THE SEMIGROUP EFFICIENCY OF GROUPS AND MONOIDS

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Abstract

A finite semigroup (respectively monoid or group) S is said to be efficient if it can be defined by a semigroup (respectively monoid or group) presentation $\langle A \mid R \rangle$ with $|R|-|A| = \mathrm{rank}(H_2(S^1))$. In this paper we show that a group is efficient as a group if and only if it is efficient as a monoid, but some efficient monoids are not efficient as semigroups. Moreover, we show that certain classes of efficient groups are efficient as semigroups.

1. Introduction

Monoid and semigroup presentations of groups have been investigated in, for example, [7] or [10]. The aim of this paper is to investigate the efficiency of groups as monoids and as semigroups. We show that any efficient group is efficient as a monoid and further that certain efficient groups are efficient as semigroups.

Let A be an alphabet. We denote by A^+ the free semigroup on A consisting of all non-empty words over A, and by A^* the free monoid $A^+ \cup \{\varepsilon\}$ where ε denotes the empty word. A *semigroup* (*monoid*) *presentation* is an ordered pair $\langle A \mid R \rangle$, where $R \subseteq A^+ \times A^+$ ($R \subseteq A^* \times A^*$). A semigroup (monoid) S is said to be *defined* by the semigroup (monoid) presentation $\langle A \mid R \rangle$ if $S \cong A^+/\rho$ ($S \cong A^*/\rho$), where ρ is the congruence on A^+ (A^*) generated by R.

The *deficiency* of a finite semigroup (monoid or group) presentation $\mathscr{P} = \langle A \mid R \rangle$ is defined to be $|R|-|A|$, and is denoted by $\mathrm{def}(\mathscr{P})$. The *semigroup deficiency* $\mathrm{def}_S(S)$ of a finitely presented semigroup S is given by

$$\mathrm{def}_S(S) = \min\{\mathrm{def}(\mathscr{P}) \mid \mathscr{P} \text{ is a finite semigroup presentation for } S\}.$$

The *monoid deficiency* def_M and the *group deficiency* def_G can be defined similarly. So a finitely presented group G has three deficiencies, namely $\mathrm{def}_G(G)$, $\mathrm{def}_M(G)$ and $\mathrm{def}_S(G)$, and a finitely presented monoid M has two deficiencies, namely $\mathrm{def}_M(M)$ and $\mathrm{def}_S(M)$.

It is well known that if S is a finite semigroup (monoid or group) then $\mathrm{def}_S(S) \geqslant 0$. A better bound for the deficiency of S is obtained by considering the second integral homology $H_2(S)$ of S. It is well known that $H_2(G)$ (which is also known as the Schur multiplier of G) of a finite group G is a finite abelian group and that $\mathrm{def}_G(G) \geqslant \mathrm{rank}(H_2(G))$ (see [9] or [12]). It has been shown by S. J. Pride (unpublished) that, for a finite monoid M, $\mathrm{def}_M(M) \geqslant \mathrm{rank}(H_2(M))$. Note that the semigroup presentations for a semigroup S are precisely monoid presentations (without the trivial relations $1 = 1$) for the monoid S^1 obtained from S by adjoining an identity, whether or not S already has one. It follows that $\mathrm{def}_S(S) = \mathrm{def}_M(S^1)$, and so $\mathrm{def}_S(S) \geqslant \mathrm{rank}(H_2(S^1))$.

We call a finite semigroup (or monoid or group) S *efficient* as a semigroup (or monoid or group) if $\mathrm{def}_S(S) = \mathrm{rank}(H_2(S^1))$ (or $\mathrm{def}_M(S) = \mathrm{rank}(H_2(S))$ or $\mathrm{def}_G(S) = \mathrm{rank}(H_2(S))$ and *inefficient* otherwise. Hence there are three potentially different notions of efficiency for a group and two for a monoid.

The purpose of this paper is to investigate the relationship between these notions. The efficiency of groups has been investigated for a long time (see [12]). The study of the efficiency of semigroups and monoids is more recent. For some early results, see [2].

2. Comparing def_S, def_M and def_G

Since every semigroup (monoid) presentation for a group G is also a group presentation for G, we have $\mathrm{def}_S(G) \geqslant \mathrm{def}_G(G)$ and $\mathrm{def}_M(G) \geqslant \mathrm{def}_G(G)$. We now prove that, in fact, $\mathrm{def}_M(G) = \mathrm{def}_G(G)$.

