PUTNAM'S INEQUALITY OF DOUBLY COMMUTING n-TUPLES FOR LOG-HYPONORMAL OPERATORS

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Abstract

In this paper, we extend Putnam's inequality to doubly commuting n-tuples of log-hyponormal operators and study a relation of the Taylor spectrum and the Xia spectrum.

1. Introduction

In [14], Xia introduced a class for semi-hyponormal tuples and the joint spectrum of such non-commuting tuples, and he proved Putnam's inequality of semi-hyponormal tuples. In [4], Chō and Huruya generalised Putnam's inequality of p-hyponormal tuples ($0 < p \leqslant 1$). Also, in [9] Duggal has shown more clearly inequality of doubly commuting n-tuples of p-hyponormal operators. Recently, Tanahashi has studied log-hyponormal operators and extended Putnam's inequality to log-hyponormal operators ([11]; [12]). In this paper, we extend Putnam's inequality to doubly commuting n-tuples of log-hyponormal operators and show a relation between the Taylor spectrum and the Xia spectrum of such tuples.

Let $\mathscr{H}$ be a complex separable Hilbert space and $B(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$. For $T \in B(\mathscr{H})$, let $\sigma(T)$ denote the spectrum of T. An operator $T \in B(\mathscr{H})$ is called log-hyponormal if T is invertible and $\log(T^*T) \geqslant \log(TT^*)$. Also T is called p-hyponormal if $(T^*T)^p \geqslant (TT^*)^p$. Since the function $\log(\cdot)$ is operator monotone, invertible p-hyponormal operators are log-hyponormal. If $p = \frac{1}{2}$, then T is called semi-hyponormal. There exists a log-hyponormal operator which is not p-hyponormal (cf. [12]). Let U be a unitary operator. If

$$\mathscr{S}_U^{\pm}(S) = s - \lim_{n \to \pm\infty} (U^{-n} S U^n)$$

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exist, then the operators $\mathscr{S}_U^{\pm}(S)$ are called the polar symbols of S with respect to U. If $T = U|T|$ is semi-hyponormal with unitary U, then $\mathscr{S}_U^{\pm}(|T|)$ exist (cf. [15]).

2. Putnam's inequality for doubly commuting *n*-tuples of log-hyponormal operators

For a commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$, the Taylor spectrum is denoted by $\sigma_T(\mathbf{T})$. A number $\mathbf{z} = (z_1, \ldots, z_n)$ is in the joint approximate point spectrum $\sigma_a(\mathbf{T})$ if and only if there exists a sequence $\{x_m\}$ such that

$$(T_j - z_j)\, x_m \to 0 \text{ as } m \to \infty \text{ for every } j = 1, \ldots, n.$$

It is well known that $\sigma_T(\mathbf{T}) = \sigma_a(\mathbf{T})$ if $\mathbf{T}$ is a commuting n-tuple of normal operators. Tanahashi showed the following theorem.

Theorem A [11, lemma 3]. *Let T be log-hyponormal. If, for a sequence $\{x_n\}$ of unit vectors,* $\lim_{n\to\infty} (T-z)\, x_n = 0$, *then* $\lim_{n\to\infty} (T-z)^* x_n = 0$.

Therefore, by [7, corollary 3.3], we have the following theorem.

Theorem 1. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of log-hyponormal operators. Then*

$$\sigma_T(\mathbf{T}) = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : (\bar{z}_1, \ldots, \bar{z}_n) \in \sigma_a(\mathbf{T}^*)\},$$

where $\mathbf{T}^* = (T_1^*, \ldots, T_n^*)$.

