$$I_0 = \bigcap_{E\in\mathfrak{A}} \operatorname{Ker}\gamma^{G_E,G_+} \quad \text{and} \quad I_1 = \bigcap_{E\in\mathfrak{A}} \operatorname{Ker}\gamma^{G_E,G_F}.$$

**Corollary 3.4.** *Suppose that $G$ is torsion-free and $(G, G_+)$ is a qpo group. Then for any $\phi \in C(\widehat{G})$, the following conditions are equivalent:*

(1) *$T_\phi^{G_+}$ is invertible in $\mathscr{T}_r^{G_+}(\widehat{G})$ modulo the ideal $I_0$;*

(2) *$\phi$ does not vanish anywhere and* $\mathrm{ind}_t(\phi) \in G_F^0$;.

(3) *$T_\phi^{G_F}$ is invertible in $\mathscr{T}_r^{G_F}(\widehat{G})$ modulo the ideal $I_1$.*

Proof. Let $A\cup\{A_\lambda \mid \lambda \in \wedge\}$ be a family of unital $C^*$-algebras and $\rho_\lambda : A \to A_\lambda, \lambda \in \wedge$ a family of unital $C^*$-algebra morphisms. Set $I = \bigcap_{\lambda\in\wedge} \operatorname{Ker}\rho_\lambda$. Let $\rho$ be the usual $C^*$-algebra morphism from $A$ to $\Pi_{\lambda\in\wedge}A_\lambda$ and $\overline{\rho}$ the induced isometric $C^*$-algebra morphism from $A/I$ to $\Pi_{\lambda\in\wedge}A_\lambda$.

For any $T \in A$, if $T + A/I$ is invertible in $A/I$, then obviously $\rho_\lambda(T)$ is invertible in $A_\lambda$ for all $\lambda \in \wedge$. On the other hand, if $\rho_\lambda(T)$ is invertible in $A_\lambda$ for all $\lambda \in \wedge$, then $\rho(T)$ is invertible in $\Pi_{\lambda\in\wedge}A_\lambda$. So $\rho(T)$ is invertible in $\rho(A)$, which implies that $T + A/I$ is invertible in $A/I$ since $\overline{\rho}$ is an isomorphism. Therefore, for any $T \in A$, $T$ is invertible modulo $I$ if and only if $\rho_\lambda(T)$ is invertible in $A_\lambda$ for all $\lambda \in \wedge$. By Theorem 3.3, we know that (1) $\Longleftrightarrow$ (2) and (2) $\Longleftrightarrow$ (3). ■

*Remark.* When $(G, G_+)$ is an ordered group, the conditions stated in the above corollary can be simplified. In fact, it is easy to show that in this case

$$G_F = \bigcap_{g\in G_+\setminus\{0\}} (G_+ - Z_+g) = G_+ - F(G_+) = G_+ \cup (-F(G_+)),$$

where

$$F(G_+) = \{\, x \mid x \in G_+, \forall y \in G_+ \setminus \{0\}, \exists n \in N, \text{ such that } ny - x \in G_+ \,\}.$$

Let $F(G) = F(G_+) \cup (-F(G_+))$; it is a subgroup of $G$ and its elements are usually called the *finite elements*. It is easy to show that $G_F^0 = F(G)$. Let $K(F(G))$ be the closed two-sided ideal of $\mathscr{T}_r^{G_+}(\widehat{G})$ generated by $\{\, 1 - T_{\varepsilon_x}^{G_+} T_{\varepsilon_{-x}}^{G_+} \mid x \in F(G_+) \,\}$.

**Corollary 3.5.** *Let $(G, G_+)$ be an ordered group. Then for any $\phi \in C(\widehat{G})$, the following conditions are equivalent:*

(1) *$T_\phi^{G_+}$ is invertible in $\mathscr{T}_r^{G_+}(\widehat{G})$ modulo the ideal $K(F(G))$;*

(2) *$T_\phi^{G_F}$ is invertible in $\mathscr{T}_r^{G_F}(\widehat{G})$;*

(3) *$\phi$ does not vanish anywhere and* $\mathrm{ind}_t(\phi) \in F(G)$.