Theorem 2.1. *Let $\mathscr{P}_G = \langle A \,|\, R \rangle$ be a finite group presentation for a group G. Consider the monoid presentation*

$$\mathscr{P}_M = \langle A, A' \,|\, R', aa'a = 1\ (a \in A) \rangle,$$

where $A' = \{a' \,|\, a \in A\}$ is a copy of A and R' is obtained from R by replacing a^{-1} (if it occurs) by aa' in every relation in R. Then $\mathscr{P}_M$ defines G as a monoid. In particular, G is efficient as a group if and only if it is efficient as a monoid.

We will need the following technical lemma.

Lemma 2.2. *Let $\mathscr{P} = \langle A \,|\, R \rangle$ be a semigroup presentation.*

(i) *If there exists a word $e \in A^+$ such that, for each $a \in A$, $ea = a$ (left identity) and $u_a a = e$ for some $u_a \in A^+$ (left inverse), then the semigroup defined by $\mathscr{P}$ is a group.*

(ii) *If the semigroup S defined by $\mathscr{P}$ is a group, then S is isomorphic to the group defined by $\mathscr{P}$ when we consider $\mathscr{P}$ as a group presentation.*

For a proof of Lemma 2.2(i) see [6, proposition I.1.3], and for a proof of Lemma 2.2(ii) see [5, theorem 3(iii)].

Proof of Theorem 2.1. *Since* $aa' = aa'(aa'a) = (aa'a)a'a = a'a$, it follows that $a^2a' = a'a^2 = aa'a = 1$. We deduce that a^2 is an inverse of a' and aa' is an inverse of a. Therefore $\mathscr{P}_M$ defines a group. It is clear that this group is isomorphic to the group G.

Since $\mathrm{def}(\mathscr{P}_G) = \mathrm{def}(\mathscr{P}_M)$, it follows that if a group G is efficient as a group then it is efficient as a monoid. For the converse, if G is efficient as a monoid, then an efficient monoid presentation is also an efficient group presentation. ■

Note that, for a general group G, we may have $\mathrm{def}_S(G) > \mathrm{def}_G(G)$. This follows immediately from the following easy result.

Theorem 2.3. *If $\langle A \mid R \rangle$ is a semigroup presentation for a group G then $|R| \geqslant |A|$.*

PROOF. If R contains a relation of the form $a = b$ where a and b are distinct generators from A, then we can eliminate one of a or b without changing the deficiency. So, without loss of generality, we may assume that R contains no such relations. Now, for an arbitrary $a \in A$, let $w_a \in A^+$ be a word representing the inverse of a in G. Then the relation $a = aw'_a a$ holds in G, i.e. $aw'_a a$ can be obtained from a by applying relations from R. We conclude that for each $a \in A$ there is a relation of the form $a = u_a$ in R. Moreover, since $u_a \notin A$, all these relations are distinct, and so $|R| \geqslant |A|$. ■

However, in the following section we will show that $\mathrm{def}_G(G) = \mathrm{def}_S(G)$ for various well-known groups.

We now turn to examining $\mathrm{def}_M(M)$ and $\mathrm{def}_S(M)$ for a monoid M. Here it is easy to see that $\mathrm{def}_M(M) \leqslant \mathrm{def}_S(M) + 1$. Indeed, if $\langle A \mid R \rangle$ is any semigroup presentation for M, and if $e \in A^+$ is any word representing 1_M, then $\langle A \mid R, e = 1 \rangle$ is a monoid presentation for M. We are able to prove the stronger, expected result $\mathrm{def}_M(M) \leqslant \mathrm{def}_S(M)$ in the following special case.

Theorem 2.4. *If M is a monoid in which every left invertible element is also right invertible* (*in particular, if M is finite*) *then* $\mathrm{def}_S(M) \geqslant \mathrm{def}_M(M)$. *Further, if M is efficient as a semigroup then M is also efficient as a monoid.*

PROOF. Denote by G the group of units of M. It is easy to see that the condition on invertible elements of M implies that $M \backslash G$ is an ideal of M. This in turn implies that any semigroup presentation $\mathscr{P} = \langle A \mid R \rangle$ of M contains a subpresentation $\mathscr{P}_1 = \langle A_1 \mid R_1 \rangle$ ($A_1 \subseteq A$, $R_1 = R \cap (A_1^+ + A_1^+)$) defining G. Since $\mathscr{P}_1$ is also a group presentation for G, it follows that there exists a monoid presentation $\mathscr{P}_2 = \langle A_2 \mid R_2 \rangle$ for G with $\mathrm{def}(\mathscr{P}_2) = \mathrm{def}(\mathscr{P}_1)$ by Theorem 2.1.