Let $\mathbf{U} = (U_1, \ldots, U_n)$ be a commuting n-tuple of unitary operators. An operator $Q_j (j = 1, \ldots, n)$ on $B(\mathscr{H})$ is defined by

$$Q_j S = S - U_j S U_j^* \ (S \in B(\mathscr{H})).$$

Let $A \in B(\mathscr{H})$ and be hermitian. An $(n+1)$-tuple $(\mathbf{U}, A)$ is called a semi-hyponormal tuple if

$$Q_{j_1} \cdots Q_{j_m} A \geqslant 0$$

for all $1 \leqslant j_1 < \cdots < j_m \leqslant n$. Let $\mathbf{U} = (U_1, \ldots, U_n)$ be a commuting n-tuple of unitary operators, A be hermitian and $0 \leqslant k \leqslant 1$. We denote

$$(\mathscr{S}_j(k))(A) = k\mathscr{S}_{U_j}^{+}(A) + (1-k)\,\mathscr{S}_{U_j}^{-}(A).$$

Further, let $0 \leqslant k_i, k_j \leqslant 1$,

$$(\mathscr{S}_i(k_i)\,\mathscr{S}_j(k_j))\,(A) = (\mathscr{S}_i(k_i))\,((\mathscr{S}_j(k_j)\,(A))).$$

For $\mathbf{k} = (k_1, \ldots, k_n) \in [0, 1]^n$, the general polar symbols $A(\mathbf{k})$ of A with respect to $\mathbf{U}$ are defined by

$$A(\mathbf{k}) = \left(\prod_{j=1}^{n} \mathscr{S}_j(k_j)\right)(A).$$

Then from [4] it follows that $(\mathbf{U}, A(\mathbf{k}))$ is a commuting $(n+1)$-tuple of normal operators for every $\mathbf{k} \in [0, 1]^n$. Hence $\sigma_a(\mathbf{U}, A(\mathbf{k}))$ is non-empty for every $\mathbf{k} \in [0, 1]^n$. The Xia spectrum $\sigma_X(\mathbf{U}, A)$ of $(\mathbf{U}, A)$ is defined by

$$\sigma_X(\mathbf{U}, A) = \bigcup_{\mathbf{k} \in [0,1]^n} \sigma_a(\mathbf{U}, A(\mathbf{k})).$$

Further, Xia proved the following theorem.

Theorem B [14, theorem 5]. *Let* $(\mathbf{U}, A)$ *be a semi-hyponormal tuple and* $A \geqslant 0$. *Then*

$$\|Q_1 \cdots Q_n A\| \leqslant m(\sigma_X(\mathbf{U}, A)),$$

where m is the product of Haar measure on $\{z \in \mathbf{C} : |z| = 1\}^n$ *and the Lebesgue measure on* $\mathbf{R}$.

Also, Duggal demonstrated the following theorem.

Theorem C [9, lemma 3]. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of semi-hyponormal operators with unitaries* U_j *of the polar decomposition* $T_j = U_j|T_j|$ *for every* $j = 1, \ldots, n$. *Also, let* $\mathbf{U} = (U_1, \ldots, U_n)$ *and* $A = \prod_{j=1}^{n} |T_j|$. *Then*

$$\| \prod_{j=1}^{n} (|T_j| - |T_j^*|) \| \leqslant m(\sigma_X(\mathbf{U}, A)).$$

Remark. Theorems B and C are extended to a doubly commuting n-tuple of p-hyponormal operators with unitaries of the polar decompositions (cf. [3]).

First we have the following theorem.

Theorem 2. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of log-hyponormal operators* T_j *with the polar decomposition* $T_j = U_j|T_j|$ $(j = 1, \ldots, n)$. *Let* $\mathbf{U} = (U_1, \ldots, U_n)$. *We assume that, for every j,* $\log|T_j| \geqslant 0$ $(j = 1, \ldots, n)$. *Also, let* $A = (\log|T_1|) \cdots (\log|T_n|)$. *Then* $(\mathbf{U}, A)$ *is a semi-hyponormal tuple*.

