A HAHN–BANACH THEOREM FOR QUADRATIC FORMS

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Abstract

We give an estimate of the best constant C with the following property: any quadratic form Q on any hyperplane of a normed space E has an extension $\widetilde{Q}$ on E with $\|\widetilde{Q}\| \leq C \cdot \|Q\|$.

1. Introduction

Let E be a normed vector space and let F be a subspace of E. The well-known Hahn–Banach theorem asserts that every continuous linear form φ on F extends to E as a continuous linear form $\widetilde{\varphi}$.

More precisely, there exists an extension $\widetilde{\varphi}$ with $\|\widetilde{\varphi}\| \leq \|\varphi\|$. In fact, this is a quite easy consequence (using Zorn's lemma or some transfinite induction) of the same result when F is a hyperplane of E. A lot of work has been done to generalise the first statement for polynomials or holomorphic functions instead of linear forms; see, for instance, [2] and [3]. Positive results are obtained for some pairs of spaces (E, F), in particular when E is the bidual of F. Some negative results are also obtained; for instance, in [2] an example is given of an embedding of $F \simeq l_p$ ($p \in \mathbb{N}, p \geq 2$) into $E \simeq l_\infty$ for which the canonical polynomial $\sum x_n^p$ does not extend to a holomorphic function on a neighbourhood of 0 in E and of course does not extend to a continuous polynomial on E.

Here we are interested in the generalisation for quadratic forms of the second statement when F is a hyperplane of E. So the problem is as follows.

Find a constant C such that, for any hyperplane H in a normed vector space E and any continuous quadratic form Q on H, there exists a continuous extension $\widetilde{Q}$ on E with $\|\widetilde{Q}\| \leq C \cdot \|Q\|$. (Here the norm of a quadratic form is the supremum of its modulus on the unit ball.) Of course $C = 1$ does not work; otherwise, using the same induction argument as in the Hahn–Banach theorem, the extension would be possible in all codimensions (finite or infinite).

It is easy to see that the problem is essentially a finite-dimensional one: more precisely, if a constant C works for finite-dimensional spaces, then the same constant works for all spaces.

For finite-dimensional space it is known that, for any hyperplane H in a normed vector space E, there exists a linear projection π from E to H with $\|\pi\| \leq 2$; so any quadratic form Q on H has an extension $\widetilde{Q} = Q \circ \pi$ with $\|\widetilde{Q}\| \leq 2^2 \|Q\|$ and the constant $C = 4$ works. Therefore the problem is to find a constant less than 4 and, if possible, the best one. It turns out that this best constant depends on the scalar field which is used, so for $K = \mathbb{R}$ or $K = \mathbb{C}$ we will denote by $C(K)$ the least constant C such that:

> For any normed K-vector space E, any hyperplane H in E and any continuous quadratic form Q on H there exists a continuous quadratic form $\widetilde{Q}$ on E extending Q with $\|\widetilde{Q}\| \leq C \cdot \|Q\|$.

Then we have the following theorem.

Theorem. $C(\mathbb{R}) = 2$ *and* $\frac{7}{3} \leq C(\mathbb{C}) \leq 2\sqrt{2}$.

2. An upper bound for $C(K)$

2.1. Notation

If E is a vector space, let us denote by $\bigotimes^2 E$ the tensor product $E \otimes E$ and by $S_2(E)$ the quotient of $\bigotimes^2 E$ by the subspace generated by the $x \otimes y - y \otimes x$ (x and y running through E). For x and y in E we will denote by xy the class of $x \otimes y$ in the quotient $S_2(E)$ and by x^2 the product xx (notice that one has $xy = yx$). Now any quadratic form Q on E defines, by polarisation, a bilinear symmetric form Q_1 on E and hence a linear form Q_2 on $\bigotimes^2 E$ which, because of the symmetry, can be factorised to define a linear form Q_3 on the quotient space $S_2(E)$. We then have $Q_1(x, y) = \langle Q_3 \mid xy\rangle$ and $Q(x) = \langle Q_3 \mid x^2\rangle$.

Remark. There is a symmetrisation map σ defined on $\bigotimes^2 E$ by $\sigma(x \otimes y) = \frac{1}{2}(x \otimes y + y \otimes x)$ (see for instance [1]). This map is a projector of $\bigotimes^2 E$ on the subspace $\bigotimes^2_s E$ of the symmetric tensors. The kernel of σ is generated by the $x \otimes y - \sigma(x \otimes y) = \frac{1}{2}(x \otimes y - y \otimes x)$; since the space $S_2(E)$ is the quotient of $\bigotimes^2 E$ by this kernel, it is isomorphic to the image of σ. So $S_2(E)$ can be canonically identified with $\bigotimes^2_s E$. Notice that, under this identification, the product xy is identified with $\frac{1}{2}(x \otimes y + y \otimes x)$, which is sometimes called the symmetric tensor product of x and y and x^2 is identified with $x \otimes x$.

