MULTILINEAR INTERPOLATION THEOREMS

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[Received 21 December 1998. Read 7 October 1999. Published 29 December 2000.]

Abstract

A necessary and sufficient condition is given for the interpolation theorem to hold for positive multilinear operators from the product of Calderón–Lozanovskiĭ spaces generated by any couples of Banach function spaces. In a particular case, if the parameters generating those spaces are the same concave functions, the condition is equivalent to supermultiplicativity of the generating function. This result is used to obtain interpolation theorems for the multilinear operators from the product of certain interpolation orbit spaces, in particular Ovchinnikov's lower method spaces, into generalised Marcinkiewcz spaces.

1. Introduction

It is well known that the results on interpolation of bilinear operators have found interesting applications in general theory of Banach spaces. For example, by using a theorem on interpolation of bilinear operators by complex method and certain Maurey's result on factorisation of operators on ℓ_p, $1 < p < 2$, Pisier [13] proved that the space of 2-summing operators on complex ℓ_p-spaces with $1 < p < 2$ is super-reflexive.

In the present paper we study interpolation of multilinear operators between Calderón–Lozanovskiĭ spaces. Unfortunately, as far as we know, the problem is not solved in general. Certain results have been proved by Astashkin [2] for positive operators under some assumptions on the range couples and the dilation indices of the concave function generating the spaces. In section 2 we prove in a quite different way a general result on interpolation of positive operators between Calderón–Lozanovskiĭ spaces. We remark that an assumption in [2] on the indices of concave function is superfluous.

In section 3 we apply the above-mentioned result to prove theorems on interpolation of multilinear operators from the product of certain interpolation orbit spaces, in particular Ovchinnikov's lower method spaces, into generalised Marcinkiewicz spaces. These results give a generalisation of the main results presented in [3].

In the paper we shall use the standard notation and notions from interpolation theory, as presented e.g. in [4] or [12]. We recall that an *interpolation method* is a functor $\mathscr{F}$ from the category of all compatible couples of Banach spaces into the category of all Banach spaces such that, for any couple $\overline{X}$, $\mathscr{F}(\overline{X})$ is a Banach space intermediate with respect to $\overline{X}$ (i.e. $X_0 \cap X_1 \subset \mathscr{F}(\overline{X}) \subset X_0 + X_1$), and

* Research supported by KBN-Grant 2 P03A 050 09.

$T : \mathscr{F}(\overline{X}) \to \mathscr{F}(\overline{Y})$ for all Banach couples $\overline{X}$, $\overline{Y}$ and every $T : \overline{X} \to \overline{Y}$. Here, as usual, we use the notation $T : \overline{X} \to \overline{Y}$, which means that $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that the restriction of T to the space X_j is a bounded operator from X_j into Y_j, $j = 0, 1$. The spaces of all operators $T : \overline{X} \to \overline{Y}$ is a Banach space equipped with the norm $\|T\|_{\overline{X} \to \overline{Y}} := \max\{\|T\|_{X_0 \to Y_0}, \|T\|_{X_1 \to Y_1}\}$.

We recall that the space $\mathscr{L}(X_1, \ldots, X_n; Z)$ of all bounded *n-multilinear* (shortly multilinear, or bilinear if $n = 2$) operators from the product $\prod_{i=1}^n X_i$ of Banach spaces into a Banach space Y is a Banach space equipped with the norm:

$$\|T\|_{\mathscr{L}(X_1, \ldots, X_n; Y)} := \sup\{\|T(x_1, \ldots, x_n)\|_Y \,;\, \|x_i\|_X \le 1,\, i = 1, \ldots, n\}.$$

Let $\overline{X}_i = (X_{i0}, X_{i1})$, $i = 1, \ldots, n$ and $\overline{Y} = (Y_0, Y_1)$ be Banach couples. Throughout the paper we write $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$ if $T \in \mathscr{L}(X_{10} + X_{11}, \ldots, X_{n0} + X_{n1}; Y_0 + Y_1)$ and $T \in \mathscr{L}(X_{1j}, \ldots, X_{nj}; Y_j)$ for $j = 0, 1$. The space of all operators $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$ is denoted by $\mathscr{L}(\overline{X}_1, \ldots, \overline{X}_n; \overline{Y})$ and is equipped with the norm

$$\|T\| := \max\{\|T\|_{\mathscr{L}(X_{10}, \ldots, X_{n0}; Y_0)}, \|T\|_{\mathscr{L}(X_{11}, \ldots, X_{n1}; Y_1)}\}.$$

2. Calderón–Lozanovskiĭ spaces

In this section we will study interpolation of positive multilinear operators in Calderón–Lozanovskiĭ spaces. We start with some basic definitions and notation.

Let $(\Omega, \mu) := (\Omega, \Sigma, \mu)$ be a measure space with μ complete σ-finite. As usual, $L^0(\mu)$ denotes the space of all equivalence classes of measurable functions on Ω with the topology of convergence in measure on μ-finite sets.

Let $E \subset L^0(\mu)$ be a Banach lattice (on (Ω, μ)). If the unit ball $B_E := \{x \in E; \|x\|_E \le 1\}$ is closed in $L^0(\mu)$, so that E has the *Fatou property*, then E is called a *maximal* Banach lattice. It is well known that E is maximal if and only if $E = E''$ isometrically, where E' denotes the *Köthe dual* of E, i.e.

$$E' := \left\{ y \in L^0(\mu); \; \|y\|_{E'} = \sup_{x \in B_E} \int_\Omega |xy| d\mu < \infty \right\}.$$

If E is a Banach lattice and $w \in L^0(\mu)$ with $w > 0$ a.e., we define the *weighted space* $E(w)$ by the condition $\|x\|_{E(w)} := \|xw\|_E < \infty$. In the sequel the couples $(\ell_\infty, \ell_\infty(2^{-n}))$ and $(c_0, c_0(2^{-n}))$ of Banach sequence lattices defined on the set of integers $\mathbb{Z}$ will be denoted by $\overline{\ell}_\infty$ and $\overline{c}_0$ respectively.

Throughout the paper the set of all positive functions $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing in each variable which are homogeneous of degree one, i.e. $\varphi(\lambda s, \lambda t) = \lambda \varphi(s, t)$ for any $\lambda \ge 0$, is denoted by Φ. If $\varphi \in \Phi$ is *concave*, we write $\varphi \in \mathscr{U}$. Clearly, if $\varphi \in \Phi$, then the function φ_* defined by $\varphi_*(s, t) := 1/\varphi(s^{-1}, t^{-1})$ for any $s, t > 0$ is also in Φ. The subset of all $\varphi \in \Phi$ for which $\varphi(s, 1) \to 0$ and $\varphi(1, t) \to 0$ as $s \to 0$ and $t \to 0$ is denoted by Φ_0.