PROOF. Since $A = \log|T_1| \cdots \log|T_n|$ and $U_j \log|T_i| = (\log|T_i|)\, U_j$ for $i \neq j$, we have

$$Q_j A = \left(\prod_{i \neq j} \log|T_i|\right)(\log|T_j| - U_j(\log|T_j|)\, U_j^*) = \left(\prod_{i \neq j} \log|T_i|\right)(\log|T_j| - \log|T_j^*|)$$

for every j $(j = 1, \ldots, n)$. Hence for every $1 \leqslant j_1 < \cdots < j_m \leqslant n$ we have

$$Q_{j_1} \cdots Q_{j_m} A = \left(\prod_{i \neq j_1, \ldots, j_m} \log|T_i|\right)\left(\prod_{k=1, \ldots, m} (\log|T_{j_k}| - \log|T_{j_k}^*|)\right) \geqslant 0,$$

because $\log|T_i|$ $(i \neq j_1, \ldots, j_m)$ and $\log|T_{j_k}| - \log|T_{j_k}^*|$ $(k = 1, \ldots, m)$ are commuting positive operators. Hence $(\mathbf{U}, A)$ is a semi-hyponormal tuple. ■

Hence we have the following theorem.

Theorem 3. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of log-hyponormal operators. Let* $T_j = U_j|T_j|$ *be the polar decomposition* $(j = 1, \ldots, n)$, $\mathbf{U} = (U_1, \ldots, U_n)$ *and* $A = (\log|T_1|) \cdots (\log|T_n|)$. *If every* $\log|T_j| \geqslant 0$ $(j = 1, \ldots, n)$, *then*

$$\left\| \prod_{j=1}^{n} (\log|T_j| - \log|T_j^*|) \right\| \leqslant m(\sigma_X(\mathbf{U}, A)).$$

Proof. Since $(\mathbf{U}, A)$ is a semi-hyponormal tuple by Theorem 2, by Theorem C we have

$$\| \prod_{j=1}^{n} (\log|T_j| - \log|T_j^*|) \| \leqslant m(\sigma_X(\mathbf{U}, A)). \quad \blacksquare$$

Finally, we show the following theorem. The proof is similar to the proof of [3, theorem 3], but for completeness we will give a proof.

Theorem 4. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of log-hyponormal operators. Let* $T_j = U_j|T_j|$ *be the polar decomposition of* T_j *for every* $j = 1, \ldots, n$. *Also let* $\mathbf{U} = (U_1, \ldots, U_n)$ *and* $A = \log|T_1| \cdots \log|T_n|$. *Then*

$$\sigma_T(\mathbf{T}) = \{(r_1e^{i\theta_1}, r_2e^{i\theta_2}, \ldots, r_ne^{i\theta_n}) : (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}, \log r_1 \cdot \log r_2 \cdots \log r_n) \in \sigma_X(\mathbf{U}, A)\}.$$

For the proof, we need the following Berberian's extension theorem.

Theorem D [1, theorem 1]. *Let* $B(\mathscr{H})$ *be the algebra of all bounded linear operators on* $\mathscr{H}$. *Then there exist an extension space* $\mathscr{K}$ *of* $\mathscr{H}$ *and a faithful *-representation of* $B(\mathscr{H})$ *into* $B(\mathscr{K}) : T \to T^\circ$ *such that*

$$\sigma_a(T_1, \ldots, T_n) = \sigma_a(T_1^\circ, \ldots, T_n^\circ) = \sigma_p(T_1^\circ, \ldots, T_n^\circ),$$

where $\sigma_p(T_1^\circ, \ldots, T_n^\circ)$ *is the joint point spectrum of* $(T_1^\circ, \ldots, T_n^\circ)$. *Moreover, if T is log-hyponormal, then* T° *is also log-hyponormal.*

Proof of Theorem 4. We prove Theorem 4 by induction. First, for a log-hyponormal operator $T = U|T|$, we prove that $re^{i\theta} \in \sigma(T)$ if and only if $(e^{i\theta}, \log r) \in \sigma_X(U, \log|T|)$. We choose a positive number c such that $\log|cT| \geqslant 0$. Let $S = U\log|cT|$. Then S is semi-hyponormal. By [11, lemma 6] we have

$$re^{i\theta} \in \sigma(T) \Leftrightarrow cre^{i\theta} \in \sigma(cT) \Leftrightarrow (\log cr)\, e^{i\theta} \in \sigma(S).$$