If E is a normed space then the norm of a quadratic form Q is defined by $\|Q\| = \sup_{\|x\|<1} |Q(x)|$. We want to define a norm v_E on $S_2(E)$ such that the norm of the quadratic form Q is equal to the norm of the linear form Q_3.

Any u in $S_2(E)$ can be written $u = \sum \alpha_p x_p^2$ with $\alpha_p \in \mathbb{C}$ and $x_p \in E$; moreover, we may suppose that $\|x_p\| \leq 1$. Let us denote by $v_E(u)$ the infimum of the sums $\sum |\alpha_p|$ for all these decompositions of u. In other words, v_E is the Minkowski functional of the balanced convex hull in $S_2(E)$ of the set of the x^2 for x running through the closed unit ball of E. It is now easy to prove that v_E is a norm on $S_2(E)$ having the required property. Notice that for $x \in E$ one has $v_E(x^2) \leq \|x\|^2$ (in fact $v_E(x^2) = \|x\|^2$).

With the norm v_E the correspondence $Q \mapsto Q_3$ is one-to-one and isometric, so we will identify the quadratic form Q and the linear form Q_3 and we will denote again by Q the form Q_3.

2.2. Using the usual Hahn–Banach theorem

Let E be a normed vector space, H a hyperplane of E, Q a continuous quadratic form on H, and C a constant; we want to find sufficient conditions (on C) to ensure the existence of an extension of Q with norm less than $C \cdot \|Q\|$. The injection $H \hookrightarrow E$ gives an injection $S_2(H) \hookrightarrow S_2(E)$; each of these spaces is endowed with a norm v_H (resp. v_E) and Q is a linear form on $S_2(H)$ with norm $\|Q\|$. The problem is now to extend this linear form by a linear form on $S_2(E)$ with norm less than $C \cdot \|Q\|$.

By the classical Hahn–Banach theorem such an extension exists if one as $N(Q) \leq C \cdot \|Q\|$, where $N(Q)$ denotes the norm of the linear form Q when $S_2(H)$ is endowed with the restriction of v_E. The problem is that this restriction is not the norm v_H (otherwise the Hahn–Banach theorem would be true for quadratic forms!). It is now clear that a sufficient condition for the existence of the extension is:

$$\text{for any } u \text{ in } S_2(H) \quad v_H(u) \leq C \cdot v_E(u).$$

(In fact this condition is also necessary to extend all the quadratic forms Q with $\|Q\| \leq C \cdot \|Q\|$.)

2.3. $C(\mathbb{R}) \leq 2,\ C(\mathbb{C}) \leq 2\sqrt{2}$

With the above notation and using the definition of v_E we have to prove the following:

$$\text{let } u = \sum \alpha_p x_p^2 \text{ with } x_p \in E,\ \|x_p\| \leq 1 \text{ and } u \in S_2(H), \text{ then } v_H(u) \leq C \cdot \sum |\alpha_p|$$
$$\text{with } C = 2 \text{ (resp. } C = 2\sqrt{2}) \text{ if the scalar field is } \mathbb{R} \text{ (resp. } \mathbb{C}).$$

Let us denote by φ a linear form on E such that $H = \operatorname{Ker}\varphi$. For any linear form ψ on E the product $Q = \varphi\psi$ is a quadratic form and $\langle Q \mid u \rangle = \sum \alpha_p \varphi(x_p)\psi(x_p)$; since the restriction of Q to H is zero and $u \in S_2(H)$ we have $\langle Q \mid u \rangle = 0$. So, if $\varphi(x_p) = \lambda_p$, for any ψ we have $\psi(\sum \alpha_p \lambda_p x_p) = 0$, hence $\sum \alpha_p \lambda_p x_p = 0$. Applying the form φ we get $\sum \alpha_p \lambda_p^2 = 0$.

From these equalities, for any index q we have:

$$\sum_p \alpha_p (\lambda_p x_q - \lambda_q x_p)^2 = \lambda_q^2 \cdot u,$$

and, if θ_q is such that $|\alpha_q \lambda_q^2| = \alpha_q \lambda_q^2 \theta_q \quad (|\theta_q| = 1)$,

$$\sum_{p,q} \alpha_p \alpha_q \theta_q (\lambda_p x_q - \lambda_q x_p)^2 = A \cdot u \text{ with } A = \sum |\alpha_q \lambda_q^2|.$$

By symmetry one gets $\sum_{p,q} \alpha_p \alpha_q (\theta_p + \theta_q)(\lambda_p x_q - \lambda_q x_p)^2 = 2A \cdot u$.