For each $a \in A_1$, let $\bar{a} \in A_2^*$ be a word representing the same element of G, and let $\overline{R \backslash R_1}$ be the set of relations obtained by replacing every occurrence of every $a \in A_1$ in every relation from $R \backslash R_1$ by the corresponding word $\bar{a}$. It is now easy to prove that $\mathscr{P}_3 = \langle (A \backslash A_1) \cup A_2 \mid \overline{R \backslash R_1} \cup R_2 \rangle$ is a monoid presentation for M and that $\mathrm{def}(\mathscr{P}_3) = \mathrm{def}(\mathscr{P})$, thus completing the proof of the first statement. The proof of the second statement follows on noting that $H_2(M) = H_2(M^1)$, which can be deduced from the fact that the bar resolution for M^1 is the same as the standard resolution for M; see [8]. ■

Combining Theorems 2.1 and 2.3, we see that the inequality in Theorem 2.4 may be strict. An alternative easy example is given below.

Example 2.5. Consider the three-element monoid M with the (efficient) monoid presentation $\langle a \mid a^3 = a^2 \rangle$. We may use techniques similar to those used in [2] to prove the inefficiency of zero-semigroups and free semilattices to show that M cannot be presented efficiently by a semigroup presentation. It may also be shown that a semigroup presentation with minimal deficiency is $\langle a, b \mid a^3 = a^2, bab = a, b^2 = b \rangle$. For details see [1].

3. The efficiency of certain groups as semigroups

Next we consider finite groups. It is still an open problem whether $\mathrm{def}_G(G) = \mathrm{def}_S(G)$ holds for all finite groups G. From now on, we demonstrate that equality holds for certain finite (efficient) groups.

For $n \geqslant 3$ odd, the following group presentation

$$\mathcal{P}_n = \langle x, y \mid y^n x^2 = 1, y^{\frac{n-1}{2}} x^3 y^{\frac{n-1}{2}} = x \rangle$$

defines the dihedral group D_{2n} of order $2n$. Indeed, from the first relation it follows that x^2 is central in the group G defined by $\mathcal{P}_n$. Abelianising G proves that $x^2 \in G'$. Since the quotient $G/\langle x^2 \rangle$ is obviously isomorphic to D_{2n}, and since D_{2n} has trivial multiplier for n odd, it follows that $x^2 = 1$ in G, and so $G \cong D_{2n}$.

Theorem 3.1. *For n even, $n \geqslant 2$, the dihedral group D_{2n} of order $2n$ is defined by the semigroup presentation*

$$\langle a, b \mid a^3 = a, a^2 = b^n, ab^{n-1}a = b \rangle,$$

and for n odd, $n \geqslant 3$, it is defined by the semigroup presentation

$$\mathcal{Q}_n = \langle x, y \mid y^n x^2 y = y, y^{\frac{n-1}{2}} x^3 y^{\frac{n-1}{2}} = x \rangle.$$

Hence D_{2n} is efficient as a semigroup.

Proof. The semigroup efficiency of D_{2n} with n even, $n \geqslant 2$, is known (see [2, theorem 2.2]). Now we consider the case when n is odd. From $y^n x^2 y = y$ and

$$y^n x^3 = y^n x^2 y^{\frac{n-1}{2}} x^3 y^{\frac{n-1}{2}} = y^{\frac{n-1}{2}} x^3 y^{\frac{n-1}{2}} = x$$

it follows that $y^n x^2$ is a left identity for the semigroup defined by $\mathcal{Q}_n$, and $y^n x$ is a left inverse of x. It is easy to show that $y^n x y^{\frac{n-1}{2}} x^3 y^{\frac{n-3}{2}}$ is a left inverse of y. It follows from Lemma 2.2(i) that $\mathcal{Q}_n$ defines a group. By Lemma 2.2(ii) and the discussion preceding the theorem, it follows that this group is D_{2n}. Therefore D_{2n} is efficient as a semigroup for each n. ■

The group efficiency of the Projective Special Linear group $PSL(2,p)$ has been studied in many papers (see for example [3], [11] and [13]). We now consider its efficiency as a semigroup.

Theorem 3.2. *The semigroup presentation*

$$\mathcal{G}_p = \langle x, y \mid x^3 = x, yxyxy = x, (xy^4 x y^{\frac{p+1}{2}})^2 y^{p+1} = y \rangle$$

defines a group for each positive odd integer p. In particular, if p is an odd prime, than $\mathcal{G}_p$ defines $PSL(2,p)$, and so $PSL(2,p)$ is efficient as a semigroup.