Since $S = U\log|cT|$ is semi-hyponormal, it follows that, by [4, theorem 3],

$$(\log cr)\, e^{i\theta} \in \sigma(S) \Leftrightarrow (e^{i\theta}, \log cr) \in \sigma_X(U, \log|cT|) \Leftrightarrow (e^{i\theta}, \log r) \in \sigma_X(U, \log|T|).$$

Hence Theorem 4 holds for $n = 1$. Next, we assume that Theorem 4 holds for such $(n-1)$-tuples. Let $(z_1, \ldots, z_n, a) \in \sigma_X(\mathbf{U}, A)$. Then we show that there exist positive numbers $a_1, \ldots, a_n$ such that

$$(z_1 a_1, \ldots, z_n a_n) \in \sigma_T(\mathbf{T}) \text{ and } a = (\log a_1) \cdots (\log a_n).$$

By the assumption there exist $\mathbf{k} = (k_1, \ldots, k_n) \in [0, 1]^n$ and a sequence $\{x_m\}$ of unit vectors such that

$$(U_j - z_j)\, x_m \to 0 \ (j = 1, \ldots, n) \text{ and } (A(\mathbf{k}) - a)\, x_m \to 0,$$

where $A(\mathbf{k}) = \left(\prod_{j=1}^{n} \mathscr{S}_j(k_j)\right)(A)$. By the proof of Theorem 3 we have

$$A(\mathbf{k}) = \prod_{j=1}^{n} A(j),$$

where $A(j) = (\mathscr{S}_j(k_j))(\log|T_j|)$. By Theorem D, let $\mathscr{K}$ be the extension space of $\mathscr{H}$. Then

$$\mathscr{M} = \mathrm{Ker}(U_1^\circ - z_1) \bigcap \cdots \bigcap \mathrm{Ker}(U_n^\circ - z_n) \bigcap \mathrm{Ker}(A(\mathbf{k})^\circ - a)$$

is a non-zero subspace of $\mathscr{K}$. Since $(U_1^\circ, \ldots, U_n^\circ, A(1)^\circ, \ldots, A(n)^\circ)$ is a commuting $2n$-tuple, $\mathscr{M}$ is an invariant subspace for $A(1)^\circ, \ldots, A(n)^\circ$. Also, since $a \in \sigma(A(\mathbf{k})^\circ_{|\mathscr{M}})$, there exist $b_1, \ldots, b_n$ and a non-zero vector $x^\circ \in \mathscr{M}$ such that

$$(A(j)^\circ - b_j)\,x^\circ = 0 \text{ for every } j\ (j = 1, \ldots, n) \text{ and } a = b_1 \cdots b_n,$$

by Theorem D and the spectral mapping theorem of the joint spectrum. Let

$$\mathscr{N} = \mathrm{Ker}(U_n^\circ - z_n) \bigcap \mathrm{Ker}(A(n)^\circ - b_n).$$

Then by the above it follows that

$$(z_1, \ldots, z_{n-1}, a_1 \cdots a_{n-1}) \in \sigma_X(\mathbf{U}', A'),$$

where $\mathbf{U}' = (U_1, \ldots, U_{n-1})$ and $A' = \prod_{j=1}^{n-1} A(j)$. By Theorem D and the assumption of induction, we have

$$(z_1 e^{b_1}, \ldots, z_{n-1} e^{b_{n-1}}) \in \sigma_T(T_1, \ldots, T_{n-1}).$$

Since $\mathbf{S} = (T^\circ_{1|\mathscr{N}}, \ldots, T^\circ_{n-1|\mathscr{N}})$ is a doubly commuting $(n-1)$-tuple of log-hyponormal operators on $\mathscr{N}$ and $(z_1 e^{b_1}, \ldots, z_{n-1} e^{b_{n-1}}) \in \sigma_T(\mathbf{S})$, by Theorems 1 and D it follows that there exists a non-zero vector y° in $\mathscr{N}$ such that

$$(T_j^\circ - z_j e^{b_j})^* y^\circ = 0 \text{ for every } j\ (j = 1, \ldots, n-1).$$

Let

$$\mathscr{L} = \bigcap_{j=1}^{n-1} \mathrm{Ker}((T_j^\circ - z_j e^{b_j})^*).$$

Then $\mathscr{N} \cap \mathscr{L}$ is a non-zero subspace of $\mathscr{K}$. Hence we have $(z_n, e^{b_n}) \in \sigma_{jp}(U^\circ_{n|\mathscr{L}}, A(n)^\circ_{|\mathscr{L}})$ and $(z_n, e^{b_n}) \in \sigma_X(U^\circ_{n|\mathscr{L}}, \log|T_{n|\mathscr{L}}|^\circ)$. Also by induction we have

$$z_n e^{b_n} \in \sigma(T^\circ_{n|\mathscr{L}}).$$

Since $T^\circ_{n|\mathscr{L}}$ is a log-hyponormal operator on $\mathscr{L}$, there exists a non-zero vector $w^\circ \in \mathscr{L}$ such that

$$(T_n^\circ - z_n e^{b_n})^* w^\circ = 0\ (= (T_j^\circ - z_j e^{b_j})^* w^\circ\ (j = 1, \ldots, n-1)).$$

Hence there exists a sequence $\{x_m\}$ of unit vectors such that

$$(T_j - z_j e^{b_j})^* x_m \to 0 \text{ for every } j\ (j = 1, \ldots, n).$$

Let $a_j = e^{b_j}$ $(j = 1, \ldots, n)$. Then we have $(z_1 a_1, \ldots, z_n a_n) \in \sigma_T(\mathbf{T})$.

Conversely, we assume that $(z_1 a_1, \ldots, z_n a_n) \in \sigma_T(\mathbf{T})$. Also we assume that the theorem holds for doubly commuting $(n-1)$-tuples of log-hyponormal operators. By Theorem 1 there exists a sequence $\{x_m\}$ of unit vectors such that

$$(T_j - z_j a_j)^* x_m \to 0 \text{ for every } j\ (j = 1, \ldots, n). \tag{1}$$

Consider the extension space $\mathscr{K}$ of $\mathscr{H}$ and let

$$\mathscr{U} = \mathrm{Ker}((T_n^\circ - z_n a_n)^*).$$

Then by Theorem D and (1) there exists $z^{o} \in \mathscr{U}$ such that

$$(T_j^{o} - z_j a_j)^* z^{o} = 0 \text{ for every } j \ (j = 1, \ldots, n-1).$$

Since $(T_1^{o}, \ldots, T_{n-1}^{o})$ is a commuting $(n-1)$-tuple of log-hyponormal operators on $\mathscr{U}$, it follows that $(z_1 a_1, \ldots, z_{n-1} a_{n-1}) \in \sigma_T(T_{1|\mathscr{U}}^{o}, \ldots, T_{n-1|\mathscr{U}}^{o})$. By the assumption of induction we have

$$(z_1, \ldots, z_{n-1}, \log a_1 \cdots \log a_{n-1}) \in \sigma_X(\mathbf{U}', A'),$$

where $\mathbf{U}' = (U_{1|\mathscr{U}}^{o}, \ldots, U_{n-1|\mathscr{U}}^{o})$ and $A' = \log|T_{1|\mathscr{U}}^{o}| \cdots \log|T_{n-1|\mathscr{U}}^{o}|$. Hence there exist $(l_1, \ldots, l_{n-1}) \in [0, 1]^{n-1}$ and a non-zero vector $u^{o} \in \mathscr{U}$ such that

$$(U_j^{o} - z_j) u^{o} = (A(1)^{o} \cdots A(n-1)^{o} - \log a_1 \cdots \log a_{n-1}) u^{o} = 0,$$

where $A(j) = (\mathscr{S}_j(l_j))(\log|T_j|)$ for every $j \ (j = 1, \ldots, n-1)$.

Next, let

$$\mathscr{V} = \bigcap_{j=1}^{n-1} \operatorname{Ker}(U_1^{o} - z_j) \bigcap \operatorname{Ker}(A(1)^{o} \cdots A(n-1)^{o} - \log a_1 \cdots \log a_{n-1}).$$

Since $\mathscr{U} \bigcap \mathscr{V}$ is a non-zero subspace, we have

$$z_n a_n \in \sigma(T_{n|\mathscr{V}}^{o}).$$

Hence by Theorem 1 there exists $0 \leqslant l_n \leqslant 1$ such that $z_n a_n \in \sigma(U_n A(n))$, where $A(n) = (\mathscr{S}_n(l_n))(\log|T_n|)$. Since $U_n A(n)$ is a normal operator, by Theorem D there exists $v^{o} \in \mathscr{V}$ such that

$$(U_n^{o} - z_n) v^{o} = (A(n)^{o} - \log a_n) v^{o} = 0.$$

Let $l = (l_1, \ldots, l_n)$ and $A(l) = \prod_{j=1}^{n} A(j)$. Since by Theorem C we have

$$(z_1, \ldots, z_n, \log a_1 \cdots \log a_n) \in \sigma_p(U_1^{o}, \ldots, U_n^{o}, A(l)^{o}),$$

it follows, from the definition of the Xia spectrum, that

$$(z_1, \ldots, z_n, \log a_1 \cdots \log a_n) \in \sigma_X(\mathbf{U}, A). \quad \blacksquare$$

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References

[1] S. K. Berberian, Approximate proper vectors, *Proceedings of the American Mathematical Society* **13** (1962), 111–14.

[2] M. Chō, Spectral properties of p-hyponormal operators, *Glasgow Mathematical Journal* **36** (1994), 117–22.

[3] M. Chō, Relations between the Taylor spectrum and the Xia spectrum, to appear in *Proceedings of the American Mathematical Society*.

[4] M. Chō and T. Huruya, Putnam's inequality for p-hyponormal n-tuples, *Glasgow Mathematical Journal* **41** (1999), 13–17.

[5] M. Chō and M. Itoh, Putnam's inequality for p-hyponormal operators, *Proceedings of the American Mathematical Society* **123** (1995), 2435–40.

[6] M. Chō and M. Itoh, On spectra of p-hyponormal operators, *Integral Equations and Operator Theory* **23** (1995), 287–93.

[7] R. Curto, On the connectedness of invertible n-tuples, *Indiana University Mathematics Journal* **29** (1980), 393–406.

[8] R. Curto, P. Muhly and D. Xia, A trace estimate for p-hyponormal operators, *Integral Equations and Operator Theory* **6** (1983), 507–14.

[9] B. Duggal, A Putnam area inequality for the spectrum of n-tuples of p-hyponormal operators, *Glasgow Mathematical Journal* **41** (1999), 385–91.

[10] C. R. Putnam, *Commutation properties of Hilbert space operators*, Springer-Verlag, Berlin, 1967.

[11] K. Tanahashi, Putnam's inequality for log-hyponormal operators, preprint, 1998.

[12] K. Tanahashi, On log-hyponormal operators, to appear in *Integral Equations and Operator Theory*.

[13] J. L. Taylor, A joint spectrum for several commuting operators, *Journal of Functional Analysis* **6** (1970), 172–91.

[14] D. Xia, On the semi-hyponormal n-tuple of operators, *Integral Equations and Operator Theory* **6** (1983), 879–98.

[15] D. Xia, *Spectral theory of hyponormal operators*, Birkhäuser Verlag, Boston, 1983.