Clearly $\lambda_p x_q - \lambda_q x_p \in \operatorname{Ker}\varphi = H$, so $v_H((\lambda_p x_q - \lambda_q x_p)^2) \leq \|\lambda_p x_q - \lambda_q x_p\|^2$.

Since $\|x_k\| \leq 1$, we have $\|\lambda_p x_q - \lambda_q x_p\|^2 \leq (|\lambda_p| + |\lambda_q|)^2 \leq 2(|\lambda_p|^2 + |\lambda_q|^2)$; therefore

$$2A v_H(u) \leq 2 \sum_{p,q} |\alpha_p||\alpha_q||\theta_p + \theta_q|(|\lambda_p|^2 + |\lambda_q|^2).$$

By symmetry this gives $Av_H(u) \leq 2\sum_{p,q} |\alpha_p||\alpha_q||\theta_p + \theta_q||\lambda_p|^2$.

So, if we prove, for any index q, $\sum_p |\alpha_p||\theta_p + \theta_q||\lambda_p|^2 \leq (C/2) \cdot A$ we obtain the desired estimate $v_H(u) \leq C \cdot \sum |\alpha_q|$ (the case $A = 0$ is obvious).

(A) Suppose that $K = \mathbb{R}$.

Then θ_k equals 1 or -1 and $|\theta_p + \theta_q| = 1 + \theta_p\theta_q$. Consequently,

$$\sum_p |\alpha_p||\theta_p + \theta_q||\lambda_p|^2 = \sum_p |\alpha_p||\lambda_p|^2 + \sum_p \theta_p\theta_q|\alpha_p||\lambda_p|^2,$$

the first sum is A and the second is

$$\theta_q \sum_p \theta_p|\alpha_p||\lambda_p|^2 = \theta_q \sum_p \theta_p^2\alpha_p\lambda_p^2 = \theta_q \sum_p \alpha_p\lambda_p^2 = 0.$$

The proof is complete with $C = 2$.

(B) Suppose that $K = \mathbb{C}$.

The Cauchy–Schwarz inequality gives

$$\sum_p |\alpha_p||\theta_p + \theta_q||\lambda_p|^2 \leq \left(\sum_p |\alpha_p||\lambda_p|^2 \cdot \sum_p |\alpha_p||\theta_p + \theta_q|^2|\lambda_p|^2\right)^{1/2}.$$

The second sum in brackets can be expanded as

$$2 \cdot \sum_p |\alpha_p||\lambda_p|^2 + \sum_p \theta_p\overline{\theta_q}|\alpha_p||\lambda_p|^2 + \sum_p \overline{\theta_p}\theta_q|\alpha_p||\lambda_p|^2.$$

The first of these sums is A, the third one is $\theta_q \sum_p \alpha_p\lambda_p^2 = 0$, and the second one is the conjugate. So the expression in brackets is $2A^2$ and the proof is complete with $C = 2\sqrt{2}$.

3. A lower bound for $C(K)$

Let us consider the following data: an integer N ($N \geq 2$),

for $1 \leq p \leq N$ $\alpha_p \in K$ such that $\sum \alpha_p = 0$,

for $1 \leq p \leq N$ and $1 \leq q \leq N$ $\epsilon_{p,q} \in K$ such that $|\epsilon_{p,q}| \leq 1$, $\epsilon_{p,q} = \epsilon_{q,p}$ and $\sum_p \alpha_p\epsilon_{p,q}$ is a constant A independent of q (which implies $\sum_{p,q} \alpha_p\alpha_q\epsilon_{p,q} = A\sum_q \alpha_q = 0$).

The space K^N with canonical basis $(e_p)_{1\leq p\leq N}$ is endowed with the norm $\|x\| = \sum |x_p|$. Select λ_p such that $\sum \lambda_p\alpha_p = A$ (for instance $\lambda_p = \epsilon_{p,1}$) and let $u = \sum \alpha_p e_p$. For $x = (x_1, \ldots, x_N)$, define $\varphi(x) = \sum x_p$ and $\psi(x) = \sum \lambda_p x_p$, so that $\varphi(u) = 0$ and $\psi(u) = A$.

Denote by E the quotient space K^N/Ku endowed with the quotient norm and by H the hyperplane $\operatorname{Ker}\varphi/Ku$.