Recall that if $\overline{X} = (X_0, X_1)$ is a couple of Banach lattices on (Ω, μ) and $\varphi \in \mathscr{U}$, then the *Calderón–Lozanovskiĭ space* $\varphi(\overline{X}) = \varphi(X_0, X_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \le \lambda \varphi(|x_0|, |x_1|)$ μ-a.e. for some $x_j \in X_j$ with $\|x_j\|_{X_j} \le 1$, $j = 0, 1$. The space

$\varphi(\overline{X})$ is a Banach lattice on (Ω, μ) equipped with the norm

$$\|x\| := \inf\{\lambda > 0;\, |x| \leq \lambda\varphi(|x_0|, |x_1|),\, x_0 \in B_{X_0},\ x_1 \in B_{X_1}\}$$

(see [8]; [9]; [14]). In the case of the power function $\varphi_\theta(s,t) = s^{1-\theta}t^\theta$ with $0 < \theta < 1$, we obtain the space $X_0^{1-\theta}X_1^\theta$ introduced by Calderón [5]. We note that simple calculations show that for any Banach lattice E and any weights w_0 and w_1 we have

$$\varphi(E(w_0), E(w_1)) = E(\varphi_*(w_0, w_1)),$$

with equivalent norms.

Let $E_1, \ldots, E_n$ and F be Riesz spaces (i.e. vector lattices). A map $T : E_1^+ \times \ldots \times E_n^+ \to F^+$ is said to be *positively multilinear* if for each x_j in positive cone $E_j^+ := \{x \in E_j\,; x_j \geq 0\}$ of E_j, $j = 1, \ldots, n$, $j \neq i$, the map $X_i^+ \ni x \longmapsto T(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$ is positively homogeneous and additive.

A map $T : E_1 \times \ldots \times E_n \to F$ is said to be a *positive multilinear* map if the restriction map $T : E_1^+ \times \ldots \times E_n^+ \to F^+$ is positively multilinear.

It is easy to see that if $T : E_1 \times \ldots \times E_n \to F$ is a positive multilinear map then

$$|T(x_1, \ldots, x_n)| \leq T(|x_1|, \ldots, |x_n|)$$

for every $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$.

If $E_1, \ldots, E_n, F$ are normed lattices, the set of all positively multilinear maps is denoted by $L^+(E_1, \ldots, E_n; F)$. On $L^+(E_1, \ldots, E_n; F)$ we introduce the functional

$$\|T\|_{L^+(E_1,\ldots,E_n;F)} := \sup\{\|T(x_1, \ldots, x_n)\|_F\,; 0 \leq x_j \in B_{X_j},\ j = 1, \ldots, n\}.$$

Since every positive linear map from a Banach lattice into a normed lattice is bounded, the Uniform Boundedness Principle immediately yields that every positive multilinear map from the product $E_1 \times \ldots \times E_n$ of Banach lattices into a normed lattice F is a bounded multilinear operator. Clearly, in this case

$$\|T\|_{\mathscr{L}(E_1,\ldots,E_n;F)} = \|T\|_{L^+(E_1,\ldots,E_n;F)}.$$

We are now in a position to prove the main result of this section.

Theorem 2.1. *Let $\overline{X} = (X_0, X_1), \overline{Y} = (Y_0, Y_1), \overline{Z} = (Z_0, Z_1)$ be couples of Banach lattices on measure spaces (Ω_0, μ_0), (Ω_1, μ_1), (Ω_2, μ_2) respectively, and let $S : (X_0 + X_1) \times (Y_0 + Y_1) \to Z_0 + Z_1$ be an operator dominated by a positively bilinear map T such that $T : X_j^+ \times Y_j^+ \to Z_j^+$, $j = 0, 1$. If $\varphi_0, \varphi_1, \varphi_2 \in \mathscr{U}$ are such that $\varphi_0(1,s)\varphi_1(1,t) \leq C\varphi_2(1, st)$ for some $C > 0$ and all $s, t > 0$, then $S : \varphi_0(\overline{X}) \times \varphi_1(\overline{Y}) \to \varphi_2(\overline{Z})$ and*

$$\|S(x,y)\|_{\varphi_2(\overline{Z})} \leq C \max\{\|T\|_{L^+(X_0,Y_0;Z_0)},\, \|T\|_{L^+(X_1,Y_1;Z_1)}\}\|x\|_{\varphi_0(\overline{X})}\|y\|_{\varphi_1(\overline{Y})}$$

for all $(x, y) \in \varphi_0(\overline{X}) \times \varphi_1(\overline{Y})$.

PROOF. Let $x \in \varphi_0(\overline{X})$ and $y \in \varphi_1(\overline{Y})$ with $\|x\|_{\varphi_0(\overline{X})} < 1$ and $\|y\|_{\varphi_1(\overline{Y})} < 1$. Then there exist $0 \leq x_j \in X_j$, $0 \leq y_j \in Y_j$ with $\|x_j\|_{X_j} < 1$ and $\|y_j\|_{Y_j} < 1$ for $j = 0, 1$ such that $|x| \leq \varphi_0(x_0, x_1)$ and $|y| \leq \varphi_1(y_0, y_1)$ a.e. Since S is dominated by a positively bilinear map T, we obtain

$$|S(x,y)| \leq T(|x|,|y|) \leq T(\varphi_0(x_0,x_1),\varphi_1(y_0,y_1)) \text{ a.e.}$$

In order to finish the proof, it suffices to show that for all $0 \leq x_j \in X_j$, $0 \leq y_j \in Y_j$, $j = 0, 1$, the following inequality holds:

$$T(\varphi_0(x_0,x_1),\varphi_1(y_0,y_1)) \leq C\varphi_2(T(x_0,y_0),T(x_1,y_1)) \text{ a.e.}$$

Let $u = x_0 + x_1$ and $v = y_0 + y_1$. Suppose first that x_0, x_1 and y_0, y_1 are simple elements with respect to u and v respectively, i.e.