Proof. (1) $\Longrightarrow$ (2) follows from the fact that $K(F(G)) \subseteq \operatorname{Ker}\gamma^{G_F,G_+}$. Since conditions (3) and (2) stated in Corollary 3.4 are equivalent, (2) $\Longrightarrow$ (3) holds. (3) $\Longrightarrow$ (1) follows from Lemma 3.1 and Lemma 3.2. ■

*Remark.* When $G$ is countable and $F(G) \neq 0$, the equivalence of (1) and (3) was proved in [4] in a very different way.

We now come to discuss Fredholm operators in Toeplitz algebras. Let $(G, G_+)$ be an ordered group. As shown in [4], $\mathscr{T}_r^{G_+}(\widehat{G})$ containing Fredholm operators of non-zero index is equivalent to saying that $G$ admits a least positive element, and in this case $F(G) \cong Z$ and $K(F(G)) = K(H^{G_+}(\widehat{G}))$, the ideal of compact operators on $H^{G_+}(\widehat{G})$. When $G$ admits a least positive element, by Corollary 3.5 we know that, for any $\phi \in C(\widehat{G})$, $T_\phi^{G_+}$ is Fredholm if and only if $\phi$ does not vanish anywhere and $\mathrm{ind}_t(\phi) \in F(G)$.

Now let $G$ be a discrete abelian group, and $(G, G_1), \ldots, (G, G_n)$ be ordered groups. Let $(G^n, G_1 * \ldots * G_n)$ be the *ℓexico-ordered group*. Then the subgroup of finite elements in $G^n$,

$$F(G^n) = \{0\} \times \ldots \times \{0\} \times \{F(G_n) \cup (-F(G_n))\}.$$

Thus we have the following corollary.

**Corollary 3.6.**

(1) *If $(G, G_n)$ does not contain a least positive element, then all Fredholm operators in $\mathscr{T}_r^{G_1*\ldots*G_n}(\widehat{G}^n)$ are of zero-index.*

(2) *If $(G, G_n)$ contains a least positive element, then for any $\phi_1, \ldots, \phi_n \in C(\widehat{G})$, $T_{\phi_1\otimes\ldots\otimes\phi_n}^{G_1*\ldots*G_n}$ is Fredholm if and only if $T_{\phi_1}^{G_1}, \ldots, T_{\phi_{n-1}}^{G_{n-1}}$ all are invertible and $T_{\phi_n}^{G_n}$ is Fredholm.*

PROOF. (1) See [4, lemma 2.2].

(2) By Lemma 3.2 and Corollary 3.5, we know that the following conditions are equivalent:

(a) $T_{\phi_1\otimes\ldots\otimes\phi_n}^{G_1*\ldots*G_n}$ is Fredholm;

(b) $\phi = \phi_1 \otimes \ldots \otimes \phi_n$ does not vanish anywhere and $\mathrm{ind}_t(\phi) \in F(G^n)$;

(c) $\phi_1, \ldots, \phi_n$ does not vanish anywhere, $\mathrm{ind}_t(\phi_1) = \ldots = \mathrm{ind}_t(\phi_{n-1}) = 0$ and $\mathrm{ind}_t(\phi_n) \in F(G_n) \cup (-F(G_n))$;

(d) $T_{\phi_1}^{G_1}, \ldots, T_{\phi_{n-1}}^{G_{n-1}}$ all are invertible and $T_{\phi_n}^{G_n}$ is Fredholm. ∎

*Remark.* The special case of the preceding corollary, where $(G, G_i) = (Z, Z_+)$ for $i = 1, 2, \ldots, n$, was obtained in [3].

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