Let us define a quadratic form P on K^N by:

$$P(x) = \sum_{p,q} \epsilon_{p,q} x_p x_q - 2\varphi(x)\psi(x).$$

For $t \in K$ we have

$$\begin{aligned} P(x+tu) = P(x) &+ 2t \sum_{p,q} \epsilon_{p,q}\alpha_p x_q + t^2 \sum_{p,q} \epsilon_{p,q}\alpha_p\alpha_q \\ &-2t\,(\varphi(u)\psi(x) + \varphi(x)\psi(u)) - 2t^2\varphi(u)\psi(u). \end{aligned}$$

Since $\varphi(u) = 0$, $\psi(u) = A = \sum_p \epsilon_{p,q}\alpha_p$ and $\sum_{p,q} \epsilon_{p,q}\alpha_p\alpha_q = 0$, we see that $P(x+tu)$ is independent of t; hence P defines a quadratic form $\dot{P}$ on the quotient space E. Denote by Q the restriction of $\dot{P}$ to H. On $\operatorname{Ker}\varphi$ we have $P(x) = \sum_{p,q} \epsilon_{p,q} x_p x_q$ and, by $|\epsilon_{p,q}| \leq 1$, we conclude that $|P(x)| \leq \|x\|^2$ and so $\|Q\| \leq 1$.

Let us consider now an extension $\widetilde{Q}$ of Q to E and denote by $\dot{e}_p$ the canonical image of e_p in E. For any vector v in E let us consider the sum

$$S = \sum_p \alpha_p \widetilde{Q}(\dot{e}_p + v) = \sum_p \alpha_p \widetilde{Q}(\dot{e}_p) + 2\sum_p \alpha_p \widetilde{Q}'(\dot{e}_p, v) + \sum_p \alpha_p \widetilde{Q}(v)$$

(where $\widetilde{Q}'$ denotes the symmetric bilinear form defined by $\widetilde{Q}$). The second sum is $\widetilde{Q}'(\sum_p \alpha_p \dot{e}_p, v)$, which is zero since $\sum_p \alpha_p \dot{e}_p = 0$; the third one is also zero since $\sum_p \alpha_p = 0$.

So we have proved that S does not depend on v. When $v = -\dot{e}_1$ the vectors $\dot{e}_p - \dot{e}_1$ are in H (because $\varphi(e_p) = \varphi(e_1) = 1$) and so S depends only on Q and not on the extension $\widetilde{Q}$. In particular, we can compute S with $v = 0$ and $\widetilde{Q} = \dot{P}$. We obtain

$$S = \sum_p \alpha_p P(e_p) = \sum_p \alpha_p \epsilon_{p,p} - 2\sum_p \alpha_p \lambda_p = \sum_p \alpha_p \epsilon_{p,p} - 2A.$$

We can also compute S with $v = 0$ and $\widetilde{Q}$ such that $\|\widetilde{Q}\| \leq C(K)\|Q\|$; since $\|Q\| \leq 1$ and $\|\dot{e}_p\| \leq 1$ we obtain

$$|S| \leq C(K)\sum_p |\alpha_p| \text{ and, if the } \alpha_p \text{ are not all zero, } C(K) \geq \frac{|\sum_p \alpha_p \epsilon_{p,p} - 2A|}{\sum_p |\alpha_p|}.$$

Example 1. $K = \mathbb{R}$. For any $n \in \mathbb{N}$ ($n \geq 1$) let us use the above construction with the following data: $N = 2n$, $\alpha_p = (-1)^p$, $\epsilon_{p,p} = -\alpha_p$ and $\epsilon_{p,q} = (\alpha_p + \alpha_q)/2$ when $p \neq q$. The hypotheses are clearly satisfied with $A = n - 2$. Now $\sum_p \alpha_p \epsilon_{p,p} = -2n$; hence $C(\mathbb{R}) \geq (4n-4)/2n = 2 - 2/n$. This must be true for all $n \geq 1$, so $C(\mathbb{R}) \geq 2$.

Example 2. $K = \mathbb{C}$. For any $n \in \mathbb{N}$ ($n \geq 1$) let us use the above construction with the following data: $N = 3n$, $\alpha_p = \mathrm{e}^{2ip\pi/3}$ and, if $\alpha_p = \alpha_q$ and $p \neq q$, then $\epsilon_{p,q} = -\alpha_p\alpha_q$;

otherwise $\epsilon_{p,q} = \alpha_p\alpha_q$. The hypotheses are clearly satisfied with $A = -2(n-1)$. Now $\sum_p \alpha_p\epsilon_{p,p} = 3n$; hence $C(\mathbb{C}) \geq (7n-4)/3n$. This must be true for all $n \geq 1$, so $C(\mathbb{C}) \geq 7/3$.

References

[1] J. M. Ansemil and K. Floret, The symmetric tensor product of a direct sum of locally convex spaces, *Studia Mathematica* **129** (1998), 285–95.

[2] R. M. Aron and P. D. Berner, A Hahn–Banach extension theorem for analytic mappings, *Bulletin de la Société Mathématique de France* **106** (1978), 3–24.

[3] I. Zalduendo, A canonical extension for analytic functions on Banach spaces, *Transactions of the American Mathematical Society* **320** (1990), 747–63.