$$x_0 = \sum_{k=1}^{n} a_k u_k \text{ and } x_1 = \sum_{k=1}^{n} b_k u_k$$
$$y_0 = \sum_{k=1}^{n} c_k v_k \text{ and } y_1 = \sum_{k=1}^{n} d_k v_k,$$

where $a_k, b_k, c_k, d_k \geq 0$, $u_k \wedge u_m = 0$ and $v_k \wedge v_m = 0$ for $k \neq m$, $k, m = 1, \ldots, n$ and $\Sigma_{k=1}^{n} u_k = u$, $\Sigma_{k=1}^{n} v_k = v$. Hence we have

$$\varphi_0(x_0,x_1) = \sum_{k=1}^{n} \varphi_0(a_k,b_k)u_k,$$
$$\varphi_1(y_0,y_1) = \sum_{k=1}^{n} \varphi_1(c_k,d_k)v_k.$$

This implies that

$$\begin{aligned} T(\varphi_0(x_0,x_1),\varphi_1(y_0,y_1)) &= T\left(\sum_{k=1}^{n} \varphi_0(a_k,b_k)u_k, \sum_{k=1}^{n} \varphi_1(c_k,d_k)v_k\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_0(a_i,b_i)\varphi_1(c_j,d_j)T(u_i,v_j). \end{aligned}$$

By the assumption on $\varphi_0, \varphi_1, \varphi_2$, it follows that $\varphi_0(s_1,t_1)\varphi_1(s_2,t_2) \leq C\varphi_2(s_1s_2,t_1t_2)$ for any $s_j, t_j > 0$, $j = 0, 1$. In consequence, by subadditivity of concave function φ_2, we obtain

$$\begin{aligned} T(\varphi_0(x_0,x_1),\varphi_1(y_0,y_1)) &\leq C\sum_{i=1}^{n}\sum_{j=1}^{n} \varphi_2(a_ic_j,b_id_j)T(u_i,v_j) \\ &\leq C\sum_{i=1}^{n} \varphi_2\left(\sum_{j=1}^{n} T(a_iu_i,c_jv_j), \sum_{j=1}^{n} T(b_iu_i,d_jv_j)\right) \end{aligned}$$

$$\leq C\varphi_2\left(\sum_{i=1}^{n}\sum_{j=1}^{n} T(a_iu_i, c_jv_j), \sum_{i=1}^{n}\sum_{j=1}^{n} T(b_iu_i, d_jv_j)\right)$$
$$= C\varphi_2(T(x_0, y_0), T(x_1, y_1)).$$

If $0 \leq x_j \in X_j$ and $0 \leq y_j \in Y_j$ $(j = 0, 1)$ are arbitrary, then it follows by $0 \leq x_j \leq u$ and $0 \leq y_j \leq v$ that there are sequences $\{x_n^j\}$ and $\{y_n^j\}$ of simple elements with respect to u and v respectively such that $0 \leq x_n^j \leq x_j$, $0 \leq y_n^j \leq y_j$ and $x_j - x_n^j \leq u/n$, $y_j - y_n^j \leq v/n$ for all $n \in \mathbb{N}$, $j = 0, 1$. Combining the above with concavity of φ_0 and φ_1 we get

$$0 \leq \varphi_0(x_0, x_1) - \varphi_0(x_n^0, x_n^1) \leq \varphi_0(x_0 - x_n^0, x_1) - \varphi_0(0, x_1)$$
$$+ \varphi_0(x_n^0, x_1 - x_n^1) - \varphi(x_n^0, 0) + \xi_n^0 u$$

and

$$0 \leq \varphi_1(y_0, y_1) - \varphi_1(y_n^0, y_n^1) \leq \varphi_1(y_0 - y_n^0, y_1) - \varphi_1(0, y_1)$$
$$+ \varphi_1(y_n^0, y_1 - y_n^1) - \varphi(y_n^0, 0) + \xi_n^1 v,$$

where $\xi_n^j = (\varphi_j(1/n, 0) - \varphi_0(0, 1) + \varphi_j(1, 1/n) - \varphi_j(1, 0))$ for $n \in \mathbb{N}$ and $j = 0, 1$. The above estimates imply that

$$\varphi_0(x_0, x_1) \leq \varphi_0(x_n^0, x_n^1) + \xi_n^0 u,$$
$$\varphi_1(y_0, y_1) \leq \varphi_1(y_n^0, y_n^1) + \xi_n^1 v.$$

By using the previous estimate for simple elements, and the positivity and bilinearity of T, we obtain

$$\begin{aligned} T(\varphi_0(x_0, x_1), \varphi_1(y_0, y_1)) &\leq T(\varphi_0(x_n^0, x_n^1), \varphi_1(y_n^0, y_n^1)) + T(\varphi_0(x_n^0, x_n^1), \xi_n^1 v) \\ &\quad + T(\xi_n^0 u_n, \varphi_1(y_n^0, y_n^1)) + T(\xi_n^0 u, \xi^1 v) \\ &\leq C\varphi_2(T(x_0, y_0), T(x_1, y_1)) + \xi_n^1\, T(\varphi_0(x_0, x_1), v) \\ &\quad + \xi_n^0\, T(u, \varphi_1(y_0, y_1)) + \xi_n^0\, \xi_n^1\, T(u, v). \end{aligned}$$

Thus the required inequality follows by $\xi_n^j \to 0$ as $n \to \infty$ for $j = 0, 1$. ∎

Similarly we show the following result.

Theorem 2.2. *Let* $\overline{X}_i = (X_{i0}, X_{i1})$, $i = 1, \ldots, n$, *and* $\overline{Y} = (Y_0, Y_1)$ *be couples of Banach lattices and let* $S : \prod_{i=1}^{n}(X_{i0} + X_{i1}) \to Y_0 + Y_1$ *be an operator dominated by a positively multilinear map* T *such that* $T : \prod_{i=1}^{n} X_{ij}^{+} \to Y_j^{+}$ *is bounded for* $j = 0, 1$. *If* $\varphi_1, \ldots, \varphi_n, \varphi \in \mathcal{U}$ *satisfy the estimate* $\varphi_1(1, s_1) \cdot \ldots \cdot \varphi_n(1, s_n) \leq C\varphi(1, s_1 \cdot \ldots \cdot s_n)$ *for some* $C > 0$ *and all* $s_1, \ldots, s_n > 0$, *then* S *is bounded from the product* $\prod_{i=1}^{n} \varphi_i(\overline{X}_i)$ *into* $\varphi(\overline{Y})$.

Corollary 2.3. *Let* $\overline{X}_i = (X_{i0}, X_{i1})$ *for* $i = 1, \ldots, n$ *and* $\overline{Y} = (Y_0, Y_1)$ *be couples of Banach lattices and let* $T : \prod_{i=1}^{n}(X_{i0} + X_{i1}) \to Y_0 + Y_1$ *be a positive multilinear* *operator such that* $T : \prod_{i=1}^{n} X_{ij} \to Y_j$ *for* $j = 0, 1$. *If* $\varphi \in \mathcal{U}$ *satisfies the estimate* $\varphi(1, s)\varphi(1, t) \leq C\varphi(1, st)$ *for some* $C > 0$ *and all* $s, t > 0$, *then* T *is bounded from the product* $\prod_{i=1}^{n} \varphi(X_{i0}, X_{i1})$ *into* $\varphi(Y_0, Y_1)$.

We note that the above result is an extension of the result presented in [2], where under the above conditions it is shown, in a quite different way, that T is bounded from $\prod_{i=1}^{n} \varphi(X_{i0}, X_{i1})$ into $\varphi(Y_0'', Y_1'') \cap (Y_0 + Y_1)$ provided that the *dilation indices* of $\rho = \varphi(1, \cdot)$ defined by $\alpha_\rho := \lim_{t\to 0}(\ln s_\rho(t)/\ln t)$, $\beta_\rho := \lim_{t\to\infty}(\ln s_\rho(t)/\ln t)$ are non-trivial, i.e. $0 < \alpha_\rho \leq \beta_\rho < 1$. Here $s_\rho(t) := \sup\{\rho(tu)/\rho(u)\,;\, u > 0\}$ for $t > 0$.

Theorem 2.4. *Let* $\varphi_{ij}, \varphi_i, \psi_j \in \mathcal{U}$ *and let* $\overline{X}_i$ *be couples of Banach lattices,* $i = 1, \ldots, n$, $j = 0, 1$. *Assume that* $T : \prod_{i=1}^{n}(\varphi_{i0}(\overline{X}_i), \varphi_{i1}(\overline{X}_i)) \to (\psi_0(\overline{Y}), \psi_1(\overline{Y}))$ *is a positive multilinear operator. If there exists* $C > 0$ *such that* $\varphi_1(1, s_1)\cdot\ldots\cdot\varphi_n(1, s_n) \leq C\psi(1, s_1\cdot\ldots\cdot s_n)$ *for all* $s_1, \ldots, s_n > 0$, *then* T *is bounded from the product* $\prod_{i=1}^{n} \varphi_i(\varphi_{i0}, \varphi_{i1})(\overline{X}_i)$ *into* $\psi(\psi_0, \psi_1)(\overline{Y})$.

Proof. Since for any $\phi_0, \phi_1, \phi \in \mathcal{U}$ and any couple $\overline{E} = (E_0, E_1)$ of Banach lattices the following reiteration formula holds (see [10]),

$$\phi(\phi_0(\overline{E}), \phi_1(\overline{E})) = \phi(\phi_0, \phi_1)(\overline{E}),$$

the result follows by Theorem 2.1. ■

Let E be a Banach lattice on (Ω, μ). It is easy to see that if ψ is an *Orlicz function* (i.e. $\psi : [0, \infty) \to [0, \infty)$ is a convex function which takes value zero only at zero) and $\varphi \in \mathcal{U}$ is defined by $\varphi(0, 0) = 0$, $\varphi(s, t) = t\psi^{-1}(s/t)$ for $s, t > 0$, then the Calderón–Lozanovskiĭ space $\varphi(E, L_\infty)$ coincides isometrically with the Banach lattice

$$E_\psi := \{x \in L^0(\mu);\ \psi(|x|/\lambda) \in E \text{ for some } \lambda > 0\}$$

equipped with the norm

$$\|x\|_{E_\psi} := \inf\{\lambda > 0;\ \|\psi(|x|/\lambda)\|_E \leq 1\}.$$

In particular, $\varphi(L_1(\mu), L_\infty(\mu))$ coincides isometrically with the Orlicz space $L_\psi(\mu)$.

Applying Theorem 2.4, we can obtain a multilinear theorem for positive multilinear operators between spaces of the type E_φ. We only formulate a result for the Orlicz spaces.

Theorem 2.5. *Let* ϕ_i, ψ_i, $i = 1, \ldots, n$, θ_0, θ_1 *be Orlicz functions and let* $\varphi_1, \ldots, \varphi_n, \varphi \in \mathcal{U}$ *be such that there exists* $C > 0$ *such that* $\varphi_1(1, s_1)\cdot\ldots\cdot\varphi_n(1, s_n) \leq C\varphi(1, s_1\cdot\ldots\cdot s_n)$ *for all* $s_1, \ldots, s_n > 0$. *If* $T : \prod_{i=1}^{n}(L_{\phi_i}(\mu_i), L_{\psi_i}(\mu_i)) \to (L_{\theta_0}(\mu), L_{\theta_1}(\mu))$ *is a positive multilinear operator, then* T *is bounded from the product* $\prod_{i=1}^{n} L_{\Phi_i}(\mu_i)$ *into* $L_\theta(\mu)$, *where* $\theta^{-1}(t) = \varphi(\theta_0^{-1}(t), \theta_1^{-1}(t))$ *and* $\Phi_i^{-1}(t) = \varphi_i(\phi_0^{-1}(t), \phi_1^{-1}(t))$ *for* $t > 0$, $i = 1, \ldots, n$.

We note that it follows from the proofs that in all results of this section as well as in the next one the norm of interpolated multilinear operator T is estimated by

$C\|T\|_{\mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})}$, where C is a positive constant depending on parameters generating intermediate spaces with respect to Banach couples $\overline{X}_1,\dots,\overline{X}_n$ and $\overline{Y}$. In fact, in the case of the Calderón–Lozanovskiĭ spaces, the above-mentioned estimate for positive operators is equivalent to the generalised C-supermultiplicativity of the involved functions as it follows from the following result.

Theorem 2.6. *Let $\varphi_1,\dots,\varphi_n,\varphi \in \mathscr{U}$ and let $\overline{X}_i$, $i = 1,\dots,n$ and $\overline{Y}$ be couples of Banach lattices. The following conditions are equivalent:*

(i) *there exists $C > 0$ such that the estimate $\varphi_1(1,s_1)\cdot\ldots\cdot\varphi_n(1,s_n) \le C\varphi(1,s_1\cdot\ldots\cdot s_n)$ holds for all $s_1,\dots,s_n > 0$;*

(ii) *there exists $C > 0$ such that for any $\overline{X}_i$, $\overline{Y}$ and any positive multilinear operator $T \in \mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})$ it holds that $\|T\|_{\mathscr{L}(\varphi_1(\overline{X}_1),\dots,\varphi_n(\overline{X}_n);\overline{Y})} \le C\|T\|_{\mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})}$;*

(iii) *for any $\overline{X}_i$, $\overline{Y}$ there exists $C > 0$ such that for any positive multilinear operator $T \in \mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})$ it holds that $\|T\|_{\mathscr{L}(\varphi_1(\overline{X}_1),\dots,\varphi_n(\overline{X}_n);\overline{Y})} \le C\|T\|_{\mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})}$.*

PROOF. (i) $\Rightarrow$ (ii) follows by Theorem 2.2 and the proof of Theorem 2.1. (ii) $\Rightarrow$ (iii) is obvious. Thus we need only show the implication (iii) $\Rightarrow$ (i). Let $\overline{X}_i = (L_1(\mathbb{R}_+,m), L_\infty(\mathbb{R}_+,m))$ for $i = 1,\dots,n$ and let $\overline{Y} = (L_1(\mathbb{R}^n_+,m_n), L_\infty(\mathbb{R}^n_+,m_n))$. Define a map $T : L^0(m)\times\ldots\times L^0(m) \to L^0(m_n)$ by

$$T(x_1,\dots,x_n) = x_1\otimes\ldots\otimes x_n,$$

where $x_1\otimes\ldots\otimes x_n(s_1,\dots,s_n) := x_1(s_1)\cdot\ldots\cdot x_n(s_n)$ for $(s_1,\dots,s_n) \in \mathbb{R}^n_+$. Clearly, $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$ and $\|T\|_{\mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})} = 1$. We have $\varphi_i(\overline{X}_i) = L_{\psi_i}(m)$ and $\varphi(\overline{Y}) = L_\psi(m_n)$ isometrically with suitable Orlicz functions ψ_i and ψ defined by $\varphi_i(s,t) = t\psi_i^{-1}(s/t)$ and $\varphi(s,t) = t\psi^{-1}(s/t)$ for $t > 0$. This easily implies that for any $s_i > 0$, $i = 1,\dots,n$, we have

$$\|\chi_{(0,s_i)}\|_{\varphi_i(\overline{X}_i)} = 1/\varphi_i(s_i^{-1},1)$$

and

$$\|\chi_{(0,s_i)}\otimes\ldots\otimes\chi_{(0,s_n)}\|_{\varphi(\overline{Y})} = 1/\varphi(s_1^{-1}\cdot\ldots\cdot s_n^{-1},1).$$

Thus, if we assume that (iii) holds, we would have

$$\|T(x_1,\dots,x_n)\|_{\varphi(\overline{Y})} \le C\|T\|_{\mathscr{L}(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})}\|x_1\|_{\varphi_1(\overline{X}_1)}\cdot\ldots\cdot\|x_n\|_{\varphi_n(\overline{X}_n)}.$$

Combining the above with $x_i = \chi_{(0,s_i)}$, $s_i > 0$ for $i = 1,\dots,n$, we obtain by homogeneity of φ_i and φ

$$\varphi_1(1,s_1)\cdot\ldots\cdot\varphi_n(1,s_n) \le C\varphi(1,s_1\cdot\ldots\cdot s_n).$$

This completes the proof. ■

3. Applications

In this section we will apply the obtained results to prove certain abstract multilinear interpolation theorems. First of all we observe that the results we have proved hold true for multilinear operators provided that the range couples are weighted L_∞-spaces. To see this, we need the following result, which is a multilinear version of the well-known classical result on the existence of the *modulus* for order-bounded linear operators between Riesz spaces. The proof is a simple modification of the argument for the classical case (see e.g. [1, theorem 1.10]) and is therefore omitted.

Proposition 3.1. *Let* $T : E_1 \times \ldots \times E_n \to F$ *be a multilinear map between Riesz spaces such that* $\sup\{|T(y_1, \ldots, y_n)| \,;\, |y_j| \le x_j,\ j = 1, \ldots, n\}$ *exists in* F *for each* $(x_1, \ldots, x_n) \in E_1^+ \times \ldots \times E_n^+$. *Then* $|T| : E_1^+ \times \ldots \times E_n^+ \to F^+$ *defined by*

$$|T|(x_1, \ldots, x_n) := \sup\{|T(y_1, \ldots, y_n)| \,;\, |y_j| \le x_j,\ j = 1, \ldots, n\}$$

is a positively multilinear map which dominates T.

Proposition 3.2. *Let* $\overline{X}_i = (X_{i0}, X_{i1})$, $i = 1, \ldots, n$, *and* $(L_\infty(w_0), L_\infty(w_1))$ *be couples of Banach lattices, and let* $T : \prod_{i=1}^n \overline{X}_i \to (L_\infty(w_0), L_\infty(w_1))$ *be a multilinear operator. If* $\varphi_1, \ldots, \varphi_n, \varphi \in \mathscr{U}$ *satisfy the estimate* $\varphi_1(1, s_1) \cdot \ldots \cdot \varphi_n(1, s_n) \le C\varphi(1, s_1 \cdot \ldots \cdot s_n)$ *for some* $C > 0$ *and all* $s_1, \ldots, s_n > 0$, *then* T *is a bounded operator from the product* $\prod_{i=1}^n \varphi_i(\overline{X}_i)$ *into* $L_\infty(\varphi_*(w_0, w_1))$.

Proof. Clearly $\sup\{|T(y_1, \ldots, y_n)| \,;\, |y_i| \le x_i,\ i = 1, \ldots, n\}$ exists in $L_\infty(w_0) + L_\infty(w_1)$ for each $(x_1, \ldots, x_n) \in (X_{10} + X_{11})^+ \times \ldots \times (X_{n0} + X_{n1})^+$. Thus we conclude by Proposition 3.1 that T is dominated by a positively multilinear operator $|T|$. To complete the proof, it suffices to apply Theorem 2.2 and the obvious observation that, by boundedness of T, we obtain that

$$|T| : {X_{1j}}^+ \times \ldots \times {X_{nj}}^+ \to L_\infty(w_j)^+$$

is bounded for $j = 0, 1$. ■

Before the presentation of the main result of this section we need some more definitions and notation.

Let $\overline{A}$ be any fixed Banach couple and let $0 \ne a \in A_0 + A_1$. We recall (see e.g. [12]) that the *interpolation orbit* $Orb_{\overline{A}}(a, \overline{X})$ of a in the couple $\overline{X}$ is the Banach space consisting of all elements of the form $x = Ta$ for some $T : \overline{A} \to \overline{X}$, equipped with the norm

$$\|x\| := \inf\{\|T\|_{\overline{A} \to \overline{X}} \,;\, x = Ta\}.$$

Clearly $\mathscr{F} = Orb_{\overline{A}}(a, \cdot)$ is an exact interpolation functor. If $a_\varphi := \{\varphi(1, 2^n)\}_{-\infty}^{\infty}$ for $\varphi \in \Phi$, then $Orb_{\overline{\ell}_\infty}(a_\varphi, \cdot)$ is the *lower Ovchinnikov's functor* denoted by $\varphi_\ell(\cdot)$ (see [12]).

Let $\varphi \in \mathscr{U}$ and let $\overline{X}$ be a Banach couple. The space $M_\varphi(\overline{X})$ of all $x \in X_0 + X_1$ such that

$$\|x\|_\varphi := \sup\{K(1, t, x; \overline{X})/\varphi(1, t) \,;\, t > 0\} < \infty$$

is called a *generalised Marcinkiewicz space*. Here K is the functional defined on $X_0 + X_1$ by

$$K(s, t, x; \overline{X}) := \inf\{s\|x_0\|_{X_0} + t\|x_1\|_{X_1} \,;\, x = x_0 + x_1\}, \quad s, t > 0.$$

Let $\overline{X} = (X_0, X_1)$ be a Banach couple and let X be a Banach space intermediate with respect to $\overline{X}$. We recall that the *Gagliardo completion* X^c of X is the space of all limits in $X_0 + X_1$ that are bounded in X. We also recall that X^c is a Banach space equipped with the norm

$$\|x\|_{X^c} = \inf_{\{x_n\}} \sup \|x_n\|_X,$$

where the infimum is taken over all bounded sequences $\{x_n\}$ in X such that $x_n \to x$ in $X_0 + X_1$.

Theorem 3.3. *Let $\overline{E}_i = (E_{i0}, E_{i1})$ be a couple of Banach lattices and let $a_i \in E_{i0} + E_{i1}$ be such that $a_i \in \varphi_i(\overline{E}_i)^c$, where $\varphi_i(s,t) \asymp K(s,t,a_i;\overline{E}_i)$ for $i = 1,\dots,n$. Assume that $\varphi \in \mathscr{U}$ is such that there exists a constant $C > 0$ such that $\varphi_1(1,s_1)\cdot\ldots\cdot\varphi_n(1,s_n) \le \varphi(1, s_1\cdot\ldots\cdot s_n)$ for all $s_1,\dots,s_n > 0$. If $\overline{X}_i = (X_{i0}, X_{i1})$, $i = 1,\dots,n$ and $\overline{Y} = (Y_0, Y_1)$ are couples of Banach spaces and $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$, then T is bounded from the product $Orb_{\overline{E}_1}(a_1, \overline{X}_1) \times \ldots \times Orb_{\overline{E}_n}(a_n, \overline{X}_n)$ into $M_\varphi(\overline{Y})$.*

PROOF. Assume that $\|T\|_{L(\overline{X}_1,\dots,\overline{X}_n;\overline{Y})} \le 1$. Let $x_i \in X_i := Orb_{E_i}(a_i, \overline{X}_i)$ with $\|x_i\|_{X_i} < 1$ for $i = 1,\dots,n$. Thus $x_i = S_i(a_i)$ for some $S_i : \overline{E}_i \to \overline{X}_i$ with $\|S_i\|_{\overline{E}_i \to \overline{X}_i} \le 1$. Let $V : \overline{Y} \to \overline{\ell}_\infty$ be any operator such that $\|V\|_{\overline{Y}\to\overline{\ell}_\infty} \le 1$. Define an operator $U = U_V$ by

$$U(y_1,\dots,y_n) := VT(S_1y_1,\dots,S_ny_n)$$

for $(y_1,\dots,y_n) \in \prod_{i=1}^n (E_{i0} + E_{i1})$. Clearly, for $j = 0, 1$ we have

$$\|U(y_1,\dots,y_n)\|_{\ell_\infty(2^{-jn})} \le \|y_1\|_{E_{1j}}\cdot\ldots\cdot\|y_n\|_{E_{nj}}.$$

This means that $U : \prod_{i=1}^n \overline{E}_i \to \overline{\ell}_\infty$. By Proposition 3.2 there exists a constant $C_1 > 0$ such that

$$\|U(y_1,\dots,y_n)\|_{\ell_\infty(\varphi_*(1,2^{-n}))} \le C_1\|y_1\|_{\varphi_1(\overline{E}_1)}\cdot\ldots\cdot\|y_n\|_{\varphi_n(\overline{E}_n)}$$

for all $(y_1,\dots,y_n) \in \prod_{i=1}^n \varphi_i(\overline{E}_i)$. Since the multilinear map $U : \prod_{i=1}^n (E_{i0} + E_{i1}) \to \ell_\infty + \ell_\infty(2^{-n})$ is continuous and $\ell_\infty(\varphi_*(1,2^{-n}))^c = \ell_\infty(\varphi_*(1,2^{-n}))$, we easily obtain that

$$\|U(y_1,\dots,y_n)\|_{\ell_\infty(\varphi_*(1,2^{-n}))} \le C_1\|y_1\|_{\varphi_1(\overline{E}_1)^c}\cdot\ldots\cdot\|y_n\|_{\varphi_n(\overline{E}_n)^c}$$

for all $(y_1,\dots,y_n) \in \prod_{i=1}^n \varphi_i(\overline{E}_i)^c$.

By the co-orbital description of the K-method spaces (see e.g. [4], [6] or [12]), we have

$$\|y\|_{M_\varphi(\overline{Y})} = \sup\{\|Vy\|_{\ell_\infty(\varphi_*(1,2^{-n}))}\,;\, \|V\|_{\overline{Y}\to\overline{\ell}_\infty} \le 1\}.$$

Since $a_i \in \varphi_i(\overline{E}_i)^c$, we conclude that

$$\begin{aligned}\|T(x_1,\dots,x_n)\|_{M_\varphi(\overline{Y})} &= \sup\{\|U_V(a_1,\dots,a_n)\|_{\ell_\infty(\varphi_*(1,2^{-n}))}\,;\, \|V\|_{\overline{Y}\to\overline{\ell}_\infty} \le 1\} \\ &\le C_1\|a_1\|_{\varphi_1(\overline{E}_1)^c}\cdot\ldots\cdot\|a_n\|_{\varphi_n(\overline{E}_n)^c} = C\end{aligned}$$

holds for all $(x_1,\dots,x_n) \in \prod_{i=1}^n X_i$ with $\|x_i\|_{X_i} < 1$. This completes the proof. ■

Theorem 3.4. *Let* $\overline{X}_i = (X_{i0}, X_{i1})$, $i = 1, \ldots, n$, *and* $\overline{Y} = (Y_0, Y_1)$ *be Banach couples and let* $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$ *be a multilinear operator. If* $\varphi_1, \ldots, \varphi_n, \varphi \in \mathscr{U}$ *satisfy* $\varphi_1(1, s_1) \cdot \ldots \cdot \varphi_n(1, s_n) \le C\varphi(1, s_1 \cdot \ldots \cdot s_n)$ *for some* $C > 0$ *and all* $s_1, \ldots, s_n > 0$, *then* T *is bounded from* $\prod_{i=1}^n \varphi_{i\ell}(\overline{X}_i)$ *into* $M_\varphi(\overline{Y})$. *If additionally* $\varphi_i \in \Phi_0$, *then* T *is bounded from* $\prod_{i=1}^n Orb_{\overline{c}_0}(a_{\varphi_i}, \overline{X}_i)$ *into* $M_\varphi(\overline{Y})$.

Proof. Let $\overline{E}_i = \overline{\ell}_\infty$ and let $a_i = \{\varphi_i(1, 2^n)\}_{-\infty}^{\infty}$ for $i = 1, \ldots, n$. By the well-known equivalence $K(s, t, a_i; \overline{\ell}_\infty) \asymp \sup_n \min\{s, t2^{-n}\}$, we have $K(s, t, a_i; \overline{\ell}_\infty) \asymp \varphi_i(s, t)$. Since

$$\varphi_i(\overline{\ell}_\infty)^c = \varphi_i(\overline{\ell}_\infty) = \ell_\infty(\varphi_{i*}(1, 2^{-n})),$$

we have $a_{\varphi_i} \in \varphi_i(\overline{\ell}_\infty)^c$. Thus the result follows by Theorem 3.3.

Now recall (see [15, pp 451, 471]) that if E is a Banach lattice on (Ω, μ) then $x \in X''$ if and only if there exists a sequence $\{x_n\}$ of elements of E, such that $0 \le x_n \uparrow |x|$ a.e. and $\sup_n \|x_n\|_E < \infty$. For $x \in E''$ we have

$$\|x\|_{E''} = \inf\left\{ \lim_{n\to\infty} \|x_n\|_E \; ; \; 0 \le x_n \uparrow |x| \text{ a.e.} \right\}.$$

Hence a routine calculation shows that if $\overline{X}$ is a couple of Banach lattices on (Ω, μ) and X is an intermediate Banach lattice with respect to $\overline{X}$ having order-continuous norm (i.e. $\|x_n\|_X \to 0$ whenever $0 \le x_n \downarrow 0$ with $x_n \in X$), then $X^c = X'' \cap (X_0 + X_1)$. Since $\varphi(\overline{c}_0) = c_0(\varphi_*(1, 2^{-n}))$ for any $\varphi \in \mathscr{U}$ and $\varphi(\overline{c}_0)'' = \ell_\infty(\varphi(1, 2^{-n}))$, we conclude that

$$\varphi(c_0, c_0(2^{-n}))^c = \ell_\infty(\varphi(1, 2^{-n}))$$

for any $\varphi \in \mathscr{U} \cap \Phi_0$.

To complete the argument, suppose that $\varphi_i \in \mathscr{U} \cap \Phi_0$ for $i = 1, \ldots, n$. Then $a_{\varphi_i} \in \varphi_i(\overline{c}_0)^c$. Since $K(s, t, a_{\varphi_i}; \overline{c}_0) \asymp \varphi_i(s, t)$, we obtain by Theorem 3.3 that T is bounded from the product $\prod_{i=1}^n Orb_{\overline{c}_0}(a_{\varphi_i}, \overline{X}_i)$ into $M_\varphi(\overline{Y})$. ■

Corollary 3.5. *Let* $\overline{X}_i$, $i = 1, \ldots, n$ *and* $\overline{Y}$ *be Banach couples and let* $\varphi \in \mathscr{U}$ *be such that* $\varphi(1, s)\varphi(1, t) \le C\varphi(1, st)$ *for some* $C > 0$ *and all* $s, t > 0$. *If* $T : \prod_{i=1}^n \overline{X}_i \to \overline{Y}$, *then* T *is bounded from* $\prod_{i=1}^n \varphi_\ell(\overline{X}_i)$ *into* $M_\varphi(\overline{Y})$. *If additionally* $\varphi \in \Phi_0$, *then* T *is bounded from* $\prod_{i=1}^n Orb_{\overline{c}_0}(a_\varphi, \overline{X}_i)$ *into* $M_\varphi(\overline{Y})$.

Applying the above corollary, we easily obtain a generalisation of the main results presented in [3] on interpolation of bilinear operators from the product of interpolation spaces generated by the Peetre functor $\langle\cdot\rangle_\rho$ or Gustavsson–Peetre functor $\langle\cdot, \rho\rangle$ determined by a parameter concave function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$. We refer the reader to [6] or [12] for the definitions of these functors.

Corollary 3.6. *Let* $\overline{X} = (X_0, X_1), \overline{Y} = (Y_0, Y_1)$ *and* $\overline{Z} = (Z_0, Z_1)$ *be Banach couples and let* $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ *be a concave function such that* $\varphi(s, t) := s\rho(t/s)$ *satisfies* $\varphi \in \Phi_0$ *and* $\varphi_* \in \Phi_0$. *If there exists* $C > 0$ *such that* $\rho(s)\rho(t) \le C\rho(st)$ *for all* $s, t > 0$ *and* $T : \overline{X} \times \overline{Y} \to \overline{Z}$ *is any bilinear operator, then* T *is bounded from the product* $\langle \overline{X}, \rho\rangle \times \langle \overline{Y}, \rho\rangle$ *into* $M_\varphi(\overline{Z})$. *In particular,* T *is bounded from* $\langle \overline{X}\rangle_\rho \times \langle \overline{Y}\rangle_\rho$ *into* $M_\varphi(\overline{Z})$.

PROOF. It is well known that if $\varphi \in \mathscr{U}$ is such that $\varphi \in \Phi_0$ and $\varphi_* \in \Phi_0$, then $Orb_{\overline{c_0}}(\overline{X}) = \langle \overline{X}, \rho \rangle$ (see [6] or [12, section 8]). Since

$$\langle \overline{X} \rangle_\rho \hookrightarrow \langle \overline{X}, \rho \rangle$$

holds with continuous inclusion for any Banach couple $\overline{X}$, where $\rho(t) = \varphi(1,t)$ for $t > 0$, the proof is finished by Theorem 3.5. ∎

We note that the above result has been proved in [3] in a quite different way under the additional assumption that the dilation indices of the function ρ are non-trivial.

Concluding this section, let us show some further applications to interpolation of multilinear operators defined on the abstract K-method spaces. We recall that if E is a *parameter* of the *K-method*, i.e. E is a Banach sequence lattice intermediate with respect to $(\ell_\infty, \ell_\infty(2^{-n}))$, then the K-space $\overline{X}_E$ consists of all $x \in X_0 + X_1$ with $\{K(1, 2^n, x; \overline{X})\}_{n=-\infty}^{\infty} \in E$. It is well known that $\overline{X}_E$ is equipped with the norm

$$\|x\| := \|\{K(1, 2^n, x; \overline{X})\}\|_E$$

is an exact interpolation space with respect to $\overline{X}$.

Let $\overline{E} = (E_0, E_1)$ be a couple of parameters of the K-method. An interpolation functor $\mathscr{F}$ is said to be *stable* on $\overline{X}$ with respect to $\overline{E}$ if the following *reiteration formula* holds:

$$\mathscr{F}(\overline{X}_{E_0}, \overline{X}_{E_1}) = \overline{X}_{\mathscr{F}(E_0, E_1)}.$$

It is well known that if E_0 and E_1 are parameters of the real method, then any interpolation functor is stable on any Banach couple $\overline{X}$ (see e.g. [4]; [12]). Recall that a Banach sequence space E is said to be a *parameter* of the *real method* if $\ell_\infty \cap \ell_\infty(2^{-n}) \hookrightarrow E \hookrightarrow \ell_1 + \ell_1(2^{-n})$ and the Calderón operator $\mathscr{P}$ defined by

$$(\mathscr{P}\xi)_n = \sum_{k=-\infty}^{\infty} \min\{1, 2^{n-k}\} \xi_k$$

is bounded in E.

We remark that it is shown in [11] that if $\overline{X} = (X_0, X_1)$ is an ordered couple (i.e. $X_0 \hookrightarrow X_1$ or $X_1 \hookrightarrow X_0$) of Banach lattices on measure space with $X_j = L_\infty(w)$, then any interpolation functor $\mathscr{F}$ is stable on $\overline{X}$ with respect to (E_0, E_1) for any parameter of the real method E_{1-j} and $E_j = \ell_\infty(2^{-jn})$ with corresponding $j = 0$ or $j = 1$ for which $X_j = L_\infty(w)$. The next result can be formulated for orbit interpolation functors satisfying the general conditions of Theorem 3.3 and couples for which we have the stability property. For simplicity, we formulate only the following result.

Theorem 3.7. *Let $\overline{A^i} = (A_{i0}, A_{i1})$ and $\overline{Y}$ be any Banach couples and let $\overline{E}_i = (E_{i0}, E_{i1})$ be a couple of parameters of the real method, $i = 1, \dots, n$. If $\varphi_1, \dots, \varphi_n, \varphi \in \mathscr{U}$ satisfy the estimate $\varphi_1(1, s_1) \cdot \ldots \cdot \varphi_n(1, s_n) \le \varphi(1, s_1 \cdot \ldots \cdot s_n)$ for some $C > 0$ and all $s_1, \dots, s_n > 0$, then any multilinear operator $T : \prod_{i=1}^n (\overline{A^i}_{E_{i0}}, \overline{A^i}_{E_{i1}}) \to \overline{Y}$ is bounded from $\prod_{i=1}^n (A_{i0}, A_{i1})_{\varphi_{i\ell}(\overline{E}_i)}$ into $M_\varphi(\overline{Y})$.*

Proof. If we put $\overline{X}_i = (\overline{A^i}_{E_{i0}}, \overline{A^i}_{E_{i1}})$ for $i = 1, \ldots, n$, then it follows by Theorem 3.4 that T is bounded from $\prod_{i=1}^{n} \varphi_{i\ell}(\overline{X}_i)$ into $M_\varphi(\overline{Y})$. Since $\overline{E}_i$ are parameters of the real method the reiteration formula applies for each functor $\varphi_{i\ell}(\cdot)$, $i = 1, \ldots, n$. This completes the proof. ■

As an application of Theorem 3.7 we present a result on interpolation of bilinear operators between generalised Marcinkiewicz spaces. We refer the reader to [7], where results on interpolation of multilinear operators are presented for the classical real method spaces.

Corollary 3.8. *Let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ be Banach couples. Assume that $\varphi_i, \psi_i, \theta_i \in \mathcal{U}$ have non-trivial indices for $i = 0, 1$. If $\varphi \in \mathcal{U}$ is such that $\varphi(1,s)\varphi(1,t) \le C\varphi(1,st)$ for some $C > 0$ and $s, t > 0$, then any operator $T : (M_{\varphi_0}(\overline{X}), M_{\varphi_1}(\overline{Y})) \times (M_{\psi_0}(\overline{X}), M_{\psi_1}(\overline{Y})) \to (M_{\theta_0}(\overline{Z}), M_{\theta_1}(\overline{Z}))$ is bounded from $M_{\varphi(\varphi_0,\varphi_1)}(\overline{X}) \times M_{\varphi(\psi_0,\psi_1)}(\overline{Y})$ into $M_{\varphi(\theta_0,\theta_1)}(\overline{Z})$.*

Proof. The result follows from Theorem 3.7. To see this observe that $M_\psi(\overline{A}) = \overline{A}_{\ell_\infty(\psi_*(1,2^{-n}))}$ for any Banach couple $\overline{A}$ and $\psi \in \mathcal{U}$. Moreover, if ψ has non-trivial indices, then $E = \ell_{\psi_*}(1, 2^{-n})$ is a parameter of the real method. We also have

$$\varphi_\ell(\ell_\infty(w_0), \ell_\infty(w_1)) = \varphi(\ell_\infty(w_0), \ell_\infty(w_1)) = \ell_\infty(\psi_*(w_0, w_1))$$

for any $\varphi \in \mathcal{U}$ and any weights w_0, w_1 (see [12]). In order to complete the proof, we need only to recall (see e.g. [4, p. 349]) that the following reiteration formula,

$$M_\varphi(M_{\theta_0}(\overline{Z}), M_{\theta_1}(\overline{Z})) = M_{\varphi(\theta_0,\theta_1)}(\overline{Z}),$$

holds for any Banach couple $\overline{Z}$ and $\varphi, \theta_0, \theta_1 \in \mathcal{U}$. ■

We remark that, taking $\overline{X} = \overline{Y} = \overline{Z} = (L_1(\mu), L_\infty(\mu))$ (or resp. $(C(0,1), C^1(0,1))$), we obtain the results on interpolation of bilinear operators between classical Marcinkiewcz symmetric spaces or Lipschitz spaces respectively.

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