PROOF. From the second relation of $\mathscr{G}_p$, we have

$$x^2y = (yxyxy)xy \equiv yx(yxyxy) = yx^2,$$

so that x^2 is central.

Next, from the second and third relations, we have

$$((xy^4xy^{\frac{p+1}{2}})^2y^p)x = (xy^4xy^{\frac{p+1}{2}})^2y^{p+1}xyxy = yxyxy = x.$$

It follows from this and the third relation that $(xy^4xy^{\frac{p+1}{2}})^2y^p$ is a left identity.

It is clear that $(xy^4xy^{\frac{p+1}{2}})^2y^{p-1}$ is a left inverse for y. Now we show that x is its own left inverse. Since $(xy^4xy^{\frac{p+1}{2}})^2y^p$ is a left identity and x^2 is central, we have

$$x^2 = ((xy^4xy^{\frac{p+1}{2}})^2y^p)x^2 = x^3y^4xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}}y^p = (xy^4xy^{\frac{p+1}{2}})^2y^p,$$

as required. Thus it follows from Lemma 2.2 that $\mathscr{G}_p$ defines a group with identity $(xy^4xy^{\frac{p+1}{2}})^2y^p$. When we consider $\mathscr{G}_p$ as a group presentation, it defines $PSL(2,p)$ for p an odd prime (see [4, theorem 3]), and so it also defines it as a semigroup presentation. Since $H_2(PSL(2,p))$ is the cyclic group of order 2, it follows that $PSL(2,p)$ is efficient as a semigroup. ■

Finally we give examples of deficiency zero efficient semigroups, namely the Special Linear groups $SL(2,p)$ with p a prime. We also prove that $SL(2,2^3)$, which is the smallest non-abelian simple group with trivial second homology, is efficient as a semigroup.

Theorem 3.3. *The semigroup presentation*

$$\mathscr{P}_{p,k} = \langle x, y \mid yxyxy = x, (xy^4xy^{\frac{p+1}{2}})^2y^px^{2k}y = y\rangle$$

defines a group for all positive integers p and k. In particular, if p is an odd prime and $k = [p/3]$ *is the integer part of* $p/3$, *then* $\mathscr{P}_{p,k}$ *defines* $SL(2,p)$, *and so* $SL(2,p)$ *is efficient as a semigroup.*

PROOF. Since, from the first and second relations of $\mathscr{P}_{p,k}$, we have

$$((xy^4xy^{\frac{p+1}{2}})^2y^px^{2k})x = (xy^4xy^{\frac{p+1}{2}})^2y^px^{2k}yxyxy = yxyxy = x,$$

it follows from the second relation that $(xy^4xy^{\frac{p+1}{2}})^2y^px^{2k}$ is a left identity.

Since $(xy^4xy^{\frac{p+2}{2}})^2y^px^{2k-1}$ is a left inverse of x and $(xy^4xy^{\frac{p+1}{2}})^2y^px^{2k-1}yxyx$ is a left inverse of y, it follows from Lemma 2.2 that $\mathscr{P}_{p,k}$ defines a group. The result now follows from [4, theorem 4]. ■

Theorem 3.4. *The semigroup presentation*

$$\mathscr{P} = \langle x, y \mid y^3x^2y = y, ((y^2xyx)^4y^2x)^2(xy)^7xy^3 = x\rangle$$

defines $SL(2,2^3)$, *and so* $SL(2,2^3)$ *is efficient as a semigroup.*

PROOF. It follows from [4] that the group defined by $\mathscr{P}$ is $SL(2,2^3)$. By Lemma 2.2, it is enough to show that $\mathscr{P}$, as a semigroup presentation, defines a group.

From the second and first relations, we have

$$(y^3x^2)x = y^3x^2((y^2xyx)^4y^2x)^2(xy)^7xy^3 = ((y^2xyx)^4y^2x)^2(xy)^7xy^3 = x.$$

It follows from the first relation that y^3x^2 is a left identity.

Since y^3x is a left inverse of x and $y^3x((y^2xyx)^4y^2x)^2(xy)^7xy^2$ is a left inverse of y, it follows from Lemma 2.2 that $\mathscr{P}$ defines a group, as required. ■

As mentioned above, it is still an open question whether one can find a finite group G for which $\mathrm{def}_S(G)$ is not equal to $\mathrm{def}_G(G)